SHL
WITHDRAWN

Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1928-1929_29

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 29

LONDON:
PRINTED BY C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.4.

1929

BIBL.
LONDIN.
UNIV.

CONTENTS.

Papers published in the *Proceedings*, pp. 1–555.

Index of Papers published in this Volume arranged according to the names of Authors, p. 556.

Index according to Subjects, p. 557.

Corrigendum, p. 557.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SOLUBLE SEXTIC EQUATIONS

By W. E. H. Berwick.

[Received 14 October, 1927.—Read 15 December, 1927.]

1. *Introduction.*

It is known that each transitive permutation group of degree n corresponds to a separate type of irrationality which can be defined by an irreducible equation of the n-th degree. There are five such groups of the fifth degree, and, on a previous occasion*, I have shown how to distinguish the five kinds of irrationality definable by irreducible quintic equations.

Transitive groups of the sixth degree are sixteen in number, and the present purpose is to distinguish as many types of sextic irrationality. When the group of an irreducible equation is unsymmetric its permutations leave unaltered in form certain unsymmetric functions of the roots, and these functions assume rational values. A function ϕ associated with a group G, of order h, is a root of a resolvent, of degree $720/h$, whose coefficients are rational functions of those of the original sextic. When the group of the equation is G, or a sub-group of G, this resolvent has a rational root in the field of the coefficients. Each transitive group of the sixth degree is a sub-group of at least one of the four maximum sub-groups of the symmetric group. The orders of these maximum sub-groups are 48, 72, 120, 360, and the degrees of the corresponding *principal* resolvents are 15, 10, 6, 2.

* W. E. H. Berwick, "The condition that a quintic equation should be soluble by radicals", *Proc. London Math. Soc.* (2), 14 (1915), 301–307.

Brioschi* has given an historical account of the earlier sextic resolvents of a quintic, with references to the work of Jacobi, Malfatti and Cayley. In the paper cited I have shown that a general quintic $F_5(x, 1) = 0$ has a sextic resolvent

$$\phi^6 + A_1\phi^5 + A_2\phi^4 + \ldots + A_6 = 0,$$

whose coefficients $A_1, A_2, \ldots, A_6$ are invariants, of degrees 4, 8, ..., 24, of the quintic form $F_5(x, y)$. For a general sextic equation it has also been found convenient to construct four principal resolvents with invariant coefficients. A similar set of resolvents can be constructed for an equation of any degree n, the coefficient of ϕ^{d-r} in a resolvent of degree d (> 2) being an invariant of degree $4r$ or $2r$, according as n is odd or even.

My friend and former colleague, Mr. C. W. Gilham, has calculated the sextic and decimic resolvents given in § 4, while the quindecimic resolvent has been obtained by Mr. J. Lamb†, a research student in the University of Leeds. I have only partially verified the calculations involved in these results.

If $\phi_1, \phi_2, \ldots, \phi_d$ are the roots of a principal resolvent for the group G, the permutations of K, a sub-group of G, leave at least one ϕ unaltered in form. In fact, the d ϕ's can be separated into sets such that the members of each set are permuted transitively by all the members of K. Taking the number of ϕ's in the sets in question to be 1, p, q, ..., the resolvent satisfied by ϕ breaks up into irreducible factors of degrees 1, p, q, ... (Table, § 5). If $\phi_1, \ldots, \phi_p$ are the zeros of an irreducible factor, each permutation P of K corresponds to a unique permutation Q of the p ϕ's, and the distinct permutations Q generate a group K' with which K is multiply isomorphic. A consideration of the groups K' in relation to K leads to the structure of the irrationalities involved in the solution of the equation. Some general theorems on such relations are proved in § 6. When, as often, the groups K, K' are simply isomorphic, the roots of the original equation are expressible rationally in terms of the p ϕ's.

There are five main classes of irreducible sextic equation. First, when no one of the principal resolvents of degree 6, 10, 15 has a rational root, the equation defines the general type of sextic irrationality with which this paper is not primarily concerned (§ 7).

---

* F. Brioschi, "Sur l'équation du sixième degré", *Acta Math.*, 12 (1889), 83–101.

† J. Lamb, "A 15-ic resolvent of the binary sextic", *Proc. Leeds Phil. Soc.*, 1 (1927), 149–151.

Next, when the sextic resolvent has linear and irreducible quintic factors, the sextic equation is effectively a sextic resolvent of an irreducible quintic (§ 8).

A sextic equation whose decimic resolvent has a rational root can be brought rationally to the form

$$(x^3+A_1x^2+A_2x+A_3)^2-m(B_1x^2+B_2x+B_3)^2=0 \qquad (\S\ 9).$$

When a further condition, appropriate to the group G_{36}, is satisfied, the roots of the sextic are expressible rationally in terms of those of two cubics. The sextic remains irreducible, however, when the field of its coefficients is enlarged by adjunction of either set of these conjugate cubic irrationalities. Equations and fields of this type have been termed *compound* (§ 11).

Again, the sextic is the product of three quadratic factors

$$a(x^2+\kappa x+\lambda)(x^2+\kappa' x+\lambda')(x^2+\kappa'' x+\lambda''),$$

in which κ, λ, etc. are members of conjugate cubic fields whenever the quindecimic resolvent has a rational root (§ 12). Certain sextics of this type also possess the character of compound equations (§ 14).

The last class arises when each of the decimic and quindecimic resolvents has a rational root, the irrationality defined by the sextic equation then being a combination of quadratic and cubic irrationalities (§ 15).

2. *Invariants of a binary sextic form.*

The binary sextic form

$$F(x,y)\equiv ax^6+6bx^5y+15cx^4y^2+20dx^3y^3+15ex^2y^4+6fxy^5+gy^6$$
$$\equiv a(x-\alpha y)(x-\epsilon y)(x-\gamma y)(x-\delta y)(x-\beta y)(x-\zeta y)$$

is known to have irreducible invariants, of degrees 2, 4, 6, 10, 15, which are connected by one syzygy giving the square of the last invariant as a polynomial in the other four*. Only the first four of these invariants enter into the coefficients of the resolvents given in § 4.

The invariant of the second degree is

$$I_2=ag-6bf+15ce-10d^2=-\frac{a^2}{240}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\zeta)^2,$$

* P. Gordan, "Beweis, dass jede Covariante und Invariante eines binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist", *Journal für Math.*, 69 (1868), 323–354.

and the catalecticant, of the fourth degree, is

$$I_4 = |aceg|.$$

Again, the quartic covariant

$$(ae-4bd+3c^2)x^4+2(af-3be+2cd)x^3y+(ag-9ce+8d^2)x^2y^2$$
$$+2(bg-3cf+2de)xy^3+(cg-4df+3e^2)y^4$$
$$\equiv k_0x^4+4k_1x^3y+6k_2x^2y^2+4k_3xy^3+k_4y^4$$

has the invariants

$$K_4 = k_0k_4-4k_1k_3+3k_2^2 = \tfrac{1}{12}I_2^2-3I_4,$$
$$K_6 = k_0k_2k_4+2k_1k_2k_3-k_0k_3^2-k_1^2k_4-k_2^3.$$

Mr. Gilham has found that the coefficients involved in the resolvents contain small prime factors when the quartic invariant, instead of I_4 or K_4, is taken to be

$$4\,.\,81\,.\,5J_4 = 4\,.\,81\,.\,5(I_2^2-100I_4) = a^4\Sigma\Delta(a\beta\gamma)\Delta(\delta\epsilon\zeta),$$

where $$\Delta(a\beta\gamma) = (\beta-\gamma)^2(\gamma-a)^2(a-\beta)^2.$$

The most suitable sextic invariant is

$$J_6 = -64I_2^3+81I_2J_4+27000K_6,$$

which gives

$$-2^3\,.\,5^3\,.\,3^4I_2J_4+2^3\,.\,3^2\,.\,5J_6 = a^6\Sigma\{\Delta(a\beta\gamma)\Delta(\delta\epsilon\zeta)\Sigma(a-\delta)^2(\beta-\epsilon)^2(\gamma-\zeta)^2\}.$$

The three summations of differences contain 15, 10, 60 terms respectively.

Finally the invariant of the tenth degree is

$$\Delta = a^{10}\Pi(a-\beta)^2.$$

(The arrangement $a\epsilon\gamma\delta\beta\zeta$ of the roots adopted above is later found to be the most convenient, $a\epsilon\gamma$, $\delta\beta\zeta$ being the roots of the cubic factors of § 9, and $a\delta$, $\epsilon\beta$, $\gamma\zeta$ those of the quadratic factors of § 12.)

3. *Transitive groups of the sixth degree.*

Groups of the sixth degree have been tabulated by Askwith* and Cayley†, whose lists include the intransitive ones as well as thirteen that

* E. H. Askwith, "On the possible groups of substitutions that can be formed with three, ..., six and seven letters respectively", *Quart. J. of Math.*, 24 (1890), 111–166 (133–151).

† A. Cayley, "On the substitution groups for two, ..., six, ..., eight letters", *Quart J. of Math.*, 25 (1891), 78–85; *Coll. Math. Papers*, 13 (1897), 125–130. [In this paper the group of 36 positive substitutions is stated to belong to the function

$$(b-c)(c-a)(a-b)(e-f)(f-d)(d-e)$$

which is unaltered by the negative substitution $(ad)(be)(cf)$. This function, incorrectly assigned by Cayley to Γ_{36}, is actually associated with G_{36}, which he overlooked.]

are transitive. The group of an irreducible equation being always transitive, it is unnecessary to examine the intransitive ones in detail here. Later, Cole* discovered three groups omitted from Cayley's list. The permutations contained in the twelve soluble groups have been tabulated more completely by Glage† than by Cayley.

In the table which follows Γ denotes a group containing even permutations only, and C a cyclical group. $H_{r!}$ is a group of the sixth degree which is simply isomorphic with the symmetric group of degree r. The symbols adopted in the earlier lists are also added.

Present	H_{720}	Γ_{360}	H_{120}	Γ_{60}	G_{72}	Γ_{36}	G_{36}	G_{18}
Cayley	720	360	120	60	72	36_2		18_3
Glage					Γ_{72}	Γ_{36}	G_{36}	Γ_{18}

Present	G_{48}	G_{24}	H_{24}	Γ_{24}	Γ_{12}	G_{12}	C_6	H_6
Cayley	48_2	24_3		24_4		12	6_1	6_3
Glage	Γ_{48}	Γ_{24}	$\mathbf{G}_{24}$	G_{24}	$\mathbf{G}_{12}$	Γ_{12}, G_{12}	$\Gamma_6, \mathbf{G}_6'$	$G_6, \mathbf{G}_6$

Γ_{360} is the alternating group of the sixth degree, and Γ_{60}, Γ_{12} are simply isomorphic with those of the fifth and fourth degrees. H_{24} and Γ_{24} are simply isomorphic with each other. The only maximum sub-groups of H_{720} are Γ_{360}, H_{120}, G_{72} and G_{48}, each of the last three having as sub-groups the groups in the same division of the table. G_{18} is a sub-group of G_{36}, and Γ_{12} of all the groups in its division. G_{12}, which has the sub-groups C_6, H_6, is itself a sub-group of G_{72}, G_{48}, G_{36} and one of the conjugates of H_{120}.

The solution of a sextic, whose group is H_{720} or Γ_{360}, is known to involve hyperelliptic modular functions. When the group is H_{120} or Γ_{60} the roots are expressible by means of elliptic modular functions, or as rational functions of the roots of an irreducible quintic. In all other cases the sextic is soluble by radicals.

4. *The four principal resolvents.*

In recording the principal resolvents of degrees 15, 10, 6 it is convenient to use numerical multiples of the invariants obtained in § 2,

* F. N. Cole, "Note on the substitution groups of six, seven, and eight letters", *Bull. New York Math. Soc.*, 2 (1893), 184–190.

† F. Glage, "Anwendungen der Gruppentheorie auf die irreducibeln Gleichungen vom sechsten Grade", *Monatshefte für Math. und Phys.*, 11 (1900), 155–169.

viz. :

$$\begin{array}{rrrr} & 15\text{-}ic & 10\text{-}ic & 6\text{-}ic \\ 2^3 i = & 2I & = c_2 & = 2^4 . 5 I_2, \\ 2^4 . 3 . 5 j = & -J & = -2^4 c_4 & = 2^4 . 3^3 . 5 J_4, \\ -2^6 (k - 2^3 ij) = & -K & = -2^3 c_6 & = 2^7 . 5 J_6, \\ 2^{10} \Delta = & L & = 2^7 c_{10} & = 2^{10} \Delta. \end{array}$$

(i) The unsymmetric functions

$$\alpha\delta + \epsilon\beta + \gamma\zeta, \quad (\alpha+\delta)(\epsilon+\beta)(\gamma+\zeta)$$

belong to the maximum sub-group

$$G_{48} \equiv \{(\alpha\delta)\}\{(\epsilon\beta)\}\{(\gamma\zeta)\}\{(\alpha\beta\gamma\delta\epsilon\zeta)\}$$

of H_{720}. Another function associated with G_{48} is

$$\psi_{56} \equiv a^2 (\alpha-\delta)^2 (\epsilon-\beta)^2 (\gamma-\zeta)^2,$$

which, from its form, is a root of an equation $\Psi(\psi) = 0$ whose coefficients are invariants of the sextic. Putting $\psi_{rs} = \psi_{sr}$, the fourteen functions conjugate to ψ_{56} are

$$\begin{array}{ll} \psi_{12} = a^2 (\alpha-\beta)^2 (\gamma-\zeta)^2 (\delta-\epsilon)^2, & \psi_{25} = a^2 (\alpha-\zeta)^2 (\gamma-\epsilon)^2 (\delta-\beta)^2, \\ \psi_{13} = a^2 (\alpha-\zeta)^2 (\beta-\epsilon)^2 (\gamma-\delta)^2, & \psi_{26} = a^2 (\alpha-\epsilon)^2 (\beta-\zeta)^2 (\gamma-\delta)^2, \\ \psi_{14} = a^2 (\alpha-\delta)^2 (\gamma-\epsilon)^2 (\beta-\zeta)^2, & \psi_{34} = a^2 (\alpha-\epsilon)^2 (\beta-\delta)^2 (\gamma-\zeta)^2, \\ \psi_{15} = a^2 (\alpha-\epsilon)^2 (\beta-\gamma)^2 (\delta-\zeta)^2, & \psi_{35} = a^2 (\alpha-\gamma)^2 (\zeta-\beta)^2 (\delta-\epsilon)^2, \\ \psi_{16} = a^2 (\alpha-\gamma)^2 (\delta-\beta)^2 (\epsilon-\zeta)^2, & \psi_{36} = a^2 (\alpha-\beta)^2 (\gamma-\epsilon)^2 (\delta-\zeta)^2, \\ \psi_{23} = a^2 (\alpha-\delta)^2 (\beta-\gamma)^2 (\epsilon-\zeta)^2, & \psi_{45} = a^2 (\alpha-\beta)^2 (\gamma-\delta)^2 (\epsilon-\zeta)^2, \\ \psi_{24} = a^2 (\alpha-\gamma)^2 (\beta-\epsilon)^2 (\zeta-\delta)^2, & \psi_{46} = a^2 (\alpha-\zeta)^2 (\delta-\epsilon)^2 (\beta-\gamma)^2. \end{array}$$

[The double suffixes have been so chosen that (*A*) ψ_{rs} is common to ϕ_r and ϕ_s in (iii), and (B) the degrees of the irreducible factors of the ϕ- and χ-resolvents in § 5 form, as far as is possible, descending sequences.]

On calculation, Mr. Lamb finds

$$\begin{aligned} \Psi(\psi) = \psi^{15} &+ 2^3 . 3 i \psi^{14} + (2^4 . 3 . 5 i^2 + 2^2 . 3 . 5 j)\psi^{13} \\ &+ (2^8 . 5 i^3 + 2^2 . 11^2 ij + 2^2 . 13 k)\psi^{12} \\ &+ (2^8 . 3 . 5 i^4 - 2^7 . 3 i^2 j + 2^8 . 3 ik - 2 . 3^2 . 5^3 j^2)\psi^{11} \\ &+ (2^{11} . 3 i^5 - 2^7 . 3^2 . 11 i^3 j + 2^7 . 3 . 11 i^2 k - 2^2 . 3 . 5 . 7 . 157 ij^2 \\ &\qquad + 2^2 . 3 . 5 . 53 jk - 2^2 . 31 \Delta)\psi^{10} \end{aligned}$$

$$+(2^{12}i^6-2^{10}.5.7i^4j+2^{11}.5i^3k-2.7.13.2371i^2j^2$$
$$+2^2.11.457ijk-2.3.173i\Delta-2^2.3^3.5^4j^3+2.311k^2)\psi^9$$
$$+(-2^{10}.3^3i^5j+2^{10}.3^2i^4k-2^3.3^2.17.1229i^3j^2$$
$$+2^4.3^2.19.29i^2jk-2^3.3.157i^2\Delta+2^2.3^2.5^2.71ij^3$$
$$+2^3.3^2.43ik^2-2^2.3^2.5^2.37j^2k+2.3.5.73j\Delta)\psi^8$$
$$+(-2^5.3^4.1031i^4j^2+2^6.3^4.37i^3jk-2^5.3.67i^3\Delta$$
$$+2^2.3^4.5^3.89i^2j^3+2^5.3^4i^2k^2-2^3.3^4.5^3.11ij^2k$$
$$+2^2.3.1361ij\Delta+3^4.5^5.13j^4+2^2.3^4.5^3jk^2-2^3.3.41k\Delta)\psi^7$$
$$+(-2^7.3^3.23^2i^5j^2+2^8.3^3.23i^4jk-2^7.3.11i^4\Delta$$
$$+2^2.3^4.11.2707i^3j^3-2^7.3^3.i^3k^2-2^2.3^4.13^2.53i^2j^2k$$
$$+2^2.3.2971i^2j\Delta+2^2.3^3.5^3.2221ij^4+2^2.3^5.193ijk^2$$
$$-2^2.3.317ik\Delta-2^2.3^3.5^3.137j^3k+2^3.3^2.5^2.7j^4\Delta$$
$$-2^2.3^3.13k^3)\psi^6$$
$$+(2^4.3^6.23^2i^4j^3-2^4.3^5.23.29i^3j^2k+2^4.3^2.7.23i^3j\Delta$$
$$+2.3^5.5^2.53.233i^2j^4+2^4.3^5.7^2i^2jk^2-2^4.3^5i^2k\Delta$$
$$-2^2.3^8.5^2.29ij^3k+2.3^2.5.2671ij^2\Delta-2^4.3^5ik^3$$
$$-2^3.3^5.5^5.7j^5+2.3^5.5^2.61j^2k^2-2^2.3^2.5.7.13jk\Delta$$
$$+2^2.31\Delta^2)\psi^5$$
$$+(2^2.3^7.5.13.23^2i^3j^4-2^2.3^7.5.7^2.23i^2j^3k+2^4.3^4.13.37i^2j^2\Delta$$
$$-2^4.3^7.5^4.13ij^5+2^2.3^7.5.59ij^2k^2-2^2.3^4.7.23ijk\Delta$$
$$+2^5.3^2i\Delta^2+2^4.3^7.5^4j^4k-2.3^6.5^3j^3\Delta-2^2.3^7.5jk^3$$
$$+2^2.3^4k^2\Delta)\psi^4$$
$$+(3^6.23^4i^4j^4-2^2.3^6.23^3i^3j^3k+2.3^4.5.23^2i^3j^2\Delta$$
$$-2^3.3^6.5^3.23^2i^2j^5+2.3^7.23^2.i^2j^2k^2-2^2.3^4.5.23i^2jk\Delta$$
$$+3.43i^2\Delta^2+2^4.3^6.5^3.23ij^4k+2^2.3^4.5^2.67ij^3\Delta$$
$$-2^2.3^6.23ijk^3+2.3^4.5ik^2\Delta+2^4.3^6.5^6j^6-2^3.3^6.5^3j^3k^2$$
$$-2^2.3^6.5^2j^2k\Delta+2^2.3.5.23j\Delta^2+3^6k^4)\psi^3$$
$$+(2.3^5.5.23^2i^2j^3-2^2.3^5.5.23ij^2k+2.3^2.101ij\Delta$$
$$-2^3.3^5.5^4j^4+2.3^5.5jk^2-2^2.3^2k\Delta)\Delta\psi^2$$
$$+3^4.5^2j^2\Delta^2\psi-\Delta^3.$$

(ii) Such unsymmetric functions as

$$a(a\epsilon\gamma+\delta\beta\zeta), \quad a(a+\epsilon+\gamma)(\delta+\beta+\zeta)$$

are associated with the group

$$G_{72} \equiv \{a\epsilon\gamma\}\{\delta\beta\zeta\}\{(a\delta)(\epsilon\beta)(\gamma\zeta)\}.$$

An invariant expression belonging to G_{72} is

$$\psi_{23}+\psi_{31}+\psi_{12}+\psi_{56}+\psi_{64}+\psi_{45}:$$

here we adopt the alternative function

$$\begin{aligned}\chi_{10} \equiv \tfrac{1}{4}a^2\{&(a-\delta)(\epsilon-\beta)(\gamma-\zeta)+(a-\delta)(\epsilon-\zeta)(\gamma-\beta)\\&+(a-\beta)(\epsilon-\zeta)(\gamma-\delta)+(a-\zeta)(\epsilon-\beta)(\gamma-\delta)\\&+(a-\zeta)(\epsilon-\delta)(\gamma-\beta)+(a-\beta)(\epsilon-\delta)(\gamma-\zeta)\}^2\\ \equiv a^2\{&3a\epsilon\gamma-(\epsilon\gamma+\gamma a+a\epsilon)(\delta+\beta+\zeta)+(a+\epsilon+\gamma)(\beta\zeta+\zeta\delta+\delta\beta)-3\delta\beta\zeta\}^2\\ \equiv \chi(&a\epsilon\gamma|\delta\beta\zeta), \text{ say.}\end{aligned}$$

The nine functions conjugate to χ_{10} are

$$\begin{aligned}&\chi_1=\chi(a\beta\epsilon|\gamma\delta\zeta), \quad \chi_2=\chi(a\beta\delta|\gamma\epsilon\zeta), \quad \chi_3=\chi(a\gamma\delta|\beta\epsilon\zeta),\\ &\chi_4=\chi(a\gamma\zeta|\beta\delta\epsilon), \quad \chi_5=\chi(a\delta\epsilon|\beta\gamma\zeta), \quad \chi_6=\chi(a\delta\zeta|\beta\gamma\epsilon),\\ &\chi_7=\chi(a\beta\zeta|\gamma\delta\epsilon), \quad \chi_8=\chi(a\beta\gamma|\delta\epsilon\zeta), \quad \chi_9=\chi(a\epsilon\zeta|\beta\gamma\delta).\end{aligned}$$

Mr. Gilham has calculated the decimic equation satisfied by the ten χ's and finds

$$\begin{aligned}X(\chi) &= (\chi-\chi_1)\ldots(\chi-\chi_{10})\\ &= \chi^{10}+2\cdot3^2I\chi^9+(3^2\cdot13I^2-2\cdot3J)\chi^8\\ &\quad+(2^3\cdot3\cdot17I^3-2\cdot3\cdot11IJ-2\cdot3\cdot11K)\chi^7\\ &\quad+(2\cdot3^2\cdot7^2I^4-2\cdot3\cdot41I^2J-2\cdot3^2\cdot23IK+J^2)\chi^6\\ &\quad+(2^2\cdot3^2\cdot5\cdot7I^5-2\cdot3\cdot71I^3J-2\cdot3^2\cdot61I^2K-2^2\cdot3^2IJ^2\\ &\qquad-2\cdot3\cdot19JK-3\cdot41L)\chi^5\\ &\quad+(2\cdot3\cdot7\cdot29I^6-2\cdot3\cdot5\cdot11I^4J-2\cdot3\cdot5\cdot53I^3K-2\cdot3^2I^2J^2\\ &\qquad-2^3\cdot3\cdot11IJK-2\cdot3\cdot47IL+2^3\cdot3J^3+3\cdot43K^2)\chi^4\\ &\quad+(2^3\cdot3^2\cdot11I^7-2\cdot3I^5J-2\cdot3^3\cdot5^2I^4K+2^2\cdot47I^3J^2\\ &\qquad-2^2\cdot3\cdot13I^2JK-2\cdot3\cdot19I^2L+2^4\cdot3IJ^3+2^2\cdot3^2\cdot11IK^2\\ &\qquad-2^4\cdot5J^2K+2\cdot47JL)\chi^3\end{aligned}$$

$$+(3^2.37I^8+2.3.29I^6J-2.3^2.37I^5K+3^3.11I^4J^2-2^3.3I^3JK$$
$$+2^3.3.5I^3L+2^3.3.5I^2J^3+2.3^2.23I^2K^2+2^2.3^3IJL$$
$$+2^4J^4+2^3.3.5JK^2-2.3.11KL)\chi^2$$
$$+(2.41I^9+2.3.19I^7J-2.3.29I^6K+2^4.3^2I^5J^2$$
$$-2.3.11I^4JK+3.23I^4L+2^5I^3J^3+2^2.3.13I^3K^2$$
$$-2^4.3^2I^2J^2K+2.3I^2JL-2^4.3IJK^2-2^2.3.5IKL$$
$$-2^5J^3K-2^3J^2L-2^6K^3)\chi$$
$$+(3^2I^{10}+2^3.3I^8J-2.3^2I^7K+2^4I^6J^2-2^4.3I^5JK$$
$$-2.3I^5L+3^2I^4K^2-2^5I^3J^2K-2^3I^3JL+2^3.3I^2JK^2$$
$$+2.3I^2KL+2^4J^2K^2+2^3JKL+L^2).$$

(iii) The third maximum sub-group of H_{720} is simply isomorphic with the symmetric group of the fifth degree. Putting

$$S=(\alpha\beta\gamma\delta),\quad T=(\alpha\xi\gamma\delta\beta),\quad U=(\alpha\beta\epsilon\gamma\xi\delta),$$

the permutations of G_{120} are*

$$S^lT^mU^n \quad \text{or} \quad U^nS^lT^m.$$

It was pointed out by Serret† that, after a change in the letters, a symmetric function of

$$\alpha\beta+\gamma\epsilon+\delta\xi,\quad \alpha\gamma+\delta\beta+\xi\epsilon,\quad \alpha\delta+\xi\gamma+\epsilon\beta,\quad \alpha\xi+\epsilon\delta+\beta\gamma,\quad \alpha\epsilon+\beta\xi+\gamma\delta$$

belongs to H_{120}. Here we adopt the invariant function

$$\phi_6=a^2\{(\alpha-\beta)^2(\gamma-\epsilon)^2(\delta-\xi)^2$$
$$+(\alpha-\gamma)^2(\delta-\beta)^2(\xi-\epsilon)^2$$
$$+(\alpha-\delta)^2(\xi-\gamma)^2(\epsilon-\beta)^2$$
$$+(\alpha-\xi)^2(\epsilon-\delta)^2(\beta-\gamma)^2$$
$$+(\alpha-\epsilon)^2(\beta-\xi)^2(\gamma-\delta)^2\},$$

whose conjugates are

$$\phi_1=\phi_6(\gamma\alpha),\quad \phi_2=\phi_6(\alpha\epsilon),\quad \phi_3=\phi_6(\epsilon\gamma),\quad \phi_4=\phi_6(\alpha\epsilon\gamma),\quad \phi_5=\phi_6(\alpha\gamma\epsilon).$$

* A. Cayley, *loc. cit.* (83–4).

† J. A. Serret, "Mémoire sur les fonctions de quatre, cinq et six lettres", *Journal de Math.* 15 (1850), 45–70.

The sextic resolvent satisfied by the six ϕ's, as calculated by Mr. Gilham, is

$$\begin{aligned}\Phi(\phi) = \phi^6 &+ 6c_2\phi^5 + (15c_2^2 + 8c_4)\phi^4 \\ &+ (20c_2^3 + 30c_2c_4 - 2c_6)\phi^3 \\ &+ (15c_2^4 + 42c_2^2c_4 - 6c_2c_6 + 16c_4^2)\phi^2 \\ &+ (6c_2^5 + 26c_2^3c_4 - 6c_2^2c_6 + 24c_2c_4^2 - 8c_4c_6 - 4c_{10})\phi \\ &+ (c_2^6 + 6c_2^4c_4 - 2c_2^3c_6 + 9c_2^2c_4^2 - 6c_2c_4c_6 - 3c_2c_{10} + c_6^2) = 0.\end{aligned}$$

(iv) Finally, as for an equation of any degree, there is a quadratic resolvent

$$T(\tau) = \tau^2 - \Delta = 0,$$

due to the alternating group.

The first three of these resolvents have been computed by the method which the author adopted previously in calculating the sextic resolvent of a quintic equation†. Writing a resolvent as

$$\Theta(\theta) \equiv \theta^p + x_1 i\theta^{p-1} + (x_2 i^2 + x_3 j)\theta^{p-2} + \ldots = 0,$$

the numerical coefficients x_r are found by means of the invariants of special sextics, not necessarily irreducible. The respective resolvents involve

$$25, \quad 84, \quad 155$$

numerical coefficients, of which

$$1, \quad 8, \quad 50$$

have been found to be zero. This, coupled with the comparatively small factors of those which remain, is an indication that these resolvents are among the simplest associated with a general sextic equation.

5. *Irreducible factors of the principal resolvents.*

Taking $\theta_1, \theta_2, \ldots, \theta_p$ (rational functions of $\alpha, \beta, \ldots, \zeta$) to be the roots of the principal resolvent $\Theta(\theta) = 0$, due to a maximum sub-group G_M, it is known from Galois' theory that θ_p is unaltered in form by the substitutions of G_M. Under the substitutions of G, a sub-group of G_M, the conjugate roots $\theta_1, \ldots, \theta_{p-1}$ of the resolvent are in general permuted

† W. E. H. Berwick, *loc. cit.* (304).

intransitively. If $\theta_a, \ldots, \theta_e$ are rearranged transitively by every substitution of G, a symmetric function of these conjugates takes rational values when the group of the equation is G. In this case $\Theta(\theta)$ has the rational irreducible factor

$$(\theta-\theta_a) \ldots (\theta-\theta_e).$$

To determine the degrees and zeros of the irreducible factors of $\Theta(\theta)$ it is sufficient to see how the p θ's are permuted under each substitution of G.

The table which follows gives the degrees of the irreducible factors of the four principal resolvents for each of the sixteen groups. It has been found possible so to arrange the notation that the suffixes follow consecutively for the roots of the first three resolvents; thus χ_7, χ_8, χ_9 are the roots of the cubic factor of $X(\chi)$ for G_{12}. The significance of the italics and asterisks is given at the end of § 6.

	$T(\tau)$	$\Phi(\phi)$	$X(\chi)$	$\Psi(\psi)$
H_{720}	2	*6*	*10*	*15*
Γ_{360}	1, 1	*6*	*10*	*15*
H_{120}	2	*5*, 1	*10*	*10*, *5*
Γ_{60}	1, 1	*5*, 1	*10*	*10*, *5*
G_{72}	2	*6*	*9*, 1	*9*, *6*
Γ_{36}	1, 1	*6*	*9*, 1	*9*, *6*
G_{36}	2	3*, 3**	*9*, 1	*9*, 3*, 3**
G_{18}	2	3*, 3**	*9*, 1	*9*, 3*, 3**
G_{48}	2**	4*, 2	6*, 4*	*8*, 6*, 1
G_{24}	2**	4*, 2	*6*, 4*	*8*, 6*, 1
H_{24}	2	*4*, 1, 1	*6*, *4*	*4*, *4*, *6*, 1
Γ_{24}	1, 1	*4*, 2	*6*, *4*	*8*, *6*, 1
Γ_{12}	1, 1	*4*, 1, 1	*6*, *4*	*4*, *4*, *6*, 1
G_{12}	2***	3*, 1, 2**	*6*, 3*, 1	*6*, 3*, 3*, 2**, 1
C_6	2**	3*, 1, 2**	*6*, 3*, 1	*6*, 3*, 3*, 2**, 1
H_6	2	*3*, 1, 1, 1	*3*, *3*, *3*, 1	*3*, *3*, *3*, *3*, 1, 1, 1

The general types of sextic associated with the groups G_{72}, G_{48}, G_{12} were noted by Luther*, who also gave the degrees of the irreducible factors of the χ and ψ resolvents in these cases.

* E. Luther, "Ueber die Factoren der algebraisch lösbaren irreducibeln Gleichungen vom sechsten Grade und ihren Resolventen", *Journal für Math.*, 37 (1848), 193–220.

6. *Structure of the roots of an equation in terms of those of its irreducible resolvents.*

Any irreducible factor of a principal resolvent of an equation will be called an irreducible resolvent. The roots of a sextic are expressible rationally in terms of those of some of its irreducible resolvents; in other cases they are rational functions of the roots of two irreducible resolvents. Such statements depend on theorems which will now be proved for an equation of any degree.

Take $\alpha, \beta, \ldots, \kappa$ to be the roots of an irreducible equation of degree n, and $\theta_1, \ldots, \theta_r$ (rational functions of $\alpha, \beta, \ldots, \kappa$) to be those of an irreducible resolvent for the group G of order nh. Each permutation P of G corresponds to a unique permutation Q of the r θ's, and the latter permutations may or may not be all distinct.

I. First, when no two of the nh permutations Q are identical, these permutations define a group G' simply isomorphic with G. The permutations of G which leave α unaltered form a group G_α, and the h equivalent permutations in the θ's define a group G'_α of the same order h. A symmetric function $\Theta(\theta_1, \ldots, \theta_r)$ of the linear forms into which

$$c_1\theta_1 + \ldots + c_r\theta_r$$

is changed by the permutations of G'_α, when expressed in terms of $\alpha, \beta, \ldots, \kappa$, is unaltered by the permutations of G_α. The function Θ can always be chosen so as to be different in *value* from its conjugates. In these circumstances α and Θ belong to the same group G_α and

$$\Theta = R_1(\alpha), \quad \alpha = R_2(\Theta) = R(\theta_1, \ldots, \theta_r),$$

a rational function unaltered in *form* by the permutations of G'_α. Applying the permutation

$$\begin{pmatrix} \alpha, & \ldots, & \theta_1, & \ldots, & \theta_r \\ \beta, & \ldots, & \theta_a, & \ldots, & \theta_c \end{pmatrix}$$

(a member of G) to the last equation, it follows that

$$\beta = R(\theta_a, \ldots, \theta_c).$$

Hence, whenever the groups G, G' are simply isomorphic, the roots $\alpha, \beta, \ldots, \kappa$ are expressible rationally in terms of the roots of the irreducible resolvent.

II. Next, when G is multiply isomorphic with G', the permutations of G which leave $\theta_1, \theta_2, \ldots, \theta_r$ unaltered in order are

$$1, \quad S_2, \quad \ldots, \quad S_{h_1},$$

forming a sub-group G_1 of G. If T' is a permutation of G not in G_1,

$$T'S_i(\theta_1, \ldots, \theta_r) = T'(\theta_1, \ldots, \theta_r) = (\theta_a, \ldots, \theta_e),$$

$$S_iT'(\theta_1, \ldots, \theta_r) = S_i(\theta_a, \ldots, \theta_e) = (\theta_a, \ldots, \theta_e).$$

Hence the set of permutations $T'G_1$ is identical with the set G_1T', though not necessarily in the same order; in fact, if $h_1k_1 = nh$, all the permutations of G fall into the scheme

$$G_1, \quad T_2'G_1, \ \ldots, \ T_{k_1}'G_1.$$

Another irreducible resolvent $\Lambda(\lambda) = 0$, for which G_2, T_j, h_2, k_2 are defined in the same way as G_1, S_i, h_1, k_1 for $\Theta(\theta)$, can often be chosen in such a way that G_1, G_2 have no common permutation except identity. In these circumstances no two of the permutations of G_1 derange the sequence $(\lambda_1, \ldots, \lambda_s)$ in the same way. For if

$$S_i(\lambda_1, \ldots, \lambda_s) = S_k(\lambda_1, \ldots, \lambda_s) = (\lambda_e, \ldots, \lambda_t),$$

then $$(S_i^{-1}S_k)(\lambda_1, \ldots, \lambda_s) = S_i^{-1}(\lambda_e, \ldots, \lambda_t) = (\lambda_1, \ldots, \lambda_s),$$

showing $S_i^{-1}S_k$ to be a member of G_2, *i.e.* $S_i = S_k$. Thus every member of G_1 permutes the λ's differently, whence

$$h_1 \leqslant k_2 = nh/h_2 \quad \text{or} \quad h_1h_2 \leqslant nh.$$

In the case in which $h_1h_2 = nh$, we have $h_2 = k_1$, and every member of the group

$$G_2 \equiv (1, \ T_2, \ \ldots, \ T_{h_2})$$

permutes the θ's differently. Hence $T_j = T_j'S_i$ and the set T_jG_1 is identical with $T_j'G_1$ or G_1T_j' or G_1T_j, *i.e.*

$$G = G_1 + T_2G_1 + \ldots + T_{h_2}G_1$$

$$= G_2 + S_2G_2 + \ldots + S_{h_1}G_2.$$

The permutation T_jS_i is in the set T_jG_1, also in the set $S_iG_2 (= G_2S_i)$. Similarly S_iT_j appears in each of these sets. The several members of T_jG_1 appear in the h_1 different sets S_iG_2, and so for the members of S_iG_2. Accordingly T_jS_i, S_iT_j can only appear in both when these permutations are identical. Hence, whenever G_1, G_2 have no common permutation except identity and the product of their orders is the order of G, the group G is the direct product of G_1 and G_2.

It is readily proved (without assuming the last theorem on the group) that $\alpha, \ldots, \kappa$ are expressible rationally in terms of θ's and λ's. The permutations of G which leave α unchanged form a group G_α of order h. Applying these permutations to a general rational function of θ's and λ's.

$$S(\theta_1, \ldots, \theta_r; \lambda_1, \ldots, \lambda_s), \text{ say,}$$

there are h such functions different in form and numerical value. A sufficiently general rational function of these belongs to the group G_α. It follows, by a known argument, that

$$\alpha = S_\alpha(\theta_1, \ldots, \theta_r; \lambda_1, \ldots, \lambda_s),$$

a rational function unaltered in form by the permutations of G_α.

More extended results can be given (i) when G_1, G_2 have no common operation except identity and $h_1 h_2 < nh$; and (ii) when G_1, G_2 have a common sub-group. However, all the irreducible resolvents (of the third or higher degree) of an irreducible sextic are covered by the two theorems here proved, and so it is unnecessary to enter into further developments on this occasion.

In the table of § 5 the figures in italics refer to irreducible resolvents covered by Theorem I, each root of the sextic being expressible rationally in terms of the roots of any one of these resolvents. Such irreducible resolvents will be called *normal*. Theorem II applies to resolvents marked with asterisks; the roots of the sextic are expressible rationally in terms of those of any resolvent marked with one asterisk combined with those of any resolvent (for the same group) marked with two or more asterisks. Two such irreducible resolvents will be said to *normalize* each other.

It will be seen that, for the sixteen possible groups :

(i) 58 irreducible resolvents are normal;

(ii) 15 irreducible resolvents, of degree > 2, are normalized by a quadratic surd; and

(iii) the two cubic ϕ (or ψ) resolvents for G_{36} and G_{18} normalize each other.

7. H_{720} *and* Γ_{360}. *The general sextic.*

When no one of the 6-ic, 10-ic, 15-ic resolvents has a rational root, the sextic is general in character, its group being H_{720} or Γ_{360}, according

as the discriminant is not or is a rational square. Such equations have been discussed by several authors, notably Klein*, Cole†, and Fricke‡.

8. *H_{120} and Γ_{60}. Sextic resolvents of a quintic.*

The sextic resolvent $\Phi(\phi)$ has a rational linear factor $\phi-\phi_6$, its remaining quintic factor being irreducible. Since H_{120} is simply isomorphic with the symmetric group $\{\phi_1\phi_2\phi_3\phi_4\phi_5\}$, Theorem I of § 6 applies. The twenty permutations which leave α unaltered are $V^m W^n$, where

$$V = (\gamma\epsilon\delta\beta\xi) = (\phi_1\phi_2\phi_5\phi_3\phi_4),$$

$$W = (\gamma\epsilon\beta\delta) = (\phi_1\phi_2\phi_3\phi_5).$$

Hence

$$\alpha = M(\phi_1,\ \phi_2,\ \phi_5,\ \phi_3,\ \phi_4),$$

a rational metacyclic function, and, on applying the permutations $U, U^2, \ldots, U^5$, where

$$U = (\alpha\beta\epsilon\gamma\xi\delta) = (\phi_1\phi_2\phi_4)(\phi_3\phi_5),$$

we have

$$\epsilon = M(\phi_4,\ \phi_1,\ \phi_3,\ \phi_5,\ \phi_2),$$

$$\gamma = M(\phi_1,\ \phi_2,\ \phi_3,\ \phi_5,\ \phi_4),$$

$$\delta = M(\phi_4,\ \phi_1,\ \phi_5,\ \phi_3,\ \phi_2),$$

$$\beta = M(\phi_2,\ \phi_4,\ \phi_5,\ \phi_3,\ \phi_1),$$

$$\xi = M(\phi_2,\ \phi_4,\ \phi_3,\ \phi_5,\ \phi_1).$$

When the quintic in ϕ has the metacyclic group $\{V, W\}$, its sextic resolvent $a(x) = 0$ has the rational root α.

An alternative quintic in terms of whose roots those of the sextic can be expressed metacyclically is

$$\Pi(\psi-\psi_{6i}) \equiv \psi^5-\phi_6\psi^4+\ldots-\Delta = 0,$$

the rational factor of the 15-ic resolvent. Since two coefficients ϕ_6, Δ of this factor are known, the whole quintic can be evaluated by rational operations.

The group Γ_{60}, which contains the even permutations of H_{120}, is simply isomorphic with the alternating group $\{\phi_1\phi_2\phi_3\phi_4\phi_5\}'$. Since

* F. Klein, "Zur Theorie der allgemeinen Gleichungen sechsten und siebenten Grades", *Math. Annalen* 28 (1887), 499–532; *Ges. Math. Abh.* 2 (1922), 439–472; "Über die Auflösung der allgemeinen Gleichungen fünften und sechsten Grades", *Math. Annalen*, 61 (1905), 50–71; *Ges. Math. Abh.* 2 (1922), 481–504.

† F. N. Cole, "A contribution to the theory of the general equation of the sixth degree", *Amer. J. of Math.*, 8 (1886), 265–286.

‡ R. Fricke, *Lehrbuch der Algebra*, 2 (1926), 295–331.

Δ is a rational square and the function

$$\Pi(\alpha-\beta)/\Pi(\phi_i-\phi_j), \quad i, j = 1, 2, 3, 4, 5,$$

is seen to be unaltered by every permutation of the group, the discriminant of the quintic is also a rational square. Hence

$$\alpha = M(\phi_1', \ \phi_2', \ \phi_5', \ \phi_3', \ \phi_4'), \quad \text{etc.},$$

metacyclic functions of the roots of a quintic whose group is alternating. Jacobian sextic equations, which have been considered by Klein*, Brioschi†, and Cayley‡, are of this type.

9–11. G_{72}, Γ_{36}, G_{36}, and G_{18}. *Bicubic sextic equations.*

9. *Cubic factors of a sextic whose decimic resolvent has a rational root.*

Each of the four functions

$$\left.\begin{aligned} &a(\alpha+\epsilon+\gamma-\delta-\beta-\zeta), \quad a(\alpha\epsilon\gamma-\delta\beta\zeta), \\ &\qquad\qquad a(\epsilon\gamma+\gamma\alpha+\alpha\epsilon-\beta\zeta-\zeta\delta-\delta\beta), \\ &\sqrt{\chi_{10}} \equiv a[3\alpha\epsilon\gamma-(\epsilon\gamma+\gamma\alpha+\alpha\epsilon)(\delta+\beta+\zeta) \\ &\qquad\qquad +(\alpha+\epsilon+\gamma)(\beta\zeta+\zeta\delta+\delta\beta)-3\delta\beta\zeta] \end{aligned}\right\} \quad (1)$$

belongs to the intransitive group $\{\alpha\epsilon\gamma\}\{\delta\beta\zeta\}$, of order 36, contained in G_{72}. The ratio of any two of them belongs to G_{72} itself and is a rational number. Since both

$$a(\epsilon\gamma+\gamma\alpha+\alpha\epsilon+\beta\zeta+\zeta\delta+\delta\beta) \quad \text{and} \quad a(\alpha\epsilon\gamma+\delta\beta\zeta)$$

belong to G_{72}, the sextic, which will here be taken to be

$$aE(x) \equiv a(ax^6+bx^5+cx^4+dx^3+ex^2+fx+g),$$

is the product of the two factors

$$ax^3+(\tfrac{1}{2}b+B_1\sqrt{m})x^2+(A_2+B_2\sqrt{m})x+(A_3+B_3\sqrt{m}) \equiv a(x-\alpha)(x-\epsilon)(x-\gamma),$$

$$ax^3+(\tfrac{1}{2}b-B_1\sqrt{m})x^2+(A_2-B_2\sqrt{m})x+(A_3-B_3\sqrt{m}) \equiv a(x-\delta)(x-\beta)(x-\zeta),$$

with A_2, A_3, B_1, B_2, B_3 rational.

---

* F. Klein, "Vorlesungen über das Ikosaeder" (1884), Kap. 4, 211–238.

† F. Brioschi, "Ueber die Auflösung der Gleichungen vom fünften Grade", *Math. Annalen*, 13 (1878), 109–160.

‡ A. Cayley, "On the Jacobian sextic equation", *Quart. J. of Math.*, 18 (1882), 52–65; *Coll. Math. Papers*, 11 (1896), 389–401; "Note on the Jacobian sextic equation", *Math. Annalen*, 30 (1887), 78–84; *Coll. Math. Papers*, 12 (1897), 493–499.

Two extraneous cases are settled immediately. In the first place χ_{10} cannot be a rational square. For, if so, the four functions (1) would also be rational, and the sextic would be a product of rational cubic factors. Secondly, the factorization, when possible, is unique, for were both

$$ax^3+(\tfrac{1}{2}b+B_1\sqrt{m})\,x^2+(A_2+B_2\sqrt{m})\,x+(A_3+B_3\sqrt{m}) \equiv a(x-\alpha)(x-\gamma)(x-\epsilon),$$

$$ax^3+(\tfrac{1}{2}b+B_1'\sqrt{n})\,x^2+(A_2'+B_2'\sqrt{n})\,x+(A_3'+B_3'\sqrt{n}) \equiv a(x-\alpha)(x-\xi)(x-\beta),$$

factors of $aE(x)$, α would be a member of the field $[\sqrt{m}, \sqrt{n}]$, and again the sextic would be reducible.

The five relations connecting A_2, A_3, B_1, B_2, B_3 obtained by equating $aE(x)$ to the product of its cubic factors are necessarily consistent with the equation given by the rational root of the resolvent

$$\sqrt{\chi_{10}} = h\sqrt{m} = (-6B_3+bB_2-2B_1A_2)\sqrt{m}. \tag{2}$$

From these equations the coefficients A_i, B_j can be expressed rationally in terms of h : it follows that the factorization is complete *algebraically*. The writer has not worked through the algebraic analysis involved in this method of determining the factors, since the whole problem of the factorization is essentially *arithmetical*, and not algebraical, in character.

To carry out the factorization arithmetically we assume that the coefficients of $E(x)$ are integral, either rational integers or complex integers of some number-field R. It is also sufficient to assume that m is an integer divisible by the square of no ideal in R. By a theorem due to Dedekind the coefficients

$$\tfrac{1}{2}b \pm B_1\sqrt{m}, \quad A_2 \pm B_2\sqrt{m}, \quad A_3 \pm B_3\sqrt{m}$$

are integral $[R, \sqrt{m}]$. So also is

$$\sqrt{\chi_{10}} \equiv h\sqrt{m} \equiv a[3a\epsilon\gamma - \ldots - 3\delta\beta\xi],$$

whence h is integral, too. Comparing coefficients of x^4, x, 1 in the product,

$$2A_2+b^2-mB_1^2 = ac, \tag{3}$$

$$2A_2A_3-2mB_2B_3 = af, \tag{4}$$

$$A_3^2-mB_3^2 = ag. \tag{5}$$

From (2) and (4)

$$A_2 = \frac{\frac{1}{2}bf+mB_3(6B_3+h)}{bA_3-2mB_1B_3},$$

whence, from (3),

$$\frac{\frac{1}{2}bf+mB_3(6B_3+h)}{\frac{1}{2}bA_3-mB_1B_3}+b^2-mB_1^2=ac, \tag{6}$$

a cubic equation for B_1 when A_3, B_3 are known. A_3, B_3 are to be determined by the methods of higher arithmetic from (5), which may have a limited or an unlimited number of solutions. Trying possible values in succession, for the correct pair the cubic (6) will have a root B_1, a half-integer in R, and the equations (2), (3), (4), linear in A_2, B_2, will be consistent. When these conditions are satisfied, it usually happens that the further conditions given by the coefficient of x^3, x^2 are also satisfied, and hence that the factorization is complete.

10. *The four types of bicubic equation.*

When $X(\chi)=0$ has a rational root χ_{10}, the group of the sextic, G_{72} or one of its sub-groups, is readily determined from the cubic factors of § 9. The groups first to be distinguished are G_{72}, G_{36}, Γ_{36}, and G_{18}, the resolvent $\Psi(\psi)=0$ then having no rational root.

G_{36} is the group of the function

$$(\epsilon-\gamma)(\gamma-\alpha)(\alpha-\epsilon)(\beta-\xi)(\xi-\delta)(\delta-\beta),$$

and

$$a^4(\epsilon-\gamma)^2(\gamma-\alpha)^2(\alpha-\epsilon)^2=d_1+d_2\sqrt{m},$$

$$a^4(\beta-\xi)^2(\xi-\delta)^2(\delta-\beta)^2=d_1-d_2\sqrt{m}.$$

Hence the group of the equation reduces to G_{36} when

$$(d_1+d_2\sqrt{m})(d_1-d_2\sqrt{m})$$

is a rational square, involving

$$d_1+d_2\sqrt{m}=l(f_1+f_2\sqrt{m})^2,$$

with l, f_1, f_2 rational. The two cubic equations

$$\left.\begin{aligned} ax^3+(\tfrac{1}{2}b+B_1\sqrt{m})x^2+(A_2+B_2\sqrt{m})x+(A_3+B_3\sqrt{m}) \\ =a(x-\alpha)(x-\epsilon)(x-\gamma)=0, \end{aligned}\right\} \tag{7}$$

$$ax^3+(\tfrac{1}{2}b-B_1\sqrt{m})x^2+(A_2-B_2\sqrt{m})x+(A_3-B_3\sqrt{m})=0 \tag{8}$$

are then normal in $[\sqrt{m},\ \sqrt{l}]$.

But, when l is itself a rational square,

$$d_1+d_2\sqrt{m}=(f_1+f_2\sqrt{m})^2,$$

and

$$(\epsilon-\gamma)(\gamma-\alpha)(\alpha-\epsilon)+(\beta-\xi)(\xi-\delta)(\delta-\beta),$$

a function associated with G_{18}, takes a rational value. The two cubics (7), (8) are then normal in $[\sqrt{m}]$ and the group of the sextic is G_{18}.

Again, the group reduces from G_{72} to Γ_{36}, the common sub-group of G_{72} and Γ_{360}, when Δ is a rational square. The functions $\sqrt{\chi_{10}}$ and

$$P \equiv (\alpha-\delta)(\gamma-\delta)(\epsilon-\delta)(\alpha-\xi)(\gamma-\xi)(\epsilon-\xi)(\alpha-\beta)(\gamma-\beta)(\epsilon-\beta)$$

are each associated with $\{\alpha\epsilon\gamma\}\{\delta\beta\xi\}$, their product with G_{72}, whence

$$P = k\sqrt{m}$$

with k rational. It follows that the rational number

$$\sqrt{\Delta} = a^6 . \sqrt{(d_1+d_2\sqrt{m})} . \sqrt{(d_1-d_2\sqrt{m})} . k\sqrt{m},$$

and that the cubics (7), (8) are rendered normal by adjunction of the respective irrationalities

$$\sqrt{(d_1+d_2\sqrt{m})}, \quad \sqrt{(d_2 m+d_1\sqrt{m})}$$

to the field of their coefficients.

11. *Compound bicubic equations.*

When the group of a sextic equation is G_{36} or G_{18}, its sextic resolvent is the product of irreducible cubic factors and symmetric functions of

$$\phi_1, \ \phi_2, \ \phi_3, \quad \text{also of} \quad \phi_4, \ \phi_5, \ \phi_6,$$

are rational. The effect of the generating permutations of G_{36} on the ϕ's is given by the table

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
$(\alpha\gamma\epsilon)$	ϕ_2	ϕ_3	ϕ_1	ϕ_6	ϕ_4	ϕ_5
$(\delta\zeta\beta)$	ϕ_2	ϕ_3	ϕ_1	ϕ_5	ϕ_6	ϕ_4
$(\alpha\delta)(\epsilon\beta)(\gamma\zeta)$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_6	ϕ_5
$(\epsilon\gamma)(\zeta\beta)$	ϕ_1	ϕ_3	ϕ_2	ϕ_4	ϕ_6	ϕ_5

Six permutations of G_{36}, viz.:

$$(1), \quad (\delta\xi\beta), \quad (\delta\beta\xi), \quad (\epsilon\gamma)(\xi\beta), \quad (\epsilon\gamma)(\beta\delta), \quad (\epsilon\gamma)(\delta\xi)$$

do not involve α: and these are the only permutations of the group which leave

$$\phi_1\phi_4+\phi_2\phi_5+\phi_3\phi_6$$

unaltered in form. It follows that

$$\alpha = R(\phi_1\phi_4+\phi_2\phi_5+\phi_3\phi_6),$$

a rational function, and, applying suitable permutations of the group,

that

$$\epsilon = R(\phi_1\phi_6+\phi_2\phi_4+\phi_3\phi_5),$$
$$\gamma = R(\phi_1\phi_5+\phi_2\phi_6+\phi_3\phi_4),$$
$$\delta = R(\phi_1\phi_4+\phi_2\phi_6+\phi_3\phi_5),$$
$$\beta = R(\phi_1\phi_5+\phi_2\phi_4+\phi_3\phi_6),$$
$$\zeta = R(\phi_1\phi_6+\phi_2\phi_5+\phi_3\phi_4).$$

The roots of the sextic are thus expressible rationally in terms of those of two irreducible cubics. Such an equation remains irreducible, however, when ϕ_1, ϕ_2, ϕ_3 are adjoined to the field of its coefficients, though its group then reduces to H_6.

A sextic equation of this kind will be termed *compound*.

Since half the permutations of G_{36} (or G_{18}) change the sign of

$$(\phi_2-\phi_3)(\phi_3-\phi_1)(\phi_1-\phi_2)(\phi_5-\phi_6)(\phi_6-\phi_4)(\phi_4-\phi_5),$$

the product of the discriminants of $[\phi_1]$ and $[\phi_4]$ cannot be a rational square. Conversely it can be shown that two cubic equations so related define a unique set of conjugate compound sextic fields. Taking the cubic fields, of discriminants D, D', to be $[\theta]$ and $[\lambda]$, the field

$$[\theta,\ \theta',\ \theta'',\ \lambda,\ \lambda',\ \lambda'']$$

is normal and of degree 36. The six-valued expression

$$\alpha \equiv \theta\lambda+\theta'\lambda'+\theta''\lambda''$$

is unaltered by the permutations of the group

$$(1),\quad (\theta\theta'\theta'')(\lambda\lambda'\lambda''),\quad (\theta\theta''\theta')(\lambda\lambda''\lambda'),$$
$$(\theta'\theta'')(\lambda'\lambda''),\quad (\theta''\theta)(\lambda''\lambda),\quad (\theta\theta')(\lambda\lambda'),$$

and any other expression invariant for this group is necessarily a rational function of α. The uniqueness of $[\alpha]$ follows. Thus

$$R_1(\theta)\,R_2(\lambda)+R_1(\theta')\,R_2(\lambda')+R_1(\theta'')\,R_2(\lambda''),$$
$$(\theta+\lambda)(\theta'+\lambda')(\theta''+\lambda''),$$

are both rational functions of α.

[When D'/D is a rational square it is easily seen that the sextic in α is the product of cubic factors whose discriminants are l^2D, n^2D.]

Since ϕ_1, ψ_{23} are unaltered by the same permutations of G_{36}, also ϕ_2, ψ_{31}, and ϕ_3, ψ_{12},

$$\phi_1 = R_1(\psi_{23}),\quad \phi_2 = R_1(\psi_{31}),\quad \phi_3 = R_1(\psi_{12}).$$

Again

$$\phi_4 = R_2(\psi_{56}),\quad \phi_5 = R_2(\psi_{64}),\quad \phi_6 = R_2(\psi_{45});$$

hence, for the present purpose, the cubic factors of $\Phi(\phi)$ can be replaced by the cubic factors of $\Psi(\psi)$.

Since none of the permutations of G_{18} changes the function

$$(\phi_5-\phi_6)(\phi_6-\phi_4)(\phi_4-\phi_5),$$

the group of the sextic is G_{18} when the cubic field $[\phi_4]$ is normal.

The distinctive property of compound equations, which first appears in G_{36}, G_{18}, G_{48}, and G_{24} sextics, is evidently a forerunner of similar peculiarities, which would repay investigation, in certain other equations of higher degree. Glage's memoir* contains a long section on G_{18}-sextics, but the present writer has failed to find previous reference to the compound character of these equations, which appears to be an essential quality of them.

12–14. G_{48}, G_{24}, H_{24}, Γ_{24}, *and* Γ_{12}. *Cubiquadratic sextic equations.*

12. *The quadratic factors of a sextic whose quindecimic resolvent has a rational root.*

When the group of a sextic equation is G_{48} or one of its sub-groups, the expressions

$$\psi_{56} \equiv a^2(a-\delta)^2(\epsilon-\beta)^2(\gamma-\zeta)^2$$

and any symmetric function of

$$S(a,\ \delta), \quad S(\epsilon,\ \beta), \quad S(\gamma,\ \zeta),$$

where S denotes symmetry, take rational values. Accordingly, the coefficients of the three quadratic factors of the sextic, which will here be written

$$E(x) \equiv x^6+bx^5+cx^4+dx^3+ex^2+fx+g,$$

viz. :

$$\left.\begin{aligned} x^2+\kappa x+\lambda &\equiv (x-a)(x-\delta) \\ x^2+\kappa' x+\lambda' &\equiv (x-\epsilon)(x-\beta) \\ x^2+\kappa'' x+\lambda'' &\equiv (x-\gamma)(x-\zeta) \end{aligned}\right\}, \tag{9}$$

lie in conjugate cubic fields $[\theta]$, $[\theta']$, $[\theta'']$, relative to the rational domain $[b, c, d, e, f, g]$.

As in § 9, the factors can be determined algebraically, but their evaluation is essentially arithmetical in character. Nothing is lost in assuming $b, \ldots, g$ to be integral, so that κ, λ are complex integers of $[\theta]$. When the coefficients are also rational it is not necessary to have the rational integral value of ψ_{56}, whose calculation from the equation

* F. Glage, *loc. cit.*, 167–169.

$\Psi(\psi) = 0$ may involve heavy numerical work, when carrying out the following method.

If $(1, \omega_2, \omega_3)$ is an integral basis of $[\theta]$ and

$$\mathfrak{d} = (\omega_2' - \omega_2'', \ \omega_3' - \omega_3''),$$

the discriminant of this field is given by*

$$(D) = (\mathfrak{d}\mathfrak{d}'\mathfrak{d}'')^2.$$

Since

$$(\gamma-\epsilon)(\gamma-\beta)(\xi-\epsilon)(\xi-\beta) = (\kappa'-\kappa'')(\kappa'\lambda''-\kappa''\lambda')+(\lambda'-\lambda'')^2$$
$$= R_1(\theta) \equiv 0 (\mathrm{mod}\ \mathfrak{d}^2),$$

and

$$(\alpha-\delta)^2 = \kappa^2-4\lambda = \mu, \quad \mu\mu'\mu'' = \psi_{56},$$

$$\Delta = \mathrm{H}(\alpha-\beta)^2 = n^2 D^2 \psi_{56}.$$

Hence ψ_{56} is a factor of Δ and Δ/ψ_{56} is an integral square; these facts limit the choice of trial values of ψ_{56} in searching for a rational root of $\Psi(\psi) = 0$, if needed. D is an integral factor of $\sqrt{(\Delta/\psi_{56})}$ and is consequently limited in choice. Also when the sextic has four or six real roots D is necessarily positive. To an assigned trial value of D there belong only a finite number of cubic fields which have been determined on a previous occasion†.

Fixing attention on one of the cubic fields, which are to be tried in rotation,

$$\psi_{56} = \xi\xi'\xi'', \quad g = \gamma\gamma'\gamma'',$$

where ξ, γ are integers of $[\theta]$ and limited in choice apart from units. When D is negative, these factorizations, when possible, can be effected, and the primitive unit ϵ_0 of the field can be determined, in a finite number of operations‡. (A field in which g is not a norm, and a trial value of ψ_{56} which is not a norm, can be rejected immediately.) Primitive units when $D > 0$ are ϵ_1, ϵ_2. Suitable trial values of λ, μ are

$$\lambda = \gamma\epsilon, \quad \mu = \xi\eta,$$

where ϵ, $\eta = \epsilon_0^r$, $D < 0$ or $\epsilon_1^s\epsilon_2^t$, $D > 0$.

---

* D. Hilbert, "Theorie der algebraischen Zahlkörper", *Jahresbericht Deutscher Math. Vereinigung*, 4 (1897), 175-546 (200-201).

† W. E. H. Berwick, "On cubic fields with a given discriminant", *Proc. London Math. Soc.* (2), 23 (1925), 359–378.

‡ W. E. H. Berwick, "The classification of ideal numbers that depend on a cubic irrationality", *Proc. London Math. Soc.* (2), 12 (1913), 393–429.

In the factorization (9)

$$\left.\begin{aligned} \Sigma\kappa = b, \quad \Sigma(\lambda+\kappa'\kappa'') = c, \\ \Sigma\lambda'\lambda''+\Sigma\lambda\kappa'\kappa'' = e, \quad \Sigma\kappa\lambda'\lambda'' = f, \\ \text{also} \qquad 4\lambda+\mu = \kappa^2, \end{aligned}\right\} \tag{10}$$

from which there follow immediately

$$\left.\begin{aligned} 2\Sigma\lambda+\Sigma\mu = b^2-2c \\ 2\Sigma\frac{1}{\lambda}+\Sigma\frac{\mu}{\lambda^2} = \frac{f^2}{g^2}-\frac{2e}{g} \\ N(4\lambda+\mu) = k^2 \end{aligned}\right\}. \tag{11}$$

These equations serve to exclude some of the trial values of λ, μ. If we take a set for which all are satisfied, the cubic equation in $4\lambda+\mu$,

$$(4\lambda+\mu)^3-A(4\lambda+\mu)^2+B(4\lambda+\mu)-k^2 = 0,$$

should agree with

$$(\kappa^3+b\kappa^2+c'\kappa \pm k)(\kappa^3-b\kappa^2+c'\kappa \mp k) = 0,$$

when λ, μ have been chosen aright.

Usually the equations (10), (11) are only satisfied by one set of the trial values and the factorization of the sextic is readily completed.

13. *The five types of cubiquadratic sextic.*

It is now possible to distinguish five kinds of sextic equation, associated with the groups G_{48}, G_{24}, H_{24}, Γ_{24}, and Γ_{12}, having one rational ψ, but no rational χ. The sextic is the product of three quadratic factors

$$(x^2+\kappa x+\lambda)(x^2+\kappa' x+\lambda')(x^2+\kappa'' x+\lambda'')$$

with coefficients in conjugate cubic fields $[\theta]$, $[\theta']$, $[\theta'']$, and the group, in general G_{48}, reduces to one of the sub-groups in special circumstances. Γ_{12}, as sub-group of each of the others, contains the permutations

$$\begin{aligned} &(1), \quad (\epsilon\beta)(\gamma\zeta), \quad (\alpha\epsilon\gamma)(\delta\beta\zeta), \quad (\alpha\gamma\epsilon)(\delta\zeta\beta), \\ &\qquad (\gamma\zeta)(\alpha\delta), \quad (\alpha\epsilon\zeta)(\delta\beta\gamma), \quad (\alpha\zeta\epsilon)(\delta\gamma\beta), \\ &\qquad (\alpha\delta)(\epsilon\beta), \quad (\alpha\beta\zeta)(\delta\epsilon\gamma), \quad (\alpha\zeta\beta)(\delta\gamma\epsilon), \\ &\qquad\qquad\qquad (\alpha\beta\gamma)(\delta\epsilon\zeta), \quad (\alpha\gamma\beta)(\delta\zeta\epsilon). \end{aligned}$$

In the first place the expression

$$\begin{aligned} \omega &\equiv (\epsilon+\beta-\gamma-\zeta)(\gamma+\zeta-\alpha-\delta)(\alpha+\delta-\epsilon-\beta) \\ &\equiv (\kappa''-\kappa')(\kappa-\kappa'')(\kappa'-\kappa), \end{aligned}$$

is unaltered by the permutations of

$$G_{24} \equiv \{\Gamma_{12}, \quad (\alpha\delta)(\epsilon\beta)(\gamma\zeta)\}.$$

Hence the group becomes G_{24} when ω is rational, D is a rational square and $[\theta]$ is a normal cubic field.

Secondly, the function

$$\frac{(\epsilon+\beta-\gamma-\zeta)(\gamma+\zeta-\alpha-\delta)(\alpha+\delta-\epsilon-\beta)}{(\alpha-\delta)(\epsilon-\beta)(\gamma-\zeta)}$$

retains its form under the permutations of

$$H_{24} \equiv \{\Gamma_{12}, \quad (\alpha\delta)(\epsilon\zeta)(\gamma\beta)\}.$$

Hence the group reduces to H_{24} when D/ψ_{56} is a rational square.

$$\Gamma_{24} \equiv \{\Gamma_{12}, \quad (\alpha\epsilon)(\delta\beta)\},$$

the third group of order 24, contains the even permutations of G_{48}. The functions

$$\sqrt{\Delta}, \quad \psi_{56}$$

are invariants for Γ_{24}, and so is the 30-valued function

$$\sqrt{\psi_{56}} \equiv (\alpha-\delta)(\epsilon-\beta)(\gamma-\zeta);$$

whence ψ_{56} is a rational square in this case.

When the group of the sextic reduces to Γ_{12}, the conditions appropriate to each group of order 24 are satisfied. Hence for such an equation the cubic field $[\theta]$ is normal and Δ, D, ψ_{56} are all rational squares.

14. *The irreducible quartic resolvents.*

Relations between the roots of a cubiquadratic sextic and those of the irreducible quartic factors of its resolvents will now be examined in the light of the theorems proved in § 6.

(i) G_{48}. This group is doubly isomorphic with the symmetric group of the quartic ϕ-resolvent which is normalized by the quadratic $T(\tau)$. A general rational function of

$$\phi_1, \quad \phi_2, \quad \phi_3, \quad \phi_4; \quad \tau_1, \quad \tau_2$$

takes 48 values under the permutations of G_{48}, the eight permutations which do not disturb α being

$$(1), \quad (\beta\gamma)(\epsilon\zeta) = (\phi_1\phi_4),$$

$$(\beta\epsilon)(\gamma\zeta) = (\phi_1\phi_4)(\phi_2\phi_3), \quad (\beta\zeta)(\gamma\epsilon) = (\phi_2\phi_3),$$

$$(\beta\gamma\epsilon\zeta) = (\phi_1\phi_2\phi_4\phi_3)(\tau_1\tau_2), \quad (\beta\epsilon) = (\phi_1\phi_3)(\phi_2\phi_4)(\tau_1\tau_2),$$

$$(\beta\zeta\epsilon\gamma) = (\phi_1\phi_3\phi_4\phi_2)(\tau_1\tau_2), \quad (\gamma\zeta) = (\phi_1\phi_2)(\phi_3\phi_4)(\tau_1\tau_2).$$

The simplest six-valued function associated with this group is

$$(\phi_1-\phi_2-\phi_3+\phi_4)(\tau_1-\tau_2).$$

Hence
$$\alpha = R[(\phi_1-\phi_2-\phi_3+\phi_4)\sqrt{\Delta}],$$

and, applying suitable permutations of the group,

$$\delta = R[(-\phi_1+\phi_2+\phi_3-\phi_4)\sqrt{\Delta}],$$
$$\epsilon = R[(-\phi_1+\phi_2-\phi_3+\phi_4)\sqrt{\Delta}],$$
$$\beta = R[(\phi_1-\phi_2+\phi_3-\phi_4)\sqrt{\Delta}],$$
$$\gamma = R[(-\phi_1-\phi_2+\phi_3+\phi_4)\sqrt{\Delta}],$$
$$\zeta = R[(\phi_1+\phi_2-\phi_3-\phi_4)\sqrt{\Delta}].$$

Since $[\sqrt{\Delta}]$ and $[\phi_i]$ are not sub-fields of $[\alpha]$ the general G_{48} sextic equation defines another type of *compound* irrationality.

If
$$\alpha_0 = (\phi_1-\phi_2-\phi_3+\phi_4)\sqrt{\Delta},$$

α_0^2 is a member of the cubic field $[\phi_1\phi_4+\phi_2\phi_3]$ and a root of

$$\alpha_0^6+A_1\alpha_0^4+A_2\alpha_0^2+A_3 = 0. \tag{12}$$

It follows that a sextic whose 15-ic resolvent has a rational root can be transformed rationally into (12).

The roots of any quartic $\Omega(\omega) = 0$ whose group is symmetric can be compounded with an assigned quadratic surd $\sqrt{n}$ to define a unique set of conjugate G_{48}-sextic fields having the discriminant N^2n. For the expression

$$\alpha_1 = (\omega_1-\omega_2-\omega_3+\omega_4)\sqrt{n}$$

is unaltered by the permutations

$$(1), \quad (\omega_1\omega_4), \quad (\omega_2\omega_3), \quad (\omega_1\omega_4)(\omega_2\omega_3)$$

and changes sign under

$$(\omega_1\omega_2\omega_4\omega_3), \quad (\omega_1\omega_3)(\omega_2\omega_4), \quad (\omega_1\omega_3\omega_4\omega_2), \quad (\omega_1\omega_2)(\omega_3\omega_4):$$

hence any other function possessing the same invariance is necessarily a rational function of α_1.

In particular the quartic ϕ-resolvent can be replaced by the quartic χ-resolvent, since, for the group G_{48},

$$\chi_{6+r} = R_1(\phi_r), \quad r = 1, 2, 3, 4.$$

(ii) G_{24}. Each of the irreducible quartic resolvents, whose groups are alternating, is normalized by $T(\tau)$, and G_{24} is the direct product of the

groups of $T(\tau)$ and the quartic. The permutations which do not disturb a are

$$(1), \quad (\beta\epsilon)(\gamma\xi), \quad (\beta\epsilon), \quad (\gamma\xi)$$

and, as in the last case, after § 6, Theorem II,

$$a = R[(\phi_1' - \phi_2' - \phi_3' + \phi_4')\sqrt{\Delta}], \quad \text{etc.},$$

the accents indicating roots of an alternating quartic.

Theorem I of § 6 applies to the other three groups, which are simply isomorphic with the groups of the respective quartic resolvents.

(iii) H_{24}. In this group the permutations which do not disturb a form a cyclical sub-group :

$$(1), \quad (\beta\gamma\epsilon\xi) = (\phi_1\phi_2\phi_4\phi_3),$$

$$(\beta\epsilon)(\gamma\xi) = (\phi_1\phi_4)(\phi_2\phi_3), \quad (\beta\xi\epsilon\gamma) = (\phi_1\phi_3\phi_4\phi_2).$$

Hence the roots of the sextic are cyclical functions of the roots of its quartic ϕ-resolvent, say

$$a = R(\phi_1^2\phi_2 + \phi_2^2\phi_4 + \phi_4^2\phi_3 + \phi_3^2\phi_1) = C(\phi_1\phi_2\phi_4\phi_3),$$

and, as before,

$$\delta = C(\phi_1\phi_3\phi_4\phi_2),$$

$$\epsilon = C(\phi_1\phi_2\phi_3\phi_4), \quad \beta = C(\phi_1\phi_4\phi_3\phi_2),$$

$$\gamma = C(\phi_1\phi_4\phi_2\phi_3), \quad \xi = C(\phi_1\phi_3\phi_2\phi_4).$$

[Each of the functions

$$\phi_1^2\phi_2 + \phi_2^2\phi_4 + \phi_4^2\phi_3 + \phi_3^2\phi_1, \quad (\phi_1 - \phi_2 - \phi_3 + \phi_4)\sqrt{\Delta}$$

belongs to the cyclical sub-group $\{(\beta\gamma\epsilon\xi)\}$ of H_{24}.]

In the cyclical expressions the roots of the ϕ-quartic can be replaced by those of the χ- or either ψ-quartic resolvent, according to the scheme

$$\phi_1, \quad \chi_7, \quad \psi_{15}, \quad \psi_{16},$$

$$\phi_2, \quad \chi_8, \quad \psi_{25}, \quad \psi_{26},$$

$$\phi_3, \quad \chi_9, \quad \psi_{35}, \quad \psi_{36},$$

$$\phi_4, \quad \chi_{10}, \quad \psi_{45}, \quad \psi_{46}.$$

(iv) Γ_{24}. Δ is here a rational square and a is undisturbed by the four permutations

$$(1), \quad (\beta\epsilon)(\gamma\xi) = (\phi_1\phi_4)(\phi_2\phi_3),$$

$$(\beta\xi)(\gamma\epsilon) = (\phi_2\phi_3), \quad (\beta\gamma)(\epsilon\xi) = (\phi_1\phi_4),$$

Hence $\alpha = S(\phi_1\phi_4)$, a symmetric function,

$$\delta = S(\phi_2\phi_3), \quad \epsilon = S(\phi_2\phi_4), \quad \beta = S(\phi_1\phi_3), \quad \gamma = S(\phi_3\phi_4), \quad \xi = S(\phi_1\phi_2).$$

(v) Γ_{12}. This being a sub-group of G_{24}, H_{24}, and Γ_{24}, the conditions appropriate to each of the latter are all satisfied. Hence

$$\alpha = S(\phi_1'\phi_4'), \quad \delta = S(\phi_2'\phi_3'), \quad \text{etc.},$$

symmetric functions of two roots, also expressible as cyclical functions of all the roots, of an alternating quartic.

15. G_{12}, C_6, and H_6. *Composite sextic equations.*

It remains to distinguish three more types of sextic equation whose decimic and quindecimic resolvents have each a rational root. G_{12}, the common sub-group of G_{72} and G_{48}, is also a sub-group of G_{36} and $(\alpha\gamma\epsilon)\,G_{120}\,(\alpha\epsilon\gamma)$, but not of Γ_{60}, Γ_{36}, G_{24}, H_{24}, Γ_{24}, or any of their conjugates. Each cubic factor of § 9 has a linear factor in common with each quadratic factor of § 12, whence

$$\alpha = R(\theta, \sqrt{m}), \quad \epsilon = R(\theta', \sqrt{m}), \quad \gamma = R(\theta'', \sqrt{m}),$$
$$\delta = R(\theta, -\sqrt{m}), \quad \beta = R(\theta', -\sqrt{m}), \quad \xi = R(\theta'', -\sqrt{m}).$$

The group, in general G_{12}, reduces to C_6, which is cyclical, when the cubic field $[\theta]$ is normal. Again, it reduces to H_6, which is simply isomorphic with the symmetric group of the third degree, when $D(\theta)/m$ is a rational square.

The roots of the sextic can be expressed in terms of those of its resolvents in the various cases.

(i) G_{12}. The irreducible sextic factors of $X(\chi)$ and $\Psi(\psi)$ also have the group G_{12}.

G_{12} is doubly isomorphic with the groups of the irreducible cubic resolvents, each of which is normalized by the quadratic factors of $\Phi(\phi)$ or $\Psi(\psi)$. The permutations of G_{12}, which do not alter

$$\alpha \equiv R_0(\phi_1, \phi_2, \phi_3; \phi_5, \phi_6)$$

are (1) and $$(\beta\xi)(\gamma\epsilon) = (\phi_2\phi_3)(\phi_5\phi_6).$$

Noting that $$(\phi_5 - \phi_6)\sqrt{D}/\sqrt{m}, \quad (\psi_{45} - \psi_{46})\sqrt{D}/\sqrt{m}$$

are rational, we have

$$\alpha = R_1(\phi_1, \sqrt{m}), \quad \epsilon = R_1(\phi_2, \sqrt{m}), \quad \gamma = R_1(\phi_3, \sqrt{m}),$$
$$\delta = R_1(\phi_1, -\sqrt{m}), \quad \beta = R_1(\phi_2, -\sqrt{m}), \quad \xi = R_1(\phi_3 - \sqrt{m}),$$

wherein the ϕ's can be replaced by χ's or ψ's according to the scheme

$$\begin{array}{cccc} \phi_1 & \chi_7 & \psi_{23} & \psi_{14} \\ \phi_2 & \chi_8 & \psi_{31} & \psi_{24} \\ \phi_3 & \chi_9 & \psi_{12} & \psi_{34} \end{array}$$

(ii) $C_6 \equiv \{(\alpha\beta\gamma\delta\epsilon\zeta)\}$. The roots are connected with those of the irreducible sextic resolvents by the relations

$$\alpha = F(\xi) = F_1(\chi_1) = F_2(\psi_{15}), \quad \beta = F(\alpha) = F_2(\chi_5) = F_2(\psi_{26}),$$

$$\gamma = F(\beta) = F_1(\chi_3) = F_2(\psi_{35}), \quad \delta = F(\gamma) = F_1(\chi_4) = F_2(\psi_{16}),$$

$$\epsilon = F(\delta) = F_1(\chi_2) = F_2(\psi_{25}), \quad \xi = F(\epsilon) = F_1(\chi_6) = F_2(\psi_{36}).$$

For the irreducible cubic factors of the resolvents, which define the same normal cubic field, relations similar to those in (i) hold good.

(iii) H_6. For this group ϕ_4, ϕ_5, ϕ_6 are all rational and

$$\alpha = R(\phi_1, \phi_2, \phi_3), \quad \epsilon = R(\phi_2, \phi_3, \phi_1), \quad \gamma = R(\phi_3, \phi_1, \phi_2),$$

$$\delta = R(\phi_1, \phi_3, \phi_2), \quad \beta = R(\phi_2, \phi_1, \phi_3), \quad \xi = R(\phi_3, \phi_2, \phi_1).$$

As before, in these relations, the ϕ's can be replaced by χ's or ψ's according to the scheme

$$\begin{array}{cccccccc} \phi_1 & \chi_1 & \chi_4 & \chi_7 & \psi_{23} & \psi_{14} & \psi_{15} & \psi_{16} \\ \phi_2 & \chi_2 & \chi_5 & \chi_8 & \psi_{31} & \psi_{24} & \psi_{25} & \psi_{26} \\ \phi_3 & \chi_3 & \chi_6 & \chi_9 & \psi_{12} & \psi_{34} & \psi_{35} & \psi_{36}. \end{array}$$

Glage's* memoir includes a discussion of the actual radicals involved in the solution of composite sextic equations.

* F. Glage, *loc. cit.*, 161–167.

RECIPROCAL INTEGRAL FORMULAE

By F. M. WOOD.

[Received 13 December, 1927; read 15 December, 1927.]

1. The reciprocal formulae

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt, \tag{1.1}$$

$$f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t-x}\,dt, \tag{1.2}$$

have been considered in different forms, and various theorems have been proved about them.

The integrals are defined as principal values at $t = x$, and it has been shown that, if $f(x)$ is such that the integral

$$\int_{-\infty}^{\infty} |f(x)|^p\,dx \quad (p > 1) \tag{1.3}$$

is convergent, then $g(x)$ satisfies the same condition, and the reciprocal formulae hold for *almost all* values of x*.

If, in addition, $f(x)$ satisfies a Lipschitz condition of order α uniformly in x, *i.e.*

$$|f(x+h)-f(x)| < A\,|h|^{\alpha} \quad (0 < \alpha < 1), \tag{1.4}$$

for $|h| < k$, then $g(x)$ satisfies the same Lipschitz condition, and (1.1) and (1.2) are true for *all* values of x†.

---

* M. Riesz, to whom this theorem is due, enunciated it in *Comptes rendus*, 178 (1924), 1464, and published a longer paper on the theorem, *Math. Zeitschrift*, 27 (1927), 218–244. E. C. Titchmarsh has given another proof, *Math. Zeitschrift*, 25 (1926), 321–347.

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 24 (1925), 109–130. This paper will be referred to as Paper I.

In a recent development in the theory of series inversion formulae*, Mr. Titchmarsh pointed out that the conditions for series analogous to (1 . 3) are by no means necessary. Supposing that

$$\lim_{q,\,q'\to\infty} \frac{\sin \pi\lambda}{\pi} \sum_{n=-q'}^{q} \frac{a_n}{m+n+\lambda} \tag{1.5}$$

exists, and is denoted by b_m, and writing

$$R_n = \sum_{m=n}^{\infty} \frac{a_m+a_{-m}}{m}, \quad R'_n = \sum_{m=n}^{\infty} \frac{a_m-a_{-m}}{m},$$

he proves that the series

$$\lim_{q\to\infty} \frac{\sin \pi\lambda}{\pi} \sum_{n=-q}^{q} \frac{b_m}{m+n+\lambda} \tag{1.6}$$

is summable $(C, 1)$ to the sum a_m; and that the necessary and sufficient condition for its convergence is

$$\lim_{q\to\infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\lambda} = 0, \quad (0 < \theta < 1); \tag{1.7}$$

further, that the necessary and sufficient condition for the summability $(C, 1)$ of the series

$$\lim_{q,\,q'\to\infty} \frac{\sin \pi\lambda}{\pi} \sum_{n=-q'}^{q} \frac{b_n}{m+n+\lambda} \tag{1.8}$$

is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{R'_n}{n+\lambda}, \tag{1.9}$$

and the necessary and sufficient conditions for the convergence of the series (1 . 8) are the conditions (1 . 7), (1 . 9), and

$$\lim_{q\to\infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R'_n}{n-q-\lambda} = 0.$$

2. These results suggested that a similar development might be possible in the theory of reciprocal integrals, and accordingly Theorems I–III of this paper deal with integrals of the kind

$$\lim_{T\to\infty} \int_0^T \{f(x)+f(-x)\}\, dx, \tag{2.1}$$

* E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1927), 1-11.

that is, with integrals corresponding to the series (1 . 6); and Theorems IV and V are concerned with integrals of the type

$$\int^{\infty} f(x)\,dx, \quad \int^{\infty} f(-x)\,dx, \tag{2 . 2}$$

which correspond to the series (1 . 8).

Throughout the paper, integrals over a finite interval are taken as existing in the Lebesgue sense; if the interval becomes infinite, then

$$\int^{\infty} f(x)\,dx$$

is said to exist, if $\lim_{X\to\infty} \int^{X} f(x)\,dx$ exists.

3. Suppose that $f(x)$ is a function such that

$$\int^{\infty} \frac{f(x)}{x}\,dx, \quad \int^{\infty} \frac{f(-x)}{x}\,dx \tag{3 . 1}$$

both exist. This is a standing hypothesis for all the theorems. The following theorems are proved:

THEOREM I. *If $f(x)$ obeys a Lipschitz condition of order α $(0<\alpha<1)$, uniformly with respect to x over any finite interval; then $g(x)$ also obeys a Lipschitz condition of order α, and*

$$-\frac{1}{\pi}\int_0^{\infty} \frac{g(x+t)-g(x-t)}{t}\,dt \tag{3 . 2}$$

is summable $(C, 1)$ to $f(x)$ for all values of x.

THEOREM II. *If $f(x)$ is a function whose p-th power is integrable over any finite interval $(p>1)$, then $g(x)$ is also a function whose p-th power is integrable over any finite interval and*

$$-\frac{1}{\pi}\int^{\infty} \frac{g(x+t)-g(x-t)}{t}\,dt$$

is summable $(C, 1)$ for all x; if the lower limit of the integration is taken to be the origin, then (3 . 2) exists for almost all x, and is summable $(C, 1)$ to $f(x)$.

It is convenient to express the rest of the theorems in terms of the functions

$$\phi(u) = \int_u^\infty \frac{f(t)+f(-t)}{t}\,dt, \quad \psi(u) = \int_u^\infty \frac{f(t)-f(-t)}{t}\,dt, \tag{3.3}$$

which exist for all positive u, and tend to zero as u tends to infinity.

THEOREM III. *If $f(x)$ obeys the condition of Theorem* II, *and if, in addition,*

$$\int_{T-1}^{T} [f(t)]^p\,dt = o(T^p), \tag{3.4}$$

as $T\to\infty$, then (3.3) *converges to the value $f(x)$, provided that*

$$\lim_{T\to\infty} \int_{(1-\theta)T}^{(1+\theta)T} \frac{\phi(u)}{u-T}\,du = 0 \quad (0<\theta<1). \tag{3.5}$$

THEOREM IV. *The necessary and sufficient condition for the summability $(C, 1)$ of each of the integrals*

$$\int_1^\infty \frac{g(x\pm t)}{t}\,dt \tag{3.6}$$

is the existence of

$$\int_0^\infty \frac{\psi(u)}{u+1}\,du. \tag{3.7}$$

THEOREM V. *The integrals* (3.6) *are each convergent if the conditions* (3.5) *and* (3.7) *are fulfilled, together with*

$$\lim_{T\to\infty} \int_{(1-\theta,T}^{(1+\theta)T} \frac{\psi(u)}{u-T}\,du = 0. \tag{3.8}$$

3.1. The following are examples of functions that satisfy the conditions of Theorems III–V:

(i) if $\psi(u) = 1/(\log^2 u)$, condition (3.7) is satisfied;

(ii) if $\phi(u) = 1/(\log u)$, condition (3.5) is satisfied,

for then

$$\int_{(1-\theta)T}^{(1+\theta)T} \frac{\phi(u)}{u-T}\,du = \int_0^{\theta T} \frac{1}{u}\left[\frac{1}{\log(u+T)} - \frac{1}{\log(T-u)}\right] du$$

$$= \int_0^\theta \frac{1}{x}\,\frac{\log(1-x)-\log(1+x)}{\log\{T(1+x)\}\log\{T(1-x)\}}\,dx,$$

and since $$\{\log(1+x)-\log(1-x)\}/x$$

is bounded in $(0, \theta)$, it follows that

$$\lim_{T\to\infty}\int_{(1-\theta)T}^{(1+\theta)T}\frac{1}{(u-T)\log u}du=0.$$

Indeed, it is now obvious that a function such as $1/(\log\log\ldots\log u)$, which tends to zero smoothly as $u\to\infty$, also satisfies (3 . 5).

(iii) If $\psi(u)=1/(\log\ldots\log u)$, condition (3 . 8) is satisfied.

4. *Preliminary lemmas.* The arguments used to establish the theorems for the series (1 . 6) cannot be used readily in the case of the integrals, since the proof of the existence of principal values is also required. Accordingly, another method is followed, based on that of Paper I, pages 119–121; and results are first established concerning the integrals (4 . 1) and (4 . 2) below, which are seen to exist for all values of x, and in certain circumstances to have (1 . 1) and (1 . 2) as limiting values.

Suppose $f(x)$ to be integrable over any finite interval, and to be such that the integrals (3 . 1) exist. Suppose that

$$F(\lambda, x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda f(t)}{\lambda^2+(t-x)^2}dt, \tag{4 . 1}$$

$$G(\lambda, x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(t-x)f(t)}{\lambda^2+(t-x)^2}dt, \tag{4 . 2}$$

where λ is positive. It is convenient to express the results in terms of the following functions:

$$\phi(\lambda, u, x)=\int_u^{\infty}\frac{t}{\lambda^2+t^2}[f(x+t)+f(x-t)]dt, \tag{4 . 3}$$

$$\chi(\lambda, u, x)=\int_u^{\infty}\frac{\lambda}{\lambda^2+t^2}[f(x+t)+f(x-t)]dt, \tag{4 . 4}$$

$$\psi(\lambda, u, x)=\int_u^{\infty}\frac{t}{\lambda^2+t^2}[f(x-t)-f(x-t)]dt, \tag{4 . 5}$$

$$\varpi(\lambda, u, x)=\int_u^{\infty}\frac{\lambda}{\lambda^2+t^2}[f(x-t)-f(x-t)]dt. \tag{4 . 6}$$

When $x=0$, the functions (4 . 3)–(4 . 6) are written shortly $\phi(\lambda, u)$, etc. The functions defined in (4 . 3)–(4 . 6) evidently exist for all values of u, and, if λ and x are fixed, they tend to zero as $u\to\infty$; that is, with

the usual meaning attached to the symbol $o(1)$, we can say (λ and x being fixed) that $\phi(\lambda, u, x) = o(1)$ as $u \to \infty$. Now, if λ varies, and any finite range of values of x is considered, the functions (4 . 3)–(4 . 6) still tend to zero as $u \to \infty$, uniformly with respect to λ and for any finite range of values of x. It is therefore convenient to extend the meaning of the symbol $o(1)$ to functions of this kind with more than one variable parameter. Similarly, as on page 35 below, a function of λ, x, and δ may tend to zero, as $\delta \to 0$, uniformly with respect to λ, for any finite range of values of x. Such a function will be said to be $o(1)$ as $\delta \to 0$. A similar extension of the use of the symbol $O(1)$ will be made when the functions are bounded.

4 . 1. Lemma a.

$$\int_0^T \frac{dt}{t} \int_{-\infty}^{\infty} \left[\frac{u-t}{\lambda^2+(u-t)^2} - \frac{u+t}{\lambda^2+(u+t)^2} \right] f(u+x)\, du$$
$$= \int_{-\infty}^{\infty} f(u+x)\, du \int_0^T \frac{1}{t} \left[\frac{u-t}{\lambda^2+(u-t)^2} - \frac{u+t}{\lambda^2+(u+t)^2} \right] dt.$$

The u-integral is uniformly convergent with respect to t in the finite interval (δ, T). Hence it is sufficient to show that

$$\lim_{\delta \to 0} \int_{-\infty}^{\infty} f(u+x)\, du \int_0^{\delta} \frac{1}{t} \left[\frac{u-t}{\lambda^2+(u-t)^2} - \frac{u+t}{\lambda^2+(u+t)^2} \right] dt = 0.$$

The t-integral is equal to

$$\int_0^{\delta} \left\{ \frac{u}{\lambda^2+u^2} \left(\frac{u-t}{\lambda^2+(u-t)^2} + \frac{u+t}{\lambda^2+(u+t)^2} \right) \right.$$
$$\left. - \frac{\lambda}{\lambda^2+u^2} \left(\frac{\lambda}{\lambda^2+(u-t)^2} + \frac{\lambda}{\lambda^2+(u+t)^2} \right) \right\} dt$$
$$= \tfrac{1}{2} \frac{u}{\lambda^2+u^2} \log \frac{\lambda^2+(u+\delta)^2}{\lambda^2+(u-\delta)^2} + \frac{\lambda}{\lambda^2+u^2} \left(\arctan \frac{u-\delta}{\lambda} - \arctan \frac{u+\delta}{\lambda} \right)$$
$$= \tfrac{1}{2} I_1 - I_2, \text{ say.}$$

Then, for a fixed U,

$$\lim_{\delta \to \infty} \int_{-U}^{U} f(u+x) [\tfrac{1}{2} I_1 - I_2]\, du = 0.$$

If $U^2 > \lambda^2+\delta^2$, the functions

$$\log \frac{\lambda^2+(u+\delta)^2}{\lambda^2+(u-\delta)^2}, \quad \arctan \frac{u+\delta}{\lambda} - \arctan \frac{u-\delta}{\lambda}$$

are decreasing functions of u for $u > |U|$; hence, by using the second

mean value theorem,

$$\int_U^\infty I_1 f(u+x)\,du = \log\frac{\lambda^2+(U+\delta)^2}{\lambda^2+(U-\delta)^2}\int_U^{U'}\frac{uf(u+x)}{\lambda^2+u^2}\,du$$
$$= O(\delta/U).O(1) = o(1);$$
$$\int_U^\infty I_2 f(u+x)\,du = \arctan\frac{2\delta\lambda}{\lambda^2+U^2-\delta^2}\int_U^{U'}\frac{\lambda}{\lambda^2+u^2}\,f(u+x)\,du$$
$$= O(\delta/U^2).O(1) = o(1).$$

Similarly, the integral

$$\int_{-\infty}^{-U}(\tfrac{1}{2}I_1-I_2)f(u+x)\,du$$

may be shown to be $o(1)$, and the change in the order of integration is justified.

4 . 2. LEMMA β. *Properties of* $\phi(\lambda, u, x)$ *and* $\chi(\lambda, u, x)$.

(i) *Given* ϵ, *arbitrarily small, a value of* T *exists, such that*

$$|\phi(\lambda, u, x)| < \epsilon, \quad |\chi(\lambda, u, x)| < \epsilon,$$

for $u > \sqrt{T}$.

(ii) *There exists a positive constant* K, *such that*

$$|\phi(\lambda, u, x)| < K, \quad |\chi(\lambda, u, x)| < K,$$

for $1 \leqslant u < \sqrt{T}$.

(iii) *There exists a positive constant* K, *such that*

$$\int_0^1 |\phi(\lambda, u, x)|\,du < K,$$
$$\int_0^1 u\,|\chi(\lambda, u, x)|\,du < K.$$

(i) and (ii) are obvious from the definitions (4 . 3) and (4 . 4).

(iii) From the second mean value theorem,

$$\phi(\lambda, u, x) = \int_u^1 \frac{t}{\lambda^2+t^2}[f(x+t)+f(x-t)]\,dt + \frac{1}{\lambda^2+1}\int_1^T \frac{f(x+t)+f(x-t)}{t}\,dt,$$

and so

$$|\phi(\lambda, u, x)| < \int_u^1 \frac{|f(x+t)+f(x-t)|}{t}\,dt + K_1,$$

$$\int_0^1 |\phi(\lambda, u, x)|\,du < \int_0^1 du\int_u^1 \frac{|f(x+t)+f(x-t)|}{t}\,dt + K_1$$
$$= \int_0^1 \frac{|f(x+t)+f(x-t)|}{t}\,dt\int_0^t du + K_1$$
$$= \int_0^1 |f(x+t)+f(x-t)|\,dt + K_1 < K.$$

The second mean value theorem also gives

$$\chi(\lambda, u, x) = \int_u^1 \frac{\lambda}{\lambda^2+t^2}[f(x+t)+f(x-t)]\,dt + \frac{\lambda}{\lambda^2+1}\int_1^T \frac{f(x+t)+f(x-t)}{t}\,dt,$$

and hence

$$\int_0^1 u\,|\chi(\lambda, u, x)|\,du < \int_0^1 u\,du \int_u^1 \frac{\lambda\,|f(x+t)+f(x-t)|}{t^2}\,dt + K_1$$

$$= \int_0^1 \frac{\lambda\,|f(x+t)+f(x-t)|}{t^2}\,dt \int_0^t u\,du + K_1 < K.$$

4.3. LEMMA γ. *For fixed x,*

$$-\frac{1}{\pi}\int_{-T}^{T} \frac{G(\lambda, x+t)}{t}\,dt$$

$$= F(\lambda, x) + \frac{1}{\pi^2}\int_{(1-\theta)T}^{(1+\theta)T} \frac{u-T}{\lambda^2+(u-T)^2}\,\phi(\lambda, u, x)\,du + o(1),$$

as $T\to\infty$, uniformly with respect to λ, where θ is any fixed number in (0, 1).

Using the work and the result of Lemma α, we have

$$\pi\int_{-T}^{T} \frac{G(\lambda, x+t)}{t}\,dt = \pi\int_0^T \frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt$$

$$= \int_0^T \frac{dt}{t}\int_{-\infty}^{\infty}\left[\frac{u-x-t}{\lambda^2+(u-x-t)^2} - \frac{u-x+t}{\lambda^2+(u-x+t)^2}\right] f(u)\,du$$

$$= \int_0^T \frac{dt}{t}\int_{-\infty}^{\infty}\left[\frac{u-t}{\lambda^2+(u-t)^2} - \frac{u+t}{\lambda^2+(u+t)^2}\right] f(u+x)\,du$$

$$= \int_{-\infty}^{\infty} \frac{u}{\lambda^2+u^2} f(u+x)\,du \int_0^T \left[\frac{u-t}{\lambda^2+(u-t)^2} + \frac{u+t}{\lambda^2+(u+t)^2}\right] dt$$

$$- \int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+u^2} f(u+x)\,du \int_0^T \left[\frac{\lambda}{\lambda^2+(u-t)^2} + \frac{\lambda}{\lambda^2+(u+t)^2}\right] dt$$

$$= \int_0^{\infty} \frac{u}{\lambda^2+u^2}[f(x+u)+f(x-u)]\,du \int_0^T \left[\frac{u-t}{\lambda^2+(u-t)^2} + \frac{u+t}{\lambda^2+(u+t)^2}\right] dt$$

$$- \int_0^{\infty} \frac{\lambda}{\lambda^2+u^2}[f(x+u)+f(x-u)]\,du$$

$$\times \int_0^T \left[\frac{\lambda}{\lambda^2+(u-t)^2} + \frac{\lambda}{\lambda^2+(u+t)^2}\right] dt$$

$$= I_3 - I_4, \text{ say.}$$

From (4 . 3),

$$\frac{\partial}{\partial u}\phi(\lambda, u, x) = -\frac{u}{\lambda^2+u^2}[f(x+u)+f(x-u)],$$

and hence, by integration by parts,

$$\begin{aligned} I_3 &= \int_0^\infty -\frac{\partial}{\partial u}\phi(\lambda, u, x)\, du \int_0^T \left[\frac{u-t}{\lambda^2+(u-t)^2} + \frac{u+t}{\lambda^2+(u+t)^2}\right] dt \\ &= \int_0^\infty \phi(\lambda, u, x)\left[\frac{u+T}{\lambda^2+(u+T)^2} - \frac{u-T}{\lambda^2+(u-T)^2}\right] du \\ &= \int_0^1 + \int_1^{\sqrt{T}} + \int_{\sqrt{T}}^{(1-\theta)T} + \int_{(1-\theta)T}^{(1+\theta)T} + \int_{(1+\theta)T}^\infty \\ &= J_1+J_2+J_3+J_4+J_5, \text{ say.} \end{aligned}$$

It will be shown that I_3 reduces to the second term of J_4 as $T\to\infty$. Using Lemma β,

$$|J_1| < \int_0^1 |\phi(\lambda, u, x)|\, O\left(\frac{1}{T}\right) du = o(1),$$

$$\begin{aligned} |J_2| &< K\int_1^{\sqrt{T}} \left[\frac{2T}{\lambda^2+T^2} + \frac{T}{\lambda^2+(T-\sqrt{T})^2}\right] du \\ &< K\int_1^{\sqrt{T}} \left[\frac{2}{T} + \frac{T}{(T-\sqrt{T})^2}\right] du = o(1), \end{aligned}$$

$$\begin{aligned} |J_3| &< \epsilon\int_{\sqrt{T}}^{(1-\theta)T} \left[\frac{(2-\theta)T}{\lambda^2+T^2} + \frac{T}{\lambda^2+\theta^2T^2}\right] du \\ &< \epsilon\int_{\sqrt{T}}^{(1-\theta)T} \left[\frac{2}{T} + \frac{1}{\theta^2 T}\right] du < \epsilon, \end{aligned}$$

$$\begin{aligned} |J_5| &< \int_{(1+\theta)T}^\infty \left[\frac{u-T}{\lambda^2+(u-T)^2} - \frac{u+T}{\lambda^2+(u+T)^2}\right] du \\ &< \epsilon\int_{(1+\theta)T}^\infty \left[\frac{1}{u-T} - \frac{1}{u+T}\right] du \\ &= \epsilon\log\frac{2+\theta}{\theta} < \epsilon. \end{aligned}$$

Also
$$\left|\int_{(1-\theta)T}^{(1+\theta)T} \phi(\lambda, u, x)\frac{u+T}{\lambda^2+(u+T)^2}\, du\right| < \epsilon\int_{(1-\theta)T}^{(1+\theta)T} \frac{(2+\theta)T}{\lambda^2+T^2}\, du$$
$$< \epsilon\int_{(1-\theta)T}^{(1+\theta)T} \frac{2+\theta}{T}\, du < \epsilon.$$

Hence
$$I_3 = -\int_{(1-\theta)T}^{(1+\theta)T} \phi(\lambda, u, x)\frac{u-T}{\lambda^2+(u-T)^2}\, du + o(1).$$

Passing to I_4, using (4 . 4), and integrating by parts, we have

$$I_4 = \int_0^\infty -\frac{\partial}{\partial u}\chi(\lambda, u, x)\,du \int_0^T \left[\frac{\lambda}{\lambda^2+(u-t)^2} + \frac{\lambda}{\lambda^2+(u+t)^2}\right] dt$$

$$= \chi(\lambda, 0, x)\int_0^T \frac{2\lambda}{\lambda^2+t^2}\,dt - \int_0^\infty \chi(\lambda, u, x)\left[\frac{\lambda}{\lambda^2+(u-T)^2} - \frac{\lambda}{\lambda^2+(u+T)^2}\right] du$$

$$= \pi F(\lambda, x).2 \arctan\frac{T}{\lambda} - \int_0^1 \chi(\lambda, u, x)\,\frac{4\lambda u T}{[\lambda^2+(u-T)^2][\lambda^2+(u+T)^2]}\,du$$

$$- \int_1^{\sqrt{T}} \chi(\lambda, u, x)\,\frac{4\lambda u T}{[\lambda^2+(u-T)^2][\lambda^2+(u+T)^2]}\,du$$

$$- \int_{\sqrt{T}}^\infty \chi(\lambda, u, x)\left[\frac{\lambda}{\lambda^2+(u-T)^2} - \frac{\lambda}{\lambda^2+(u+T)^2}\right] du$$

$$= \pi F(\lambda, x).2 \arctan\frac{T}{\lambda} + J_6 + J_7 + J_8.$$

From Lemma β,

$$|J_6| < \int_0^1 u\,|\chi(\lambda, u, x)|\,O\left(\frac{1}{T^3}\right) du = o(1),$$

$$|J_7| < KT\int_1^{\sqrt{T}} O\left(\frac{1}{T^4}\right) du < \epsilon,$$

$$|J_8| < \epsilon\int_{-\infty}^\infty \left[\frac{\lambda}{\lambda^2+(u-T)^2} + \frac{\lambda}{\lambda^2+(u+T)^2}\right] du = \pi\epsilon.$$

Hence $$I_4 = -2\pi\left[\tfrac{1}{2}\pi + o(1)\right]F(\lambda, x) + o(1),$$

and so

$$-\frac{1}{\pi}\int_{-T}^T \frac{G(\lambda, x+t)}{t}\,dt$$

$$= F(\lambda, x) + \frac{1}{\pi^2}\int_{(1-\theta)T}^{(1+\theta)T} \frac{u-T}{\lambda^2+(u-T)^2}\,\phi(\lambda, u, x)\,du + o(1).$$

4 . 4. LEMMA δ. *For fixed x,*

$$-\frac{1}{\pi}\cdot\frac{1}{y}\int_0^y dT\int_{-T}^T \frac{G(\lambda, x+t)}{t}\,dt = F(\lambda, x) + o(1)$$

as $y\to\infty$, uniformly for $\lambda \geqq 0$.

In view of Lemma γ, it is sufficient to show that

$$\lim_{y\to\infty}\frac{1}{y}\int_0^y dT\int_{(1-\theta)T}^{(1+\theta)T} \frac{T-u}{\lambda^2+(u-T)^2}\,\phi(\lambda, u, x)\,du = 0.$$

Now

$$\int_0^y dT \int_{(1-\theta)T}^{(1+\theta)T} \frac{T-u}{\lambda^2+(u-T)^2}\,\phi(\lambda, u, x)\,du$$

$$= \int_0^{(1-\theta)y} \phi(\lambda, u, x)\,du \int_{u/(1+\theta)}^{u/(1-\theta)} \frac{T-u}{\lambda^2+(u-T)^2}\,dT$$

$$+ \int_{(1-\theta)y}^{(1+\theta)y} \phi(\lambda, u, x)\,du \int_{u/(1+\theta)}^{y} \frac{T-u}{\lambda^2+(u-T)^2}\,dT$$

$$= \int_0^{(1-\theta)y} \phi(\lambda, u, x)\,du \int_{(-\theta u)/(1+\theta)}^{\theta u/(1-\theta)} \frac{v\,dv}{\lambda^2+v^2}$$

$$+ \int_{(1-\theta)y}^{(1+\theta)y} \phi(\lambda, u, x)\,du \int_{(-\theta u)/(1+\theta)}^{y-u} \frac{v}{\lambda^2+v^2}\,dv$$

$$= J_1+J_2.$$

$$|J_1| = \left| \int_0^{(1-\theta)y} \phi(\lambda, u, x)\,du \int_{\theta u/(1+\theta)}^{\theta u/(1-\theta)} \frac{v}{\lambda^2+v^2}\,dv \right|$$

$$< \int_0^{(1-\theta)y} |\phi(\lambda, u, x)|\,du \int_{\theta u/(1+\theta)}^{\theta u/(1-\theta)} \frac{1}{v}\,dv$$

$$< \int_0^1 |\phi(\lambda, u, x)| \log\frac{1+\theta}{1-\theta}\,du + \int_1^{\sqrt{y}} K \log\frac{1+\theta}{1-\theta}\,du + \int_{\sqrt{y}}^{(1-\theta)y} \epsilon \log\frac{1+\theta}{1-\theta}\,du$$

$$< K+K\sqrt{y}+\epsilon y,$$

and $|J_1|/y < \epsilon$.

$$|J_2| = \int_{(1-\theta)y}^{[(1+\theta)/(1+2\theta)]\,y} \phi(\lambda, u, x)\,du \int_{\theta u/(1+\theta)}^{y-u} \frac{v}{\lambda^2+v^2}\,dv - \int_{[(1+\theta)/(1+2\theta)]\,y}^{y} \int_{y-u}^{\theta u/(1+\theta)}$$

$$- \int_y^{(1+\theta)y} \int_{u-y}^{\theta u/(1+\theta)}$$

$$= J_{2.1}+J_{2.2}+J_{2.3}.$$

$$|J_{2.1}| < \epsilon \int_{(1-\theta)y}^{[(1+\theta)/(1+2\theta)]\,y} du \int_{\theta u/(1+\theta)}^{y-u} \frac{1+\theta}{\theta(1-\theta)y}\,dv < \epsilon \int_{(1-\theta)y}^{[(1+\theta)/(1+2\theta)]\,y} K du < \epsilon y,$$

$$|J_2| < \epsilon \int_{[(1+\theta)/(1+2\theta)]\,y}^{y} du \int_{y-u}^{\theta y/(1+\theta)} \frac{1}{v}\,dv < \epsilon \int_{y/(1+\theta)}^{y} du \int_{y-u}^{\theta y/(1+\theta)} \frac{1}{v}\,dv$$

$$= \epsilon \int_0^{\theta y/(1+\theta)} \frac{1}{v}\,dv \int_{y-v}^{y} dv < \epsilon y,$$

$$|J_{2.3}| < \epsilon \int_y^{(1+\theta)y} du \int_{u-y}^{\theta y} \frac{1}{v}\,dv = \epsilon \int_0^{\theta y} \frac{dv}{v} \int_y^{y+v} dv < \epsilon y.$$

This proves the result.

4.5. Lemma ϵ. *If the integral*

$$-\frac{1}{\pi}\int_0^\infty \frac{G(\lambda, t)+G(\lambda, -t)}{t}\,dt$$

is summable $(C, 1)$ *to the sum* $F(\lambda, 0)$, *then the integral*

$$-\frac{1}{\pi}\int_0^\infty \frac{G(\lambda, x+t)+G(\lambda, x-t)}{t}\,dt$$

is summable $(C, 1)$ *to the sum* $F(\lambda, x)$ *for any fixed* x.

Write $G(\lambda, x+t) = H(\lambda, t)$.

Then
$$H(\lambda, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u-x-t}{\lambda^2+(u-x-t)^2} f(u)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v-t}{\lambda^2+(v-t)^2}\phi(v)\,dv, \text{ say.}$$

Now $\phi(v)$ is integrable over any finite interval, since $f(v)$ is, and the integrals

$$\int^\infty \frac{\phi(\pm v)}{v}\,dv$$

exist, since the integrals (3.1) exist. Hence we have established the existence of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda\phi(v)}{\lambda^2+(v-t)^2} = \Phi(\lambda, t), \text{ say.}$$

Hence the integral

$$-\frac{1}{\pi}\int_0^\infty \frac{H(\lambda, t)+H(\lambda, -t)}{t}\,dt$$

is summable to the sum $\Phi(\lambda, 0)$. Now

$$\Phi(\lambda, 0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda\phi(v)}{\lambda^2+v^2}\,dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda f(u)}{\lambda^2+(u-x)^2}\,du = F(\lambda, x).$$

Hence the result.

5. *Proof of Theorem* I.

Lemmas.

(5.1) $$|F(\lambda, x)-f(x)| < A\lambda^\alpha,$$

(5.2) $$|G(\lambda, x)-g(x)| < A\lambda^\alpha.$$

These lemmas are proved in Paper I in the case when $f(x)$ is of integrable square over $(-\infty, \infty)$. The method of proof with the con-

ditions of the present theorem is precisely the same, except that the second mean value theorem is used. In Paper I, A is independent of x; here, A is not uniform in x.

By using (5 . 2) it follows that

$$\lim_{\lambda\to 0}\int_{\delta}^{T}\frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt=\int_{\delta}^{T}\frac{g(x+t)-g(x-t)}{t}\,dt \quad (\delta>0);$$

and so, by an argument similar to that of Paper I, page 123, the same result follows; that is

$$|G(\lambda, x+h)-G(\lambda, x-h)|<Ah\lambda^{\alpha-1},$$

and $g(x)$ satisfies a Lipschitz condition of order α. Hence, as in Paper I, page 129, it follows that

$$\lim_{\lambda\to 0}\int_{0}^{\delta}\frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt=\int_{0}^{\delta}\frac{g(x+t)-g(x-t)}{t}\,dt,$$

and hence that

$$(5\,.\,3)\qquad \lim_{\lambda\to 0}\int_{0}^{T}\frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt=\int_{0}^{T}\frac{g(x+t)-g(x-t)}{t}\,dt.$$

Using (5 . 2) and (5 . 3),

$$\begin{aligned}\int_{0}^{y}dT\int_{0}^{T}&\frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt\\ &=y\int_{0}^{y}\frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt-\int_{0}^{y}[G(\lambda, x+t)-G(\lambda, x-t)]\,dt\\ &=y\int_{0}^{y}\frac{g(x+t)-g(x-t)}{t}\,dt-\int_{0}^{y}[g(x+t)-g(x-t)]\,dt;\end{aligned}$$

hence, using Lemma 8 and (5 . 1),

$$-\frac{1}{\pi}\cdot\frac{1}{y}\int_{0}^{y}dT\int_{0}^{T}\frac{g(x+t)-g(x-t)}{t}\,dt=\lim_{\lambda\to 0}F(\lambda, x)=f(x);$$

that is, under the conditions stated, the integral

$$-\frac{1}{\pi}\int_{0}^{\infty}\frac{g(x+t)-g(x-t)}{t}\,dt$$

is summable $(C, 1)$ to the sum $f(x)$.

6. *Proof of Theorem II.*

Let $$f(x)=f_1(x)+f_2(x),$$

where

$$(6\,.\,1)\qquad f_1(x)=f(x)\ (|x|<X);\ f_1(x)=0\ (|x|>X).$$

Let $$g_1(x)=\frac{1}{\pi}\int_{0}^{\infty}\frac{f_1(x+t)-f_1(x-t)}{t}\,dt$$

and $$G_1(\lambda, x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{\lambda^2+(t-x)^2} f_1(t)\,dt;$$

let $g_2(x)$ and $G_2(\lambda, x)$ be the same functions of f_2 as $g_1(x)$ and $G_1(\lambda, x)$ are of f_1.

Then $$g(x) = g_1(x)+g_2(x)$$

and $$G(\lambda, x) = G_1(\lambda, x)+G_2(\lambda, x).$$

6.1. LEMMA ζ.

$$\lim_{\lambda\to 0}\int_{-\infty}^{\infty} |G_1(\lambda, x)-g_1(x)|^p\,dx = 0.$$

For $\{f(x)\}^p$ is integrable over any finite interval by the conditions of the theorem, and then (6.1) shows that $\{f_1(x)\}^p$ is integrable over $(-\infty, \infty)$. Now $g_1(x)$ is the conjugate of $f_1(x)$, so that $\{g_1(x)\}^p$ is integrable over $(-\infty, \infty)$. Moreover, it is readily verified that $(t-x)/\{\lambda^2+(t-x)^2\}$ is the conjugate of $\lambda/\{\lambda^2+(t-x)^2\}$; hence

$$G_1(\lambda, x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+(t-x)^2} g_1(t)\,dt.$$

Now $$G_1(\lambda, x)-g_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+t^2}[g_1(x+t)-g_1(x)]\,dt.$$

Hence* $$|G_1(\lambda, x)-g_1(x)|^p$$

$$= \left|\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\lambda}{\lambda^2+t^2}\right)^{(p-1)/p}\left(\frac{\lambda}{\lambda^2+t^2}\right)^{1/p}[g_1(x+t)-g_1(x)]\,dt\right|^p$$

$$\leqslant \left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{\lambda^2+t^2}\,dt\right)^{p-1}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{\lambda^2+t^2}|g_1(x+t)-g_1(x)|^p\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{\lambda^2+t^2}|g_1(x+t)-g_1(x)|^p\,dt.$$

Hence

$$\int_{-\infty}^{\infty}|G_1(\lambda, x)-g_1(x)|^p\,dx \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{\lambda^2+t^2}\,dt\int_{-}^{\infty}|g_1(x+t)-g_1(x)|^p\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda h(t)}{\lambda^2+t^2}\,dt \text{ (say)},$$

where $h(t)$ is bounded for all t, and is $o(1)$ as $t\to 0$.†

* For the method of proof of this lemma, cf. E. W. Hobson, *Theory of functions of a real variable*, 2 (2nd ed. 1926), 612, where it is proved that $\lim_{n\to\infty}\int_{-\pi}^{\pi}|f(x)-f_n(x)|^p\,dx = 0$, where $f_n(x)$ denotes a partial sum of the Fourier series for $f(x)$.

† Cf. Hobson, 1 (2nd ed. 1921), 584.

Hence

$$\int_{-\infty}^{\infty} |G_1(\lambda, x)-g_1(x)|^p\,dx = \frac{1}{\pi}\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} \frac{\lambda}{\lambda^2+t^2}\,h(t)\,dt$$
$$\leqslant \frac{1}{\pi}\left[\left|\int_{-\infty}^{-\delta}\right| + \left|\int_{-\delta}^{\delta}\right| + \left|\int_{\delta}^{\infty}\right|\right].$$

The first and third integrals are $O(\lambda)$, and the second integral

$$= O\left(\int_0^{\delta} \frac{\lambda}{\lambda^2+t^2}\,K\,dt\right) = o(1),$$

as $\delta \to 0$. Hence the result.

6.2. LEMMA η. *For fixed T, with the conditions of Theorem II,*

$$\lim_{\lambda\to 0}\int_0^T \frac{G(\lambda, x+t)-G(\lambda, x-t)}{t}\,dt = \int_0^T \frac{g(x+t)-g(x-t)}{t}\,dt$$

for almost all x, and

$$\lim_{\lambda\to 0}\int_0^T [G(\lambda, x+t)-G(\lambda, x-t)]\,dt = \int_0^T [g(x+t)-g(x-t)]\,dt.$$

6.21. Now

$$\int_0^{\infty} \frac{G_1(\lambda, x+t)-G_1(\lambda, x-t)}{t}\,dt$$
$$= \frac{1}{\pi}\int_0^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty}\left[\frac{u-x-t}{\lambda^2+(u-x-t)^2} - \frac{u-x+t}{\lambda^2+(u-x+t)^2}\right] f_1(u)\,du$$
$$= \frac{2}{\pi}\int_{-\infty}^{\infty} f_1(u)\,du \int_0^{\infty}\frac{1}{t}\left[\frac{u-x-t}{\lambda^2+(u-x-t)^2} - \frac{u-x+t}{\lambda^2+(u-x+t)^2}\right] dt$$
$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-\lambda\pi}{\lambda^2+(u-x)^2} f_1(u)\,du = -\pi F_1(\lambda, x),$$

the inversion of the order of the integrations being justified by the fact that $f_1(u) = 0$ when $u > U$, so that the u-integral is actually taken over a finite range only, and the t-integral converges uniformly with respect to u over this finite range.

Now $F_1(\lambda, x) \to f_1(x)$ almost everywhere, and

$$f_1(x) = \int_0^{\infty} \frac{g_1(x+t)-g_1(x-t)}{t}\,dt$$

almost everywhere; hence, for almost all x,

$$\lim_{\lambda\to 0}\int_0^{\infty} \frac{G_1(\lambda, x+t)-G_1(\lambda, x-t)}{t}\,dt = \int_0^{\infty} \frac{g_1(x+t)-g_1(x-t)}{t}\,dt \tag{6.21}$$

Also

$$\left|\int_T^\infty \frac{G_1(\lambda, x+t)-g_1(x+t)}{t}\,dt\right| \leqslant \left(\int_T^\infty \frac{1}{t^{p/(p-1)}}\,dt\right)^{(p-1)/p}\left(\int_T^\infty |G_1(\lambda, x+t)-g_1(x+t)|^p\,dt\right)^{1/p};$$

hence, by Lemma ζ,

$$\lim_{\lambda\to 0}\int_T^\infty \frac{G_1(\lambda, x+t)-g_1(x+t)}{t}\,dt = 0.$$

Similarly for the integral involving $G_1(\lambda, x-t)$ and $g_1(x-t)$, so that

$$(6.22)\quad \lim_{\lambda\to 0}\int_T^\infty \frac{G_1(\lambda, x+t)-G_1(\lambda, x-t)}{t}\,dt = \int_T^\infty \frac{g_1(x+t)-g_1(x-t)}{t}\,dt.$$

Combining (6.21) and (6.22),

$$\lim_{\lambda\to 0}\int_0^T \frac{G_1(\lambda, x+t)-G_1(\lambda, x-t)}{t}\,dt = \int_0^T \frac{g_1(x+t)-g_1(x-t)}{t}\,dt.$$

Again,

$$\int_0^T |G_1(\lambda, x+t)-g_1(x+t)|\,dt \leqslant \left(\int_0^T dt\right)^{(p-1)/p}\left(\int_0^T |G_1(\lambda, x+t)-g_1(x+t)|^p\,dt\right)^{1/p},$$

hence, by Lemma ζ again,

$$\lim_{\lambda\to 0}\int_0^T |G_1(\lambda, x+t)-g_1(x+t)|\,dt = 0,$$

and hence

$$\lim_{\lambda\to 0}\int_0^T [G_1(\lambda, x+t)-G_1(\lambda, x-t)]\,dt = \int_0^T [g_1(x+t)-g_1(x-t)]\,dt.$$

6.22. Now

$$f_2(x) = 0 \quad (|x| < X)$$
$$= f(x) \quad (|x| > X),$$

hence, if $X > 2T$, $f_2(x)$ satisfies the conditions of Theorem I with $\alpha = 1$, and it follows, as in Theorem I, that

$$\lim_{\lambda\to 0}\int_0^T \frac{G_2(\lambda, x+t)-G_2(\lambda, x-t)}{t}\,dt = \int_0^T \frac{g_2(x+t)-g_2(x-t)}{t}\,dt$$

and

$$\lim_{\lambda\to 0}\int_0^T [G_2(\lambda, x+t)-G_2(\lambda, x-t)]\,dt = \int_0^T [g_2(x+t)-g_2(x-t)]\,dt.$$

Combining the results for G_1 and G_2, we see that

$$\lim_{\lambda\to 0}\int_0^T \frac{G(\lambda, x+t)-G(\lambda, x-t)]}{t}\,dt = \int_0^T \frac{g(x+t)-g(x-t)}{t}\,dt$$

and

$$\lim_{\lambda\to 0}\int_0^T [G(\lambda, x+t)-G(\lambda, x-t)]\,dt = \int_0^T [g(x+t)-g(x-t)]\,dt.$$

6 . 3. Reasoning similar to the last part of the proof of Theorem I then shows that

$$\lim_{y\to\infty}\frac{1}{\pi}\cdot\frac{1}{y}\int_0^y dT\int_0^T \frac{g(x+t)-g(x-t)}{t}\,dt = -f(x).$$

7. *Proof of Theorem III.*

The result of this theorem follows from that of Lemma β, if it can be shown that

$$\lim_{\lambda\to 0}\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}\,du\int_u^\infty \frac{t}{\lambda^2+t^2}f(x+t)\,dt = \int_{(1-\theta)T}^{(1+\theta)T}\frac{du}{u-T}\int_u^\infty \frac{f(t)}{t}\,dt+o(1)$$

as $T\to\infty$, with a similar result for $f(-t)$.

7 . 1. LEMMA θ. *The integral*

$$\int_{T-1}^{T+1}\frac{du}{u-T}\int_u^\infty \frac{f(t)}{t}\,dt = o(1)$$

as $T\to\infty$, *if condition* (3 . 4) *is satisfied.*

The integral may be written

$$\int_{-1}^1 \frac{du}{u}\int_{u+T}^\infty \frac{f(t)}{t}\,dt = \lim_{\epsilon\to 0}\Big[\int_{-1}^{-\epsilon}\frac{du}{u}\int_{u+T}^T \frac{f(t)}{t}\,dt+\int_{-1}^{-\epsilon}\frac{du}{u}\int_T^{T-u}\frac{f(t)}{t}\,dt$$
$$+\int_{-1}^{-\epsilon}\frac{du}{u}\int_{T-u}^\infty+\int_\epsilon^1\frac{du}{u}\int_{T+u}^\infty\Big]$$
$$= \lim_{\epsilon\to 0}\Big[\int_{-1}^{-\epsilon}\frac{du}{u}\int_{u+T}^T \frac{f(t)}{t}\,dt+\int_{-1}^{-\epsilon}\frac{du}{u}\int_T^{T-u}\frac{f(t)}{t}\,dt\Big]$$
$$= \lim_{\epsilon\to 0}\Big[\int_{T-1}^{T-\epsilon}\frac{f(t)}{t}\,dt\int_{-1}^{t-T}\frac{du}{u}+\int_{T+\epsilon}^{T+1}\frac{f(t)}{t}\,dt\int_{-1}^{T-t}\frac{du}{u}+\int_{T-\epsilon}^{T+\epsilon}\frac{f(t)}{t}\,dt\int_1^\epsilon\frac{du}{u}\Big].$$

The last integral is $O(1/T)\,.\,\epsilon\log\epsilon$, and it tends to zero with ϵ. Hence

$$\Big|\int_{-1}^1\frac{du}{u}\int_{u+T}^\infty\frac{f(t)}{t}\,dt\Big| \leqslant \Big|\int_{T-1}^T\frac{f(t)}{t}\,dt\int_1^{T-t}\frac{du}{u}\Big|+\Big|\int_T^{T+1}\frac{f(t)}{t}\,dt\int_1^{t-T}\frac{du}{u}\Big|$$
$$= O\Big(\frac{1}{T}\Big)\Big(\int_{T-1}^{T+1}|f(t)|^p\,dt\Big)^{1/p}\Big(\int_{T-1}^{T+1}(\log|t-T|)^{p/(p-1)}\,dt\Big)^{1-(1/p)} = o(1).$$

7 . 2. Now

$$\int_u^\infty\frac{t}{\lambda^2+t^2}f(x+t)\,dt = \int_u^\infty\frac{f(t)}{t}\,dt+\int_u^\infty\frac{f(x+t)-f(t)}{t}\,dt-\int_u^\infty\frac{\lambda^2 f(x+t)}{t(\lambda^2+t^2)}\,dt,$$

and hence

$$\int_{(1-\theta)T}^{(1+\theta)T}\frac{1}{u-T}du\int_u^\infty\frac{f(t)}{t}dt-\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}du\int_u^\infty\frac{t}{\lambda^2+t^2}f(x+t)\,dt$$

$$=\int_{(1-\theta)T}^{(1+\theta)T}\frac{\lambda^2}{(u-T)[\lambda^2+(u-T)^2]}du\int_u^\infty\frac{f(t)}{t}dt$$

$$-\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}du\int_u^\infty\frac{f(x+t)-f(t)}{t}dt$$

$$+\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}du\int_u^\infty\frac{\lambda^2 f(x+t)}{t(\lambda^2+t^2)}dt$$

$$=J_1+J_2+J_3.$$

$$|J_1|=\left|\int_{(1-\theta)T}^{T-1}+\int_{T-1}^{T+1}+\int_{T+1}^{(1+\theta)T}\frac{\lambda^2}{(u-T)\{\lambda^2+(u-T)^2\}}du\int_u^\infty\frac{f(t)}{t}dt\right|$$

$$=J_{1.1}+J_{1.2}+J_{1.3}.$$

The integrations in $J_{1.1}$ and $J_{1.3}$ may be inverted, since the t-integral is uniformly convergent with respect to u for all positive values of λ. Hence

$$|J_{1.1}|<\left|\int_u^\infty\frac{f(t)}{t}dt\int_{(1-\theta)T}^{T-1}\frac{1}{(T-u)^3}du\right|<\epsilon,$$

$$|J_{1.3}|<\left|\int_u^\infty\frac{f(t)}{t}dt\int_{T+1}^{(1+\theta)T}\frac{1}{(u-T)^3}du\right|<\epsilon,$$

$$|J_{1.2}|<\left|\int_{T-1}^{T+1}\frac{1}{u-T}du\int_u^\infty\frac{f(t)}{t}dt\right|<\epsilon,$$

using Lemma θ.

The integrations in J_2 and J_3 may be inverted, since the t-integral is uniformly convergent with respect to u for all positive values of λ. Hence

$$|J_2|=\left|\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}du\int_{u+x}^\infty\frac{f(t)}{(t-x)t}dt\right|$$

$$+\left|\int_{(1-\theta)T}^{(1+\theta)T}\frac{u-T}{\lambda^2+(u-T)^2}du\int_u^{u+x}\frac{f(t)}{t}dt\right|$$

$$=\left|\int_{T-1}^{T+1}\frac{u-T}{\lambda^2+(u-T)^2}du\int_{u+x}^\infty\frac{f(t)}{(t-x)t}dt\right|$$

$$+\left|\int_{T-1}^{T+1}\frac{u-T}{\lambda^2+(u-T)^2}du\int_u^{u+x}\frac{f(t)}{t}dt\right|+o(1)$$

$$=\left|\int_0^1\frac{u}{\lambda^2+u^2}du\int_{T+x+u}^{T+x-u}\frac{f(t)}{t(t-x)}dt\right|$$

$$+\left|\int_{-1}^1\frac{u}{\lambda^2+u^2}du\int_{u+T}^{u+T+x}\frac{f(t)}{t}dt\right|+o(1).$$

The first term is $o(1)$ by the work of Lemma θ, and the second term vanishes. Hence

$$|J_2|=o(1).$$

$|J_3|$ behaves in the same way as $|J_1|$, and consequently it is $o(1)$ as $T \to \infty$. Hence the result.

8. *Proof of Theorem IV.*

In view of Lemma ϵ, it is sufficient to discuss the case $x = 0$. Let

$$g'(t) = \frac{1}{\pi}\int_0^\infty \frac{f(u)}{u-t}\,du,$$

$$g''(t) = \frac{1}{\pi}\int_{-\infty}^0 \frac{f(u)}{u-t}\,du = -\frac{1}{\pi}\int_0^\infty \frac{f(-u)}{u+t}\,du.$$

These integrals certainly exist if $f(x)$ obeys a Lipschitz condition; if $f(x)$ is such that its p-th power is integrable, these integrals exist except in the single case when $t = 0$. We have then

$$g(t) = g'(t)+g''(t),$$

and, for t sufficiently large,

$$g''(t)+g'(-t) = \frac{1}{\pi}\int_0^\infty \frac{f(u)-f(-u)}{u+t}\,du = o(1).$$

Hence

$$\int_1^T \frac{t}{T}\cdot\frac{g''(t)+g'(-t)}{t}\,dt = o(1) \tag{8.1}$$

and

$$\pi\int_1^T \frac{g''(t)+g'(-t)}{t}\,dt = \int_1^T \frac{dt}{t}\int_0^\infty \frac{-u\psi'(u)}{u+t}\,du \tag{8.2}$$

$$= \int_1^T \frac{dt}{t}\int_0^\infty \frac{t\psi(u)}{(u+t)^2}\,du = \int_0^\infty \psi(u)\left[\frac{1}{u+1}-\frac{1}{u+T}\right]du$$

$$= \int_0^T \frac{\psi(u)}{u+1}\,du - \int_0^T \frac{\psi(u)}{u+T}\,du + \int_T^\infty \psi(u)\left[\frac{1}{u+1}-\frac{1}{u+T}\right]du$$

$$= \int_0^T \frac{\psi(u)}{u+1}\,du + \int_0^T o\left(\frac{1}{T}\right)du + (T-1)\int_T^\infty o\left(\frac{1}{u^2}\right)du$$

$$= \int_0^T \frac{\psi(u)}{u+1}\,du + o(1).$$

The inversion of the integrations in

$$\int_1^T dt\int_0^\infty \frac{\psi(u)}{(u+t)^2}\,du$$

is justified by the fact that the u-integral is uniformly convergent with respect to t in the interval $(1, T)$. Adding (8.1) and (8.2), we have

$$\int_1^T \left(1-\frac{t}{T}\right)\frac{g''(t)+g'(-t)}{t}\,dt = \frac{1}{\pi}\int_0^T \frac{\psi(u)}{u+1}\,du + o(1). \tag{8.3}$$

Now, by Theorems I and II,

$$\int_1^T \left(1-\frac{t}{T}\right)\frac{g'(t)-g'(-t)}{t}\,dt = A+o(1); \tag{8.4}$$

hence, adding (8.3) and (8.4),

$$\int_1^T \left(1-\frac{t}{T}\right)\frac{g(t)}{t}\,dt = A+\frac{1}{\pi}\int_0^T \frac{\psi(u)}{u+1}\,du+o(1)$$
$$= B+o(1),$$

provided that the integral (3.7) exists.

Hence the condition is necessary. Conversely, if it is satisfied, it is seen that $\int^\infty \frac{g(t)}{t}\,dt$ is summable $(C, 1)$.

Further, Theorems I and II show that

$$\int_1^T \left(1-\frac{t}{T}\right)\frac{g(t)-g(-t)}{t}\,dt = C+o(1).$$

Hence
$$\int_1^T \left(1-\frac{t}{T}\right)\frac{g(-t)}{t}\,dt = B+C+o(1),$$

and the theorem is established.

9. *Proof of Theorem V.* If the integrals

$$\int_1^\infty \frac{g(\pm t)}{t}\,dt$$

are convergent, then

$$\lim_{T\to\infty}\int_1^T \frac{g'(t)+g''(t)}{t}\,dt, \tag{9.1}$$

$$\lim_{T\to\infty}\int_1^T \frac{g'(-t)+g''(-t)}{t}\,dt \tag{9.2}$$

both exist. It follows from the work of § 8 that the existence of

$$\lim_{T\to\infty}\int_1^T \frac{g''(t)+g'(-t)}{t}\,dt \tag{9.3}$$

is a necessary condition; hence, by subtraction, the existence of both of the limits

$$\lim_{T\to\infty}\int_1^T \frac{g'(t)-g'(-t)}{t}\,dt, \quad \lim_{T\to\infty}\int_1^T \frac{g''(t)-g''(-t)}{t}\,dt$$

is a necessary condition; then Theorems III and IV applied to these integrals show that (3.8) is necessary, together with (3.5) and (3.7). By reversing the steps of the reasoning, it is seen that these conditions are sufficient.

ON CONJUGATE FUNCTIONS

By E. C. TITCHMARSH.

[Received 17 January, 1928.—Read 16 February, 1928.]

1. Let $f(x)$ be a function which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$, and defined outside this interval by periodicity; and let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1.1}$$

The "allied" series is

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx). \tag{1.2}$$

The "conjugate" function, associated with this series, is

$$g(x)=\frac{1}{2\pi}P\int_{-\pi}^{\pi}f(t)\cot\tfrac{1}{2}(t-x)\,dt, \tag{1.3}$$

the symbol P indicating that the integral is a principal value in Cauchy's sense, *i.e.* that it is defined as

$$\lim_{\epsilon\to 0}\left(\int_{-\pi}^{x-\epsilon}+\int_{x+\epsilon}^{\pi}\right).$$

Although it is known that the series (1.2) is summable $(C, 1)$ to the sum (1.3) for almost all values of x*, $g(x)$ is not necessarily integrable, and (1.2) is not necessarily a Fourier series. The classical example of this is

$$f(x)=\sum_{n=2}^{\infty}\frac{\cos nx}{\log n},\qquad g(x)=-\sum_{n=2}^{\infty}\frac{\sin nx}{\log n}. \tag{1.4}$$

If we suppose that $\{f(x)\}^2$ is integrable, then the principal value (1.3) exists for almost all values of x, the function $g(x)$ so defined is

* See G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 211–246. References to many previous writings on the subject will be found in this paper.

integrable, together with its square, over $(-\pi, \pi)$, and

$$\int_{-\pi}^{\pi} \{g(x)\}^2 dx = \int_{-\pi}^{\pi} \{f(x)\}^2 dx - \tfrac{1}{2}\pi a_0^2. \tag{1.5}$$

Also (1.2) is now the Fourier series of $g(x)$. Further, if $f_1(x)$ satisfies the same conditions as $f(x)$, and $g_1(x)$ is its conjugate, then

$$\int_{-\pi}^{\pi} f_1(x)\, g(x)\, dx = -\int_{-\pi}^{\pi} f(x)\, g_1(x)\, dx. \tag{1.51}$$

These results have recently been extended by M. Riesz* to the case when $f(x)$ belongs to the Lebesgue class L^p $(p > 1)$, *i.e.* $|f(x)|^p$ is integrable over $(-\pi, \pi)$. Then the same is true of $g(x)$, and

$$\int_{-\pi}^{\pi} |g(x)|^p dx < M_p^p \int_{-\pi}^{\pi} |f(x)|^p dx, \tag{1.52}$$

where M_p depends on p only; while (1.51) still holds if $f_1(x)$ and $g_1(x)$ belong to $L^{p/(p-1)}$.

The existence of $g(x)$ in the general case appears to have been first proved by Privaloff† ; but the first proof which is generally accessible was given by Plessner‡, and depends on complex function-theory. Later a proof depending only on the theory of sets of points was given by Besicovitch§, and it is this proof which is the starting point of the present paper. We develop what is, as it stands, merely an existence theorem into a proof of several properties of $g(x)$.

It is known that $g(x)$ is not necessarily integrable in the sense of Lebesgue, and we can even construct examples in which $g(x)$ is not integrable over any interval. On the other hand, it was proved by Kolmogoroff, in his paper already referred to, that $|g(x)|^{1-\epsilon}$ is integrable if $\epsilon > 0$, and

$$\int_{-\pi}^{\pi} |g(x)|^{1-\epsilon} dx < \frac{A}{\epsilon} \left\{ \int_{-\pi}^{\pi} |f(x)|\, dx \right\}^{1-\epsilon}. \tag{1.6}$$

This is an immediate consequence of the result that the measure of the set where $|g(x)| > R$ is less than

$$\frac{A}{R} \int_{-\pi}^{\pi} |f(x)|\, dx. \tag{1.61}$$

---

* M. Riesz, *Math. Zeitschrift*, 27 (1927), 218–244.

† M. Privaloff, *L'intégrale de Cauchy* (Saratoff, 1919) (in Russian). See A. Kolmogoroff, *Fundamenta Math.*, 7 (1925), 24–29.

‡ A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen* (Giessen, 1922).

§ A. S. Besicovitch, *Journal London Math. Soc.*, 1 (1926), 120–128.

Much simpler proofs of (1 . 6) were given later by Littlewood* and by Hardy†.

We begin by giving a new proof of Kolmogoroff's results. Our method also leads to the result that $g(x)$ is integrable provided that

$$f(x)\log^{1+\epsilon}\{2+|f(x)|\}$$

is integrable for some positive value of ϵ. All that was previously known in this direction was that the integrability of $|f(x)|^{1+\epsilon}$ is a sufficient condition—an obvious consequence of Riesz's theorem noted above.

We next consider the problem of integrating $g(x)$. It has already been noticed‡ that, in the case (1 . 4), the integral of $g(x)$ may be defined as a principal value in Cauchy's sense. This is so in many cases where $g(x)$ has a single infinity, but it is not universally true even in that case. On the other hand, it is true if we replace the ordinary limit which defines the principal value by the corresponding "arithmetic mean". In the general case the difficulty is that the "principal-value points" of $g(x)$ are, so to speak, distributed all along the interval, and the isolation demanded by Cauchy's definition is impossible. We may now define the integral of $g(x)$ over (a, b), for almost all values of a and b, as

$$\lim_{n\to\infty}\int_{(a,b)-e_n} g(x)\,dx,$$

where e_n is a set whose measure tends to zero as $n\to\infty$. The value of the integral is the same as would be obtained by a formal calculation, namely $G(b)-G(a)$, where

$$G(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\log\left|\sin\tfrac{1}{2}(t-x)\right|dt. \tag{1.7}$$

This is a sort of extension of Cauchy's definition. It has, however, one disadvantage. The construction of the sets e_n does not depend directly on $g(x)$, but on $f(x)$. Hence the process cannot be applied to all functions, but only to those which are defined as conjugates.

We therefore give another definition, which does not, at any rate, have this defect. Let

$$\begin{aligned}[g(x)]_n &= g(x) \quad \text{if} \quad |g(x)| \leqslant n,\\ &= n \quad\;\; \text{if} \quad g(x) > n,\\ &= -n \quad \text{if} \quad g(x) < -n.\end{aligned}$$

* J. E. Littlewood, *Journal London Math. Soc.*, 1 (1926), 229-231.

† G. H. Hardy, *Journal London Math. Soc.*, 3 (1928), 166-169.

‡ E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 23 (1924), Records for March.

Then we prove that, for almost all values of a and b, the limit

$$\lim_{n\to\infty} \int_a^b [g(x)]_n \, dx$$

exists, and is equal to $G(b)-G(a)$. We denote an integral of this kind by

$$Q\int_a^b g(x)\,dx.$$

It is a kind of principal value. We consider some of the properties of Q-integrals, and their relation to P-integrals, at the end of the paper.

The Fourier coefficients of $g(x)$, and other integrals involving $g(x)$, may also be defined as Q-integrals.

We have seen that $G(x)$ is, in a certain sense, the integral of $g(x)$. Conversely $g(x)$ plays, to a certain extent, the part of a derived function of $G(x)$. In fact it is, for almost all values of x, what has been called the "approximate derivative"* of $G(x)$.

Our last problem is that of the formula

$$f(x)-\tfrac{1}{2}a_0 = -\frac{1}{2\pi} P\int_{-\pi}^{\pi} g(t)\cot\tfrac{1}{2}(t-x)\,dt, \tag{1.8}$$

which is reciprocal to (1.3). This holds for almost all values of x if $f(x)$ belongs to L^p $(p>1)$. In the general case we prove that the formula obtained by a formal integration by parts is true for almost all values of x.

2. THEOREM 1. *Let E be the set where $|g(x)|\geq R$, R being any positive number. Then*

$$m(E).R < A\int_{-\pi}^{\pi} |f(x)|\,dx.$$

2.1. Throughout the paper A denotes a positive absolute constant, not always the same one; and the symbol O implies an absolute constant. K denotes a constant which may depend on some of the parameters or functions which occur.

In order to simplify some of the details, we consider, instead of $g(x)$, the function

$$h(x) = P\int_{-3\pi}^{3\pi} \frac{f(t)}{t-x}\,dt \quad (-\pi<x<\pi). \tag{2.11}$$

* See E. W. Hobson, *Theory of functions of a real variable*, 1 (2nd ed. 1921), 656.

The results for $g(x)$ follow from those for $h(x)$. For

$$\pi g(x) = P\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{t-x}+\frac{1}{t-x-2\pi}+\frac{1}{t-x+2\pi}+C(t-x)\right\}dt$$
$$= h(x)+j(x),$$

where $C(t-x)$ is continuous for $-2\pi \leqslant t-x \leqslant 2\pi$, so that $j(x)$ is a continuous function of x, and

$$|j(x)| < A\int_{-\pi}^{\pi} |f(t)|\,dt. \tag{2.12}$$

Also

$$\int_{-\pi}^{\pi} \{j(x)\}^2 dx < A\int_{-\pi}^{\pi} \{f(t)\}^2 dt, \tag{2.13}$$

if the latter integral exists. Then it follows, from (1.5), that

$$\int_{-\pi}^{\pi} \{h(x)\}^2 dx < A\int_{-\pi}^{\pi} \{f(t)\}^2 dt, \tag{2.14}$$

if the latter integral exists.

2.2. In proving the theorem it is, as usual, sufficient to consider the case where $f(x) \geqslant 0$. We write

$$f(x) = f_1(x)+\phi(x)$$

where
$$f_1(x) = f(x), \quad \phi(x) = 0 \quad \text{if} \quad f(x) \leqslant n,$$
$$f_1(x) = 0, \quad \phi(x) = f(x) \quad \text{if} \quad f(x) > n.$$

Let
$$\int_0^{2\pi} \phi(x)\,dx = \alpha,$$

so that $\alpha = \alpha(n, f) \to 0$ when $n \to \infty$. We may thus suppose that $\alpha < 1$. Let

$$\psi(x) = \int_0^x \phi(t)\,dt,$$

so that
$$\psi(0) = 0, \quad \psi(2\pi) = \alpha.$$

Let
$$\theta(x) = \psi(x) - \tfrac{1}{2}nx.$$

(It is in the introduction of this factor $\frac{1}{2}n$ that our construction differs from that of Besicovitch.)

Let x_0 be the greatest value of x for which $\theta(x) = 0$. Then

$$x_0 = \frac{2}{n}\,\psi(x_0) \leqslant \frac{2}{n}\,\psi(2\pi) = \frac{2\alpha}{n}.$$

Consider the interval $(x_0, 1)$. Let L_1 be the set of points x_1 of this interval such that

$$\theta(x_1) < \max \theta(x) \quad (x_1 \leqslant x \leqslant 1).$$

Let L be the set consisting of L_1 together with the points of the open interval $(0, x_0)$. Then L is an open set. Let $l_1, l_2, \ldots$ be its intervals.

Every point x at which $\psi'(x) > \frac{1}{2}n$ belongs to L, since at such a point

$$\theta'(x) = \psi'(x) - \tfrac{1}{2}n > 0.$$

But $\psi'(x) = \phi(x)$ almost everywhere, and $\phi(x)$ is either zero or greater than $\frac{1}{2}n$. Thus almost all values of x for which $\phi(x)$ is different from zero belong to L, and consequently

$$\int_L \phi(x)\,du = a, \quad \int_{(0,\,2\pi)-L} \phi(x)\,dx = 0.$$

Let a_i, b_i, be the end points of l_i. Then, as in Besicovitch's paper, $\theta(a_i) = \theta(b_i)$, *i.e.*

$$\int_{a_i}^{b_i} \phi(x)\,dx = \psi(b_i) - \psi(a_i) = \tfrac{1}{2}n(b_i - a_i).$$

Hence

$$\int_L \phi(x)\,dx = \tfrac{1}{2}n \Sigma l_i,$$

so that

$$\Sigma l_i = \frac{2a}{n}.$$

We now define $\phi_1(x)$ as follows. Let

$$\phi_1(x) = \tfrac{1}{2}n$$

if x belongs to L, and otherwise $\phi_1(x) = 0$. Then

$$\int_{a_i}^{b_i} \phi_1(x)\,dx = \tfrac{1}{2}n(b_i - a_i) = \int_{a_i}^{b_i} \phi(x)\,dx.$$

Next we construct a new system L' of intervals l_i' satisfying the following conditions:

(i) $$\Sigma l_i' < 12a/n.$$

(ii) l_i' and l_i have a common centre, and

$$l_i' > 5l_i.$$

(iii) $l_i'/l_i \to \infty$ as $i \to \infty$.

The intervals of L' may overlap.

So far we have merely considered the interval $(0, 2\pi)$. Outside this interval all the above sets and functions are defined by periodicity.

2.3. We write

$$h(x) = P\int_{-3\pi}^{3\pi} \frac{f_1(t)}{t-x}\,dt + P\int_{-3\pi}^{3\pi} \frac{\phi_1(t)}{t-x}\,dt + P\int_{-3\pi}^{3\pi} \frac{\phi(t)-\phi_1(t)}{t-x}\,dt$$
$$= h_1(x)+h_2(x)+h_3(x),$$

say. Then, by (2.14),

$$\int_{-\pi}^{\pi} \{h_1(x)\}^2 dx < A\int_{-\pi}^{\pi} \{f_1(x)\}^2 dx \tag{2.31}$$
$$< An\int_{-\pi}^{\pi} f_1(x)\,dx$$
$$< An\int_{-\pi}^{\pi} f(x)\,dx.$$

Similarly

$$\int_{-\pi}^{\pi} \{h_2(x)\}^2 dx < A\int_{-\pi}^{\pi} \{\phi_1(x)\}^2 dx < An\int_{-\pi}^{\pi} \phi_1(x)\,dx \tag{2.32}$$
$$< An\,a.$$

Now consider $h_3(x)$. We write

$$J_i = \int_{l_i} \frac{\phi(t)-\phi_1(t)}{t-x}\,dt,$$

where x is not a point of L'. Then, as in Besicovitch's paper,

$$J_i = \frac{1}{t^0-x}\int_{l_i} \phi(t)\,dt - \frac{1}{t^1-x}\int_{l_i} \phi_1(t)\,dt$$
$$= \left(\frac{1}{t^0-x} - \frac{1}{t^1-x}\right)\tfrac{1}{2}nl_i,$$

where t^0, t^1 are points of l_i. Hence

$$|J_i| < \frac{An\,l_i^2}{(c_i-x)^2} \quad [c_i = \tfrac{1}{2}(a_i+b_i)].$$

Now, as in Besicovitch's paper,

$$\int_{(-\pi,\,\pi)-L} \frac{l_i^2}{(c_i-x)^2}\,dx < A\,l_i.$$

Hence

$$\int_{(-\pi,\pi)-L'} |h_3(x)|\,dx \leqslant \int_{(-\pi,\pi)-L'} \Sigma |J_i|\,dx \tag{2.33}$$

$$< An\Sigma l_i = Aa.$$

2.4. Let E_1 be the set where $|h_1(x)| > \frac{1}{3}R$, E_2 the set where $|h_2(x)| > \frac{1}{3}R$, E_3 the set where $|h_3(x)| > \frac{1}{3}R$. Then, from (2.31), (2.32), (2.33) we deduce

$$m(E_1) < \frac{An}{R^2}\int_{-\pi}^{\pi} f(x)\,dx, \tag{2.41}$$

$$m(E_2) < \frac{Ana}{R^2}, \tag{2.42}$$

$$m(E_3) < m(L') + \frac{Aa}{R} \tag{2.43}$$

$$< \frac{Aa}{n} + \frac{Aa}{R}.$$

Now if E be the set where $|h(x)| > R$, every point of E belongs to at least one of E_1, E_2, E_3. Hence

$$m(E) \leqslant m(E_1) + m(E_2) + m(E_3) \tag{2.44}$$

$$< \frac{An}{R^2}\int_{-\pi}^{\pi} f(x)\,dx + \frac{Aa}{n} + \frac{Aa}{R}.$$

Taking $n = R$, the result follows.

2.5. We may remark finally that, for any particular function $f(x)$,

$$m(E) = o(1/R).$$

This follows from (2.44) on taking $n = \epsilon R$, where ϵ is arbitrarily small.

2.6. THEOREM 2. *If $f(x)\log^{1+\epsilon}\{2+|f(x)|\}$ is integrable for some positive value of ϵ, then $g(x)$ is integrable, and*

$$\int_{-\pi}^{\pi} |g(x)|\,dx < K\int_{-\pi}^{\pi} |f(x)|\log^{1+\epsilon}\{2+|f(x)|\}\,dx, \tag{2.61}$$

where K depends on ϵ only.

Let us denote the integral on the right of (2 . 61) by J.
We have, as before,

$$\int_{-\pi}^{\pi} \{h_1(x)\}^2 dx < A \int_{-\pi}^{\pi} \{f_1(x)\}^2 dx.$$

Let S be the set where $f_1(x) \geqslant \sqrt{n}$, CS its complement. Then

$$\int_S \{f_1(x)\}^2 dx \leqslant \int_S n f_1(x) \left\{\frac{\log f_1(x)}{\log \sqrt{n}}\right\}^{1+\epsilon} dx < \frac{AJn}{(\log n)^{1+\epsilon}},$$

$$\int_{CS} \{f_1(x)\}^2 dx \leqslant \sqrt{n} \int_{CS} f_1(x)\, dx < J\sqrt{n}.$$

Hence
$$\int_{-\pi}^{\pi} \{h_1(x)\}^2 dx < \frac{AJn}{(\log n)^{1+\epsilon}},$$

and consequently
$$m(E_1) < \frac{AJn}{R^2(\log n)^{1+\epsilon}}.$$

Also
$$\alpha = \int_{-\pi}^{\pi} \{f(x) - f_1(x)\}\, dx$$

$$< (\log n)^{-1-\epsilon} \int_{-\pi}^{\pi} \{f(x) - f_1(x)\} \log^{1+\epsilon} |f(x)|\, dx,$$

$$< (\log n)^{-1-\epsilon} J.$$

Hence
$$m(E_2) < \frac{AJn}{R^2(\log n)^{1+\epsilon}},$$

$$m(E_3) < \frac{AJ}{(\log n)^{1+\epsilon}} \left(\frac{1}{n} + \frac{1}{R}\right).$$

Taking $n = R$, as before, we obtain

$$m(E) < \frac{AJ}{R(\log R)^{1+\epsilon}}.$$

Let $g_k(x) = g(x)$ if $0 < g(x) \leqslant k$, $g_k(x) = k$ if $g(x) > k$, and $g_k(x) = 0$ if $g(x) \leqslant 0$. Let μ_r be the measure of the set when $g(x) > r-1$. Then

$$\int_{-\pi}^{\pi} g_k(x)\, dx \leqslant \sum_{r=1}^{k} r(\mu_r - \mu_{r+1}) + k\mu_{k-1}$$

$$= \sum_{r=1}^{k} \mu_r < \sum_{r=1}^{k} \frac{AJ}{r(\log r)^{1+\epsilon}} < KJ$$

for all values of k. A similar argument may be applied to the set where $g(x)$ is negative. Hence $g(x)$ is integrable, and (2 . 61) holds.

2 . 7. We may obviously replace the factor

$$\log^{1+\epsilon}(2+|f|)$$

of Theorem 2 by factors such as

$$\log(2+|f|)\{\log\log(3+|f|)\}^{1+\epsilon} \quad (\epsilon > 0).$$

It is not possible to replace it by

$$\log(2+|f|)\{\log\log(3+|f|)\}^{-\epsilon}.$$

For suppose that

$$f(x) = \sum_{n=3}^{\infty} \frac{\cos nx}{\log n \log\log n}, \qquad g(x) = -\sum_{n=3}^{\infty} \frac{\sin nx}{\log n \log\log n}.$$

Then*, as $x \to 0$,

$$f(x) = O\left\{\frac{d}{dx}\frac{1}{\log(1/x)\log\log(1/x)}\right\} = O\left\{\frac{1}{x\log^2(1/x)\log\log(1/x)}\right\}.$$

Hence
$$f\log(2+|f|)\{\log\log(3+|f|)\}^{-\epsilon}$$

is integrable. But

$$\Sigma \frac{1}{n\log n\log\log n}$$

is divergent, so that the allied series is not a Fourier series, and $g(x)$ is not integrable.

3. THEOREM 3. *There is a sequence of sets E_n in $(-\pi, \pi)$ such that*

$$\lim_{n\to\infty} m(E_n) = 2\pi,$$

while, for every positive integer k,

$$\lim_{n\to\infty} \int_{E_n} g(x)\cos kx\,dx = \pi b_k,$$

$$\lim_{n\to\infty} \int_{E_n} g(x)\sin kx\,dx = -\pi a_k.$$

* W. H. Young, *Proc. London Math. Soc.* (2), 12 (1913), 433–452 (442).

We use the construction of § 2, and write

$$g(x) = \frac{1}{2\pi} P\int_{-\pi}^{\pi} f_1(t) \cot \tfrac{1}{2}(t-x)\, dt + \frac{1}{2\pi} P\int_{-\pi}^{\pi} \phi_1(t) \cot \tfrac{1}{2}(t-x)\, dt$$

$$+ \frac{1}{2\pi} P\int_{-\pi}^{\pi} \{\phi(t)-\phi_1(t)\} \cot \tfrac{1}{2}(t-x)\, dt$$

$$= g_1(x)+g_2(x)+g_3(x).$$

Since $f_1(t)$ is bounded, the allied series of the Fourier series of f_1 is the Fourier series of g_1. Hence

$$\int_{-\pi}^{\pi} g_1(x) \cos kx\, dx = \int_{-\pi}^{\pi} f_1(x) \sin kx\, dx$$

$$= \pi b_k - \int_{-\pi}^{\pi} \{f(x)-f_1(x)\} \sin kx\, dx$$

$$= \pi b_k + o(1)$$

as $n \to \infty$. Also

$$\left|\int_{L'} g_1(x) \cos kx\, dx\right| \leqslant \left\{\int_{L'} \cos^2 kx\, dx \int_{L} \{g_1(x)\}^2\, dx\right\}^{\frac{1}{2}}$$

$$\leqslant \left\{m(L') \int_{-\pi}^{\pi} \{g_1(x)\}^2\, dx\right\}^{\frac{1}{2}}$$

$$< A \left\{\frac{a}{n} \int_{-\pi}^{\pi} \{f_1(x)\}^2\, dx\right\}^{\frac{1}{2}}$$

$$< A \left\{a \int_{-\pi}^{\pi} f_1(x)\, dx\right\}^{\frac{1}{2}} = o(1).$$

Hence $$\int_{(-\pi,\,\pi)-L'} g_1(x) \cos kx\, dx = \pi b_k + o(1).$$

Similarly $$\int_{(-\pi,\,\pi)-L'} g_2(x) \cos kx\, dx = o(1).$$

Finally

$$\left|\int_{(-\pi,\,\pi)-L'} g_3(x) \cos kx\, dx\right| \leqslant \int_{(-\pi,\,\pi)-L'} |g_3(x)|\, dx = o(1),$$

as in § 2. The result then follows, the set E_n being the complement of L'. Similarly for the integral involving $\sin kx$.

3.1. COROLLARY. *If $g(x)$ is integrable, then its Fourier series is the allied series of the Fourier series of $f(x)$.*

4. THEOREM 4. *If $f(x)$ is bounded in any interval not containing $x=0$, then*

$$P_1\int_{-a}^{a} g(x)\,dx = \lim_{\epsilon\to 0}\int_{\epsilon}^{a}\{g(x)+g(-x)\}\left(1-\frac{\epsilon}{x}\right)dx$$

exists, and is equal to $G(a)-G(-a)$. But

$$P\int_{-a}^{a} g(x)\,dx$$

does not necessarily exist.

4.1. We require the following lemma.

LEMMA. *If $f(x)$ belongs to L^p $(p>1)$, then (1.3) or (2.11) may be integrated with respect to x under the integral sign between any limits.*

We have to prove that

$$\int_a^b g(x)\,dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\log\left|\frac{\sin\frac{1}{2}(t-b)}{\sin\frac{1}{2}(t-a)}\right|dt.$$

This is a case of (1.51). For if

$$f_1(x)=1\,(a\leqslant x\leqslant b);\ f_1(x)=0\,(x<a \text{ or } x>b),$$

then
$$g_1(x)=\frac{1}{\pi}\{\log|\sin\tfrac{1}{2}(b-x)|-\log|\sin\tfrac{1}{2}(a-x)|\}$$

and the result follows. The result for $h(x)$ follows from this.

4.2. We consider $h(x)$ again. Now, for $\epsilon>0$,

$$\int_\epsilon^a h(x)\,dx=\int_\epsilon^a dx\int_{-3\pi}^{\frac{1}{2}\epsilon}\frac{f(t)}{t-x}\,dt+\int_\epsilon^a dx\,P\int_{\frac{1}{2}\epsilon}^{3\pi/2}\frac{f(t)}{t-x}\,dt+\int_\epsilon^a dx\int_{3\pi/2}^{3\pi}\frac{f(t)}{t-x}\,dt.$$

We may invert the order of the first and third integrals, since the associated double integrals exist; and we may invert the order of the second integral by the lemma, since $f(t)$ is bounded for $\frac{1}{2}\epsilon<t<3\pi/2$. Hence

$$\int_\epsilon^a h(x)\,dx=\int_{-3\pi}^{3\pi} f(t)\log\left|\frac{t-\epsilon}{t-a}\right|dt,$$

and similarly
$$\int_\epsilon^a h(-x)\,dx=\int_{-3\pi}^{3\pi} f(t)\log\left|\frac{t+a}{t+\epsilon}\right|dt.$$

Hence

$$\int_\epsilon^a\{h(x)+h(-x)\}\,dx=\int_{-3\pi}^{3\pi} f(t)\log\left|\frac{t+a}{t-a}\right|dt-\int_{-3\pi}^{3\pi} f(t)\log\left|\frac{t+\epsilon}{t-\epsilon}\right|dt.$$

The principal value exists if the last term tends to zero. But this is not necessarily the case. Suppose, for example, that

$$f(t) = n^{-2}e^{n^3}, \quad \left(\frac{1}{n} < t < \frac{1}{n} + e^{-n^3}\right),$$

for $n = 1, 2, \ldots$, and $f(t) = 0$ outside these intervals. Then $f(t)$ is integrable. But for $\epsilon = 1/n$

$$\int_{-\pi}^{\pi} f(t) \log \left|\frac{t+\epsilon}{t-\epsilon}\right| dt = \sum_{m=1}^{\infty} \frac{e^{m^3}}{m^2} \int_{1/m}^{1/m+e^{-m^3}} \log \left|\frac{t+1/n}{t-1/n}\right| dt$$

$$> \frac{e^{n^3}}{n^2} \int_{1/n}^{1/n+e^{-n^3}} \log \left(\frac{t+1/n}{t-1/n}\right) dt > \frac{e^{n^3}}{n^2} \int_{1/n}^{1/n+e^{-n^3}} \log \frac{1}{nt-1}\, dt$$

$$= n^{-2}e^{n^3}(e^{-n^3}n^3 + e^{-n^3} - \log n \,.\, e^{-n^3}) > An.$$

The remainder of the integral tends to zero. Hence there is no limit in this case.

4.3. On the other hand, we have

$$\int_{\epsilon}^{a} \{h(x)+h(-x)\}\left(1-\frac{\epsilon}{x}\right) dx = \int_{-3\pi}^{3\pi} f(t) \log \left|\frac{t+a}{t-a}\right| dt$$

$$+\epsilon \int_{-3\pi}^{3\pi} \frac{f(t)}{t} \log \left|1-\frac{t^2}{a^2}\right| dt$$

$$-\int_{-3\pi}^{3\pi} \left\{\left(1+\frac{\epsilon}{t}\right) \log \left|1+\frac{\epsilon}{t}\right| - \left(1-\frac{\epsilon}{t}\right) \log \left|1-\frac{\epsilon}{t}\right|\right\} f(t)\, dt,$$

and the last term tends to zero by Lebesgue's convergence theorem, since the function in brackets tends to zero when $\epsilon \to 0$ for fixed t, and is bounded for all values of ϵ and t. This proves the theorem for $h(x)$, and so for $g(x)$.

5. We now return to the general case.

THEOREM 5. *There is a set E of measure 2π with the following property. If a and b are any two points of E, a sequence of sets e_n included in (a, b) can be found, such that*

$$\lim_{n\to\infty} m(e_n) = 0,$$

$$\lim_{n\to\infty} \int_{(a,\, b)-e_n} g(x)\, dx = G(b) - G(a).$$

Let E be the "Lebesgue set" of $f(x)$, *i.e.* the set where $|f(x)-q|$ is the derivative of its integral for all values of q. Then the measure of E is 2π. Let a and b be points of E.

We prove the corresponding result for $h(x)$. Let e_n be the part of the set L' which lies in (a, b). Then we shall prove that

$$\lim \int_{(a,b)-e_n} h(x)\,dx = -\int_{-3\pi}^{3\pi} f(t) \log \left| \frac{t-b}{t-a} \right| dt. \tag{5.1}$$

The integral on the right exists. For, by the property of the Lebesgue set,

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_a^{a+\delta} |f(t)|\,dt = |f(a)|, \tag{5.2}$$

and the existence of the integral in the neighbourhood of $t = a$ follows from this. Similarly for $t = b$.

We write $$h(x) = h_1(x)+h_2(x)+h_3(x)$$

as in § 2. Since $f_1(x)$ is bounded, it follows from § 4.1 that

$$\int_a^b h_1(x)\,dx = -\int_{-3\pi}^{3\pi} f_1(t) \log \left| \frac{t-b}{t-a} \right| dt. \tag{5.3}$$

Since $f_1(t) \to f(t)$ for almost all t, and $|f_1(t)| \leqslant |f(t)|$, it follows, from Lebesgue's convergence theorem, that the right-hand side of (5.3) tends to the right-hand side of (5.1) when $n \to \infty$. Also

$$\begin{aligned} \left| \int_{e_n} h_1(x)\,dx \right| &\leqslant \left\{ m(e_n) \int_{e_n} \{h_1(x)\}^2\,dx \right\}^{\frac{1}{2}} \\ &< \left\{ m(L') \int_{-\pi}^{\pi} \{h_1(x)\}^2\,dx \right\}^{\frac{1}{2}} < A \left\{ \frac{a}{n} \int_{-\pi}^{\pi} \{f_1(x)\}^2\,dx \right\}^{\frac{1}{2}} \\ &< A \left\{ a \int_{-\pi}^{\pi} f(x)\,dx \right\}^{\frac{1}{2}} = o(1). \end{aligned}$$

Hence

$$\lim \int_{(a,b)-e_n} h_1(x)\,dx = -\int_{-3\pi}^{3\pi} f(t) \log \left| \frac{t-b}{t-a} \right| dt. \tag{5.4}$$

Again

$$\begin{aligned} \int_{(a,b)-e_n} h_2(x)\,dx &= \int_a^b h_2(x)\,dx - \int_{e_n} h_2(x)\,dx \\ &= -\int_{-3\pi}^{3\pi} \phi_1(t) \log \left| \frac{t-b}{t-a} \right| dt + o(1) \end{aligned}$$

by the argument applied to h_1. We have to prove that the last integral

tends to zero. Consider the part of the integral over $[a, \frac{1}{2}(a+b)]$. It is

$$\int_a^{\frac{1}{2}(a+b)} \phi_1(t) \log\left|\frac{t-b}{t-a}\right| dt = \Sigma' \int_{a_i}^{l_i} \phi_1(t) \log\left|\frac{t-b}{t-a}\right| dt$$

$$= \Sigma' \tfrac{1}{2}n \int_{a_i}^{b_i} \log\left|\frac{t-b}{t-a}\right| dt,$$

where Σ' denotes a sum taken over those intervals l_i (or parts of such intervals) which lie in $[a, \frac{1}{2}(a+b)]$. Now, if a_i, b_i is any such interval,

$$\tfrac{1}{2}n(b_i-a_i) = \int_{a_i}^{b_i} \phi(x)\,dx \leqslant \int_{a_i}^{b_i} f(x)\,dx \leqslant \int_a^{b_i} f(x)\,dx,$$

and it follows from (5 . 2) that this is less than

$$K(b_i-a),$$

where K is a constant depending on f only. Hence

$$\frac{b_i-a}{a_i-a} < \frac{n}{n-2K} < 2 \quad (n > n_0).$$

Hence, for $n > n_0$,

$$\tfrac{1}{2}n \int_{a_i}^{b_i} \log\left|\frac{t-b}{t-a}\right| dt < \tfrac{1}{2}n(b_i-a_i) \log\frac{b-a}{a_i-a}$$

$$< \tfrac{1}{2}n(b_i-a_i) \log\frac{2(b-a)}{b_i-a} < \int_{a_i}^{b_i} \phi(t) \log\frac{2(b-a)}{t-a}\,dt,$$

and hence

$$\int_a^{\frac{1}{2}(a+b)} \phi_1(t) \log\left|\frac{t-b}{t-a}\right| dt < \int_a^{\frac{1}{2}(a+b)} \phi(t) \log\frac{2(b-a)}{t-a}\,dt.$$

This tends to zero as before. It follows that

$$\lim \int_{(a,\,b)-e_n} h_2(x)\,dx = 0. \tag{5 . 5}$$

Finally

$$\lim \int_{(a,\,b)-e_n} h_3(x)\,dx = 0 \tag{5 . 6}$$

by (2. 33). The result now follows from (5 . 4), (5 . 5), and (5 . 6).

COROLLARY. *If $g(x)$ is integrable over a partial interval (α, β) of $(-\pi, \pi)$, then the series (1 . 2) is an ordinary restricted Fourier series* in this interval, and $g(x)$ is the function associated with it.*

* W. H. Young, *Proc. London Math. Soc.* (2), 17 (1918), 195-236. See E. W. Hobson, *Theory of functions of a real variable*, 2 (2nd ed. 1926), 686.

For the integrated series of (1 . 2) is the Fourier series of $G(x)$, which is now an integral in (α, β), viz. the integral of $g(x)$.

6. THEOREM 6. *If a and b are any two points of E, then*

$$Q\int_a^b g(x)\,dx = G(b)-G(a).$$

In view of Theorem 5, we have to prove that

$$\lim\left\{\int_{(a,\,b)-e_n} g(x)\,dx-\int_a^b [g(x)]_n\,dx\right\} = 0.$$

Now $$\left|\int_{e_n}[g(x)]_n\,dx\right| \leqslant nm(e_n) \leqslant nm(L') < A\alpha \to 0.$$

Hence it is sufficient to prove that

$$\lim\int_{(a,\,b)-e_n}\{g(x)-[g(x)]_n\}\,dx = 0.$$

Now, by § 2 . 5, $g(x) = [g(x)]_n$ except in a set σ_n of measure $o(1/n)$. Let τ_n be the set of points common to σ_n and $(a,\ b)-e_n$. Then

$$\left|\int_{(a,\,b)-e_n}\{g(x)-[g(x)]_n\}\,dx\right| \leqslant \int_{\tau_n}|g(x)|\,dx$$

$$\leqslant \int_{\sigma_n}|g_1(x)|\,dx+\int_{\sigma_n}|g_2(x)|\,dx+\int_{(a,\,b)-e_n}|g_3(x)|\,dx.$$

Now $$\int_{\sigma_n}|g_1(x)|\,dx \leqslant \left\{m(\sigma_n)\int_{\sigma_n}\{g_1(x)\}^2\,dx\right\}^{\frac{1}{2}}$$

$$< A\left\{m(\sigma_n)\int_{-\pi}^{\pi}\{f_1(x)\}^2\,dx\right\}^{\frac{1}{2}}$$

$$< A\left\{nm(\sigma_n)\int_{-\pi}^{\pi}f(x)\,dx\right\}^{\frac{1}{2}}$$

which tends to zero. Similarly the integral involving g_2 tends to zero, and the integral involving g_3 tends to zero, as in previous cases. This proves the theorem.

7. THEOREM 7. *Let $\lambda(x)$ be the integral of a function $\lambda'(x)$ belonging to the class L^p $(p>1)$. Then*

$$\int_a^b g(x)\lambda(x)\,dx = G(b)\lambda(b)-G(a)\lambda(a)-\int_a^b G(x)\lambda'(x)\,dx,$$

a and b being points of E, and the integral on the left being defined

either as
$$\lim \int_{(a,\,b)-e_n} g(x)\,\lambda(x)\,dx$$

or as
$$\lim \int_a^b [g(x)]_n\,\lambda(x)\,dx$$

or as
$$\lim \int_a^b [g(x)\,\lambda(x)]_n\,dx = Q\int_a^b g(x)\,\lambda(x)\,dx.$$

We write, as usual,

$$g(x) = g_1(x)+g_2(x)+g_3(x).$$

Now

$$\int_a^b g_1(x)\,\lambda(x)\,dx = G_1(b)\,\lambda(b)-G_1(a)\,\lambda(a)-\int_a^b G_1(x)\,\lambda'(x)\,dx,$$

where

$$G_1(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f_1(t)\log\left|\sin\tfrac{1}{2}(t-x)\right|dt\,; \tag{7.1}$$

for, by § 4.1, $G_1(x)$ is the integral of $g_1(x)$. As in previous cases, $G_1(a)\to G(a)$, $G_1(b)\to G(b)$, since the integral (7.1) exists for $x=a$ and $x=b$. Also

$$\int_a^b \{G(x)-G_1(x)\}\,\lambda'(x)\,dx$$
$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(t)-f_1(t)\}\,dt\int_a^b \lambda'(x)\log\left|\sin\tfrac{1}{2}(t-x)\right|dx$$

and

$$\left|\int_a^b \lambda'(x)\log\left|\sin\tfrac{1}{2}(t-x)\right|dx\right|^p$$
$$\leqslant \int_a^b |\lambda'(x)|^p\,dx\left\{\int_a^b\left|\log\left|\sin\tfrac{1}{2}(t-x)\right|\right|^{p/(p-1)}dx\right\}^{p-1} < K.$$

Hence

$$\left|\int_a^b \{G(x)-G_1(x)\}\,\lambda'(x)\,dx\right| < K\int_{-\pi}^{\pi}\{f(t)-f_1(t)\}\,dt\to 0.$$

Secondly,

$$\int_a^b g_2(x)\,\lambda(x)\,dx = G_2(b)\,\lambda(b)-G_2(a)\,\lambda(a)-\int_a^b G_2(x)\,\lambda'(x)\,dx.$$

The method of § 5 shows that the integrated terms tend to zero, and the above argument shows that the last integral tends to zero. Finally, as in previous cases,

$$\int_{e_n} \{g_1(x)+g_2(x)\}\,\lambda(x)\,dx \to 0,$$

$$\int_{(a,\,b)-e_n} g_3(x)\,\lambda(x)\,dx \to 0.$$

This proves the first result, and the other two follow by the argument of § 6.

8. *An example in which $G(x)$ is unbounded in every interval.* The point will be sufficiently illustrated by considering the interval (0, 1). Let

$$f_n(t) = 1/\{t\log^2 t\} \quad (0 < t \leqslant 1/n),$$

$$f_n(t+1/n) = f_n(t).$$

Then
$$\int_0^1 f_n(t)\,dt = n\int_0^{1/n} \frac{dt}{t\log^2 t} = \frac{n}{\log n}.$$

Hence the series
$$\sum_{n=2}^{\infty} \frac{1}{n^3}\int_0^1 f_n(t)\,dt$$

is convergent, and consequently

$$\sum_{n=2}^{\infty} \frac{f_n(t)}{n^3}$$

is convergent almost everywhere in (0, 1), and its sum, $f(t)$ say, is an integrable function. Let

$$H(x) = \int_0^1 f(t)\log|x-t|\,dt, \tag{8.1}$$

the integral existing for almost all values of x. Then

$$H(x) = \sum_{n=2}^{\infty} \frac{1}{n^3}\int_0^1 f_n(t)\log|x-t|\,dt \tag{8.2}$$

for all values of x for which the integral (8.1) exists, since

$$n^{-3} f_n(t)\log|x-t|$$

is negative for all n, x, and t.

Let μ/ν be any rational fraction. Then, for $(\mu-1)/\nu < x < \mu/\nu$,

$$\begin{aligned}
|H(x)| &> \frac{1}{\nu^3}\int_0^1 f_\nu(t)\log\frac{1}{|x-t|}\,dt \\
&> \frac{1}{\nu^3}\int_{\mu/\nu}^{(\mu+1)/\nu}\left(t-\frac{\mu}{\nu}\right)^{-1}\log^{-2}\left(t-\frac{\mu}{\nu}\right)\log\frac{1}{t-x}\,dt \\
&= \frac{1}{\nu^3}\int_0^{1/\nu}\log\left(u+\frac{\mu}{\nu}-x\right)^{-1}\frac{du}{u\log^2 u} \\
&> \frac{1}{\nu^3}\int_{\mu/\nu-x}^{1/\nu}\log\frac{1}{2u}\,\frac{du}{u\log^2 u} \\
&> \frac{1}{2\nu^3}\log\log\frac{1}{\mu/\nu-x},
\end{aligned}$$

if x is sufficiently near to μ/ν. Hence $H(x)$ is unbounded in the neighbourhood of μ/ν, and so in every partial interval of (0, 1). The same thing is, of course, true of $G(x)$.

It follows also that the conjugate function of $g(x)$ is not integrable over any partial interval of (0, 1). For, if it were, by Theorem 5 its integral over the interval would be $G(x)$, and so $G(x)$ would be bounded in the interval.

9. THEOREM 8. *For almost all values of x, the set of values of δ for which*

$$\left|\frac{G(x+\delta)-G(x)}{\delta}-g(x)\right| < \epsilon$$

has unit density at $\delta = 0$, however small ϵ may be.

As usual, we prove the corresponding relation between $h(x)$, defined by (2 . 11), and

$$H(x) = \int_{-3\pi}^{3\pi} f(t)\log|t-x|\,dt.$$

We have

$$\frac{1}{\delta}\{H(x+\delta)-H(x)\}-h(x) = P\int_{-3\pi}^{3\pi}\left\{\frac{1}{\delta}\log\left|\frac{t-x}{t-x-\delta}\right|-\frac{1}{t-x}\right\}f(t)\,dt.$$

We suppose, as usual, that $f(t) \geqq 0$. Then, as in Besicovitch's paper, given any positive number η, we can write

$$f(t) = \{f_1(t)+\phi_1(t)\}+\{\phi(t)-\phi_1(t)\}$$

where f_1 and ϕ_1 are bounded, while the limits

$$\lim_{\epsilon\to 0}\int_{-3\pi}^{x-\epsilon}\frac{\phi(t)-\phi_1(t)}{t-x}\,dt, \quad \lim_{\epsilon\to 0}\int_{x+\epsilon}^{3\pi}\frac{\phi(t)-\phi_1(t)}{t-x}\,dt,$$

both exist in a set of measure $2\pi-\eta$. It then follows, from a theorem of Egoroff*, that there is a set S of measure $2\pi-2\eta$ in which each of these integrals converges uniformly. Thus, given $\zeta>0$, we can choose ϵ so small that

$$\left|\int_{x-\beta}^{x-\alpha}\frac{\phi(t)-\phi_1(t)}{t-x}\,dt\right|<\zeta, \quad \left|\int_{x+\alpha'}^{x+\beta'}\frac{\phi(t)-\phi_1(t)}{t-x}\,dt\right|<\zeta,$$

provided that $0\leqq\alpha<\beta<\epsilon$, $0\leqq\alpha'<\beta'<\epsilon$, and that x is in S. We may suppose also that $\epsilon<\zeta$.

Let S_1 be the set of points of S at which the density of S is unity. Then the measure of S_1 also is $2\pi-2\eta$.

Let x be a point of S_1, and let δ take such values that $x+\delta$ is a point of S. It is sufficient to consider positive values of δ. We suppose also that $\delta<\epsilon^3$. We write

$$\int_{-3\pi}^{3\pi}\left\{\frac{1}{\delta}\log\left|\frac{t-x}{t-x-\delta}\right|-\frac{1}{t-x}\right\}\{\phi(t)-\phi_1(t)\}\,dt$$

$$=\int_{-3\pi}^{x-\delta^{\frac{1}{4}}}+\int_{x-\delta^{\frac{1}{4}}}^{x}+\int_{x}^{x+\frac{1}{2}\delta}+\int_{x+\frac{1}{2}\delta}^{x+\delta}+\int_{x+\delta}^{x+\delta^{\frac{1}{4}}}+\int_{x+\delta^{\frac{1}{4}}}^{3\pi}$$

$$=J_1+J_2+J_3+J_4+J_5+J_6.$$

Now, for $t>x+\delta$,

$$0<\frac{1}{\delta}\log\frac{t-x}{t-x-\delta}-\frac{1}{t-x}=\frac{1}{\delta}\int_{t-x-\delta}^{t-x}\frac{t-x-u}{u(t-x)}\,du$$

$$<\frac{1}{\delta(t-x)(t-x-\delta)}\int_{t-x-\delta}^{t-x}(t-x-u)\,du=\frac{\delta}{2(t-x)(t-x-\delta)}.$$

Hence

$$|J_6|<\frac{\delta}{2\delta^{\frac{1}{4}}(\delta^{\frac{1}{4}}-\delta)}\int_{x+\delta^{\frac{1}{4}}}^{3\pi}|\phi(t)-\phi_1(t)|\,dt<K\delta^{\frac{1}{4}}<K\zeta,$$

where K depends on f only. Similarly

$$|J_1|<K\zeta.$$

* See Hobson's *Theory of functions of a real variable*, 2 (2nd ed. 1926), 145.

Again $$J_2 = \int_{x-\delta^{\frac{1}{3}}}^{x} \left\{1 - \frac{x-t}{\delta} \log \frac{x+\delta-t}{x-t}\right\} \frac{\phi(t)-\phi_1(t)}{x-t}\, dt.$$

Now, in this range,

$$\frac{d}{dt}\left\{1 - \frac{x-t}{\delta} \log \frac{x+\delta-t}{x-t}\right\} = \frac{1}{\delta} \int_{x-t}^{x+\delta-t} \frac{u-x+t}{u^2}\, du > 0,$$

so that the function in brackets is positive increasing, and its greatest value is unity. Hence, by the second mean value theorem, there is a number ξ between $x-\delta^{\frac{1}{3}}$ and x such that

$$J_2 = \int_{\xi}^{x} \frac{\phi(t)-\phi_1(t)}{x-t}\, dt.$$

Hence $$|J_2| < \xi.$$

Again $$J_3 = \int_{x}^{x+\frac{1}{2}\delta} \left\{\frac{t-x}{\delta} \log \frac{t-x}{x+\delta-t} - 1\right\} \frac{\phi(t)-\phi_1(t)}{t-x}\, dt.$$

Here

$$\frac{d^2}{dt^2}\left\{\frac{t-x}{\delta} \log \frac{t-x}{x+\delta-t} - 1\right\} = \frac{\delta}{(x+\delta-t)^2(t-x)} \geqslant 0,$$

and the function in brackets is -1 at both ends of the range. Hence it decreases from -1 to a minimum at $t = x+\theta\delta$, say $(0 < \theta < \frac{1}{2})$, and then increases. Hence the second mean value theorem gives

$$J_3 = \left(\theta \log \frac{\theta}{\theta-1} - 1\right) \int_{\xi}^{\xi'} \frac{\phi(t)-\phi_1(t)}{t-x}\, dt,$$

where $x < \xi < \xi' < x+\frac{1}{2}\delta$. Hence

$$|J_3| < A\xi.$$

Similar considerations apply to the other two integrals. We write

$$J_4 = \int_{x+\frac{1}{2}\delta}^{x+\delta} \left\{\frac{x+\delta-t}{\delta} \log \frac{t-x}{x+\delta-t} - \frac{x+\delta-t}{t-x}\right\} \frac{\phi(t)-\phi_1(t)}{x+\delta-t}\, dt.$$

Here the function in brackets increases from -1 to a positive maximum (which is an absolute constant), and then decreases to zero; and the second mean value theorem gives

$$|J_4| < A\xi.$$

Similarly, $$|J_5| < A\xi.$$

We have thus proved that, however small ζ may be, the set where

$$\left|\int_{-3\pi}^{3\pi}\left\{\frac{1}{\delta}\log\left|\frac{t-x}{t-x-\delta}\right|-\frac{1}{t-x}\right\}\{\phi(t)-\phi_1(t)\}\,dt\right|<A\zeta$$

has unit density at $\delta=0$, provided that x is a point of S_1. In other words (with the notation of § 2 . 3) $h_3(x)$ is the approximate derivative of $H_3(x)$, the function obtained from it by formal integration, at all points of S_1. But $h_1(x)$ and $h_2(x)$ are the derivatives, in the ordinary sense, of $H_1(x)$ and $H_2(x)$, for almost all values of x; for, f_1 and ϕ_1 being bounded, it follows from § 4 . 1 that H_1 and H_2 are the integrals of h_1 and h_2. Hence $h(x)$ is the approximate derivative of $H(x)$ at almost all points of S_1. Since the measure of S_1 may be as near to 2π as we please, the result follows.

10. THEOREM 9. *For almost all values of x*

$$f(x)-\tfrac{1}{2}a_0=\frac{1}{4\pi}P\int_{-\pi}^{\pi}\{G(t)-G(x)\}\operatorname{cosec}^2\tfrac{1}{2}(t-x)\,dt.$$

10 . 1. We first note that this is true if $f(t)$ belongs to L^p for $p>1$. For then $G(t)$ is the integral of $g(t)$, and

$$\left\{\int_{-\pi}^{x-\epsilon}+\int_{x+\epsilon}^{\pi}\right\}g(t)\cot\tfrac{1}{2}(t-x)\,dt$$
$$=\{2G(x)-G(x+\epsilon)-G(x-\epsilon)\}\cot\tfrac{1}{2}\epsilon$$
$$+\tfrac{1}{2}\left\{\int_{-\pi}^{x-\epsilon}+\int_{x+\epsilon}^{\pi}\right\}\{G(t)-G(x)\}\operatorname{cosec}^2\tfrac{1}{2}(t-x)\,dt.$$

The first term on the right tends to zero with ϵ wherever $G(x)$ has a derivative, *i.e.* almost everywhere. Hence the result.

10 . 2. In the general case we use our standard construction and notation. Since f_1 and ϕ_1 are bounded,

$$f_1(x)+\phi_1(x)-\tfrac{1}{2}a_0'$$
$$=\frac{1}{4\pi}P\int_{-\pi}^{\pi}\{G_1(t)+G_2(t)-G_1(x)-G_2(x)\}\operatorname{cosec}^2\tfrac{1}{2}(t-x)\,dt,$$

where a_0' is the first Fourier coefficient of $f_1+\phi_1$.

Instead of considering G_3 directly, we consider first

$$\chi(x)=P\int_{-\pi}^{\pi}\{H_3(t)-H_3(x)\}\frac{dt}{(t-x)^2},$$

and we shall prove that this principal value exists for almost all values of x in $(-\pi+\delta, \pi-\delta)-L'$, if $\delta>0$, and that

$$\int_{(-\pi+\delta,\ \pi-\delta)-L'} |\chi(x)|\,dx < K\alpha. \tag{10.21}$$

We write

$$\chi(x) = \int_{-\pi}^{x-\delta} + P\int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\pi} = \chi_1+\chi_2+\chi_3,$$

so that

$$\chi_2(x) = \lim_{\epsilon\to 0}\int_\epsilon^\delta \frac{ds}{s^2}\int_L \{\phi(u)-\phi_1(u)\}\log\left|\frac{(x+s-u)(x-s-u)}{(x-u)^2}\right| du,$$

if this limit exists. We write the integral as

$$\sum_i \int_\epsilon^\delta \frac{ds}{s^2}\int_{a_i}^{b_i} \{\phi(u)-\phi_1(u)\}\log\left|\frac{(x+s-u)(x-s-u)}{(x-u)^2}\right| du$$

$$= \Sigma J_i = \Sigma' J_i + \Sigma'' J_i,$$

where Σ' contains terms for which $b_i<x$, Σ'' terms for which $a_i>x$. Since x does not belong to L', if $b_i<x$, then also $b_i+l_i<x$, while if $a_i>x$, then $a_i+l_i>x$.

In Σ' we write

$$J_i = \int_\epsilon^{x-b_i-l_i} + \int_{x-b_i-l_i}^{x-a_i+l_i} + \int_{x-a_i+l_i}^{\delta} = J_{i1}+J_{i2}+J_{i3},$$

if $\epsilon < x-b_i-l_i$; if $x-b_i-l_i \leqslant \epsilon < x-a_i+l_i$, J_{i1} disappears, and the lower limit in J_{i2} is ϵ; if $\epsilon \geqslant x-a_i+l_i$, J_{i2} also disappears, and the lower limit in J_{i3} is ϵ. Similar modifications are required if $x-a_i+l_i>\delta$, etc.

10.3. Consider first J_{i1} (supposing that it exists). Since $\phi(u)$ is positive, and the logarithmic factor is continuous, we may write

$$\int_{a_i}^{b_i}\phi(u)\log\left|\frac{(x+s-u)(x-s-u)}{(x-u)^2}\right| du = \int_{a_i}^{b_i}\phi(u)\,du\log\left|\frac{(x+s-u_i)(x-s-u_i)}{x-u_i}\right.$$

$$= \tfrac{1}{2}nl_i\log\left|\frac{(x+s-u_i)(x-s-u_i)}{(x-u_i)^2}\right|,$$

where u_i is a point of l_i. Similarly

$$\int_{a_i}^{b_i}\phi_1(u)\log\left|\frac{(x+s-u)(x-s-u)}{(x-u)^2}\right| du = \tfrac{1}{2}n\,l_i\log\left|\frac{(x+s-u_i')(x-s-u_i')}{(x-u_i')^2}\right|,$$

where u_i' is also a point of l_i. Hence

$$\int_{a_i}^{b_i} \{\phi(u)-\phi_1(u)\} \log\left|\frac{(x+s-u)(x-s-u)}{(x-u)^2}\right| du$$
$$= \tfrac{1}{2} n\, l_i \int_{u_i'}^{v_i} \frac{2s^2\, du}{(x+s-u)(x-s-u)(x-u)},$$

which is less in absolute value than

$$\frac{n l_i^2 s^2}{(x+s-b_i)(x-s-b_i)(x-b_i)}.$$

Hence

$$|J_{i1}| < \frac{n\, l_i^2}{x-b_i} \int_{\epsilon}^{x-b_i-l_i} \frac{ds}{(x+s-b_i)(x-s-b_i)}$$
$$< \frac{n\, l_i^2}{x-b_i} \int_{0}^{x-b_i-l_i} \frac{ds}{(x+s-b_i)(x-s-b_i)} = \frac{n\, l_i^2}{2(x-b_i)^2} \log\frac{2x-2b_i-l_i}{l_i}.$$

Now
$$\Sigma' \frac{l_i^2}{(x-b_i)^2} \log\frac{2x-2b_i-l_i}{l_i}$$

is convergent for almost all values of x in $(-\pi, \pi)-L'$, for

$$\int_{(-\pi,\pi)-L'} \left\{\Sigma' \frac{l_i^2}{(x-b_i)^2} \log\frac{2x-2b_i-l_i}{l_i}\right\} dx$$
$$< \Sigma\, l_i^2 \int_{b_i+l_i}^{\infty} \log\frac{2x-2b_i-l_i}{l_i} \frac{dx}{(x-b_i)^2}$$
$$= \sum_i l_i \int_1^{\infty} \log(2\xi-1)\, \frac{d\xi}{\xi^2} < A \Sigma\, l_i < \frac{A\alpha}{n};$$

and this would be impossible if the series were divergent in a sub-set of $(-\pi, \pi)-L'$ of positive measure. Hence $\Sigma' J_{i1}$ converges uniformly with respect to ϵ almost everywhere in $(-\pi, \pi)-L'$. Since

$$\lim_{\epsilon\to 0} J_{i1}$$

exists for every i and x in $(-\pi, \pi)-L'$, it follows that

$$\lim_{\epsilon\to 0} \Sigma' J_{i1}$$

exists almost everywhere in $(-\pi, \pi)-L'$ and also that

$$\int_{(-\pi,\pi)-L'} \left|\lim_{\epsilon\to 0} \Sigma' J_{i1}\right| dx < A\alpha.$$

A similar result may be obtained for J_{i3}.

10 . 4. In dealing with J_{i2} we require the following lemma :

LEMMA. *If* $0<a\leqslant u<v\leqslant b$, $a\leqslant x<y\leqslant b$, *then*

$$\left|\int_u^v \log\left|\frac{y-t}{x-t}\right|\frac{dt}{t^2}\right|<A\,\frac{b-a}{a^2}.$$

It is sufficient to prove that

$$\left|\int_u^v \log\left|\frac{y-t}{x-t}\right|dt\right|<A(b-a),$$

since the result then follows from the second mean value theorem. Since

$$\int_u^v \log\left|\frac{y-t}{x-t}\right|dt=\int_x^y \log\left|\frac{v-t}{u-t}\right|dt$$

it is sufficient to consider the case where $y-x\leqslant v-u$. Now, if $y<u-(v-u)$ or if $x>v+(v-u)$, we have

$$\tfrac{1}{2}<\left|\frac{y-t}{x-t}\right|<2 \qquad (u<t<v),$$

and the result follows for such values. In the remaining case we may write

$$x=u+\xi(v-u) \quad (-2\leqslant\xi\leqslant 2),$$

$$y=u+\eta(v-u) \quad (-1\leqslant\eta\leqslant 3).$$

Then

$$\int_u^v \log|x-t|\,dt=(v-u)\{\log(v-u)+(1-\xi)\log|1-\xi|+\xi\log|\xi|-1\},$$

and similarly for the integral involving y; and, subtracting, we have the result.

10 . 5. We write

$$\begin{aligned}J_{i2}&=\int_{a_i}^{b_i}\{\phi(u)-\phi_1(u)\}\,du\int_\epsilon^{x-a_i+l_i}\log|x+s-u|\,s^{-2}\,ds\\&\quad+\int_{a_i}^{b_i}\{\phi(u)-\phi_1(u)\}\,du\int_\epsilon^{x-a_i+l_i}\log|x-s-u|\,s^{-2}\,ds\\&\quad-2\int_{a_i}^{b_i}\{\phi(u)-\phi_1(u)\}\log|x-u|\,du\int_\epsilon^{x-a_i+l_i}s^{-2}\,ds\\&=I_1+I_2+I_3,\end{aligned}$$

say, where $x-b_i-l_i \leqslant \epsilon < x-a_i+l_i$. As before, there are points u_i, u_i' in l_i such that

$$I_1 = \tfrac{1}{2}nl_i \int_\epsilon^{x-a_i+l_i} \log \left| \frac{x+s-u_i}{x+s-u_i'} \right| \frac{ds}{s^2}.$$

Hence
$$|I_1| < \tfrac{1}{2}nl_i \log \frac{2x-a_i+l_i-a_i}{2x-b_i-l_i-b_i} \int_{x-b_i-l_i}^{x-a_i+l_i} \frac{ds}{s^2}$$

$$< \frac{Anl_i^2}{(x-b_i)^2} \log \frac{2(x-b_i)+3l_i}{2(x-b_i)-l_i}$$

$$< \frac{Anl_i^3}{(x-b_i)^3},$$

and a result similar to that for J_{i1} is obtained.

In I_2, since the s-integral is a continuous function of u, we have

$$I_2 = \tfrac{1}{2}nl_i \int_\epsilon^{x-a_i+l_i} \log \left| \frac{x-s-u_i}{x-s-u_i'} \right| \frac{ds}{s^2},$$

and it follows from the lemma that

$$|I_2| < \frac{Anl_i^2}{(x-b_i)^2}.$$

This leads to the same result for I_2. Finally I_3 behaves like I_1.

We have thus proved that the limit which defines $\chi_2(x)$ exists almost everywhere outside L', and that

$$\int_{(-\pi+\delta,\ \pi-\delta)-L'} |\chi_2(x)|\, dx < Aa.$$

10.6. We have next

$$|\chi_1(x)| < \int_{-\pi}^{x-\delta} \frac{dt}{(x-t)^2} \int_{-3\pi}^{3\pi} |\phi(u)-\phi_1(u)| \left| \log \left| \frac{u-t}{u-x} \right| \right| du$$

$$< K \int_{-3\pi}^{3\pi} |\phi(u)-\phi_1(u)| \left| \log |u-t| \right| du,$$

so that

$$\int_{-\pi+\delta}^{\pi-\delta} |\chi_1(x)|\, dx < K \int_{-3\pi}^{3\pi} |\phi(u)-\phi_1(u)|\, du < Ka.$$

A similar result holds for χ_3, and (10.21) now follows.

10.7. We can now prove that

$$\omega(x) = \frac{1}{4\pi} P \int_{-\pi}^{\pi} \{G_3(t)-G_3(x)\} \operatorname{cosec}^2 \tfrac{1}{2}(t-x)\, dt$$

has properties similar to those $\chi(x)$. We have

$$\pi\omega(x)-\chi(x) = \int_{-\pi}^{\pi} \{G_3(t)-G_3(x)\}\{\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}(t-x)-(t-x)^{-2}\}\,dt$$

$$+P\int_{-\pi}^{\pi}\{G_3(t)-G_3(x)-H_3(t)-H_3(x)\}(t-x)^{-2}\,dt$$

$$= \omega_1(x)+\omega_2(x).$$

Now $$\int_{-\pi+\delta}^{\pi-\delta}|\omega_1(x)|\,dx < K\int_{-\pi+\delta}^{\pi-\delta}dx\int_{-\pi}^{\pi}|G_3(t)-G_3(x)|\,dt$$

$$< K\int_{-\pi}^{\pi}|\phi(u)-\phi_1(u)|\,du < K\alpha.$$

Also

$$\omega_2(x) = P\int_{-\pi}^{\pi}\frac{dt}{(t-x)^2}\int_{-\pi}^{\pi}\{\phi(u)-\phi_1(u)\}\{\lambda(u-t)-\lambda(u-x)\}\,du,$$

where $$\lambda(u-t) = \lambda(u-x)+(t-x)\lambda'(u-x)+O\{(t-x)^2\}.$$

Hence

$$\omega_2(x) = P\int_{-\pi}^{\pi}\frac{dt}{t-x}\int_{-\pi}^{\pi}\{\phi(u)-\phi_1(u)\}\lambda'(u-x)\,du+O(\alpha),$$

$$\int_{-\pi}^{\pi}|\omega_2(x)|\,dx < K\alpha.$$

The result for $\omega(x)$ then follows. In fact, we now see that

$$\Omega(x) = \tfrac{1}{4}\pi P\int_{-\pi}^{\pi}\{G(t)-G(x)\}\operatorname{cosec}^2\tfrac{1}{2}(t-x)\,dt$$

exists for almost all x in $(-\pi+\delta,\ \pi-\delta)-L'$, and that

$$\int_{(-\pi+\delta,\ \pi-\delta)-L'}|f(x)-\tfrac{1}{2}a_0-\Omega(x)|\,dx < A\alpha.$$

Since α and δ may be as small as we please, the theorem follows.

11. *Theorems on Q-integrals.*

11.1. THEOREM 10. *For a monotonic function, the P and Q definitions are equivalent.*

Suppose, for example, that $f(x)$ decreases steadily from zero at $x=-1$ to negative infinity at $x=0$, and then from positive infinity to zero at $x=1$.

The inverse function $x=\phi(y)$ of $y=f(x)$ may be defined as follows: for a given y, either $f(x)=y$ for a single value x_1 of x, in which case

$\phi(y) = x_1$; or there is a single x_1 such that $f(x_1+0) \leqslant y \leqslant f(x_1-0)$, in which case $\phi(y) = x_1$; or $f(x) = y$ throughout the interval (x_1, x_2), in which case we may put $\phi(y) = \frac{1}{2}(x_1+x_2)$.

Suppose, first, that the P-integral exists, say

$$\lim_{\epsilon \to 0} \int_\epsilon^1 \{f(x)-g(x)\}\, dx = L,$$

where $g(x) = -f(-x)$. Then the Q-integral exists and is equal to the P-integral if

$$\lim_{n \to \infty} \int_0^n \{\phi(y)-\psi(y)\}\, dy = L,$$

where $\psi(y) = -\phi(-y)$ is the function inverse to $g(x)$.

Suppose that $\phi(n) \geqslant \psi(n)$. Then

$$\begin{aligned}\int_0^n \{\phi(y)-\psi(y)\}\, dy &= n\,\phi(n)+\int_{\phi(n)}^1 f(x)\, dx - n\,\psi(n) - \int_{\psi(n)}^1 g(x)\, dx \\ &= \int_{\psi(n)}^1 \{f(x)-g(x)\}\, dx - \int_{\psi(n)}^{\phi(n)} \{f(x)-n\}\, dx \\ &\leqslant \int_{\psi(n)}^1 \{f(x)-g(x)\}\, dx,\end{aligned}$$

since $f(x) \geqslant n$ for $x \leqslant \phi(n)$. Similarly

$$\int_0^n \{\phi(y)-\psi(y)\}\, dy \geqslant \int_{\phi(n)}^1 \{f(x)-g(x)\}\, dx.$$

If $\phi(n) < \psi(n)$ the inequalities are reversed. The two inequalities together give the result.

The same inequalities, written in the form

$$\int_0^{g(\epsilon)} (\phi-\psi)\, dy \leqslant \int_\epsilon^1 (f-g)\, dx \leqslant \int_0^{f(\epsilon)} (\phi-\psi)\, dy$$

prove the converse theorem.

11.2. THEOREM 11. *In general* P does not imply Q, nor does Q imply P, and the P and Q integral may both exist without being equal.*

(i) Let

$$f(x) = n\left(\frac{1}{n} < x < \frac{1}{n}+\frac{1}{2n^2}\right), \quad = -n^2\left(-\frac{1}{n}-\frac{1}{2n^3} < x < -\frac{1}{n}\right),$$

* We write "P" for "the existence of the P integral", etc.

for $n = 2, 3, \ldots$, and $f(x) = 0$ elsewhere. Then the P-integral is zero. But

$$\int_{-1}^{1} [f(x)]_N \, dx = \tfrac{1}{2} \sum_{n=2}^{N} \frac{1}{n} + \tfrac{1}{2} \sum_{n=N+1}^{\infty} \frac{N}{n^3} - \tfrac{1}{2} \sum_{n \leqslant \sqrt{N}} \frac{1}{n} - \tfrac{1}{2} \frac{1}{n} \sum_{n > \sqrt{N}} \frac{N}{n^3}$$

$$\sim \tfrac{1}{4} \log N,$$

so that the Q-integral does not exist.

(ii) Define $f(x)$ as before for $x > 0$, and let $f(x) = -n^2$ for $-n^{-1} - n^{-3} < x < -n^{-1}$. It is easily seen that in this case the Q-integral exists, but not the P-integral.

(iii) Define $f(x)$ as before for $x > 0$, and let

$$f(x) = -2n \quad \left(-\frac{1}{n} - \frac{1}{4n^2} < x < -\frac{1}{n}\right).$$

It is easily seen that

$$P\int_{-1}^{1} f(x)\, dx = 0, \qquad Q\int_{-1}^{1} f(x)\, dx = \tfrac{1}{2} \log 2 + \tfrac{1}{4}.$$

11 . 3. THEOREM 12. *It is not necessarily true that*

$$Q\int_a^b f(x)\, dx + Q\int_a^b g(x)\, dx = Q\int_a^b \{f(x)+g(x)\}\, dx,$$

even if all the integrals exist.

Let

$$f(x) = n\left(\frac{1}{n} < x < \frac{1}{n} + \frac{1}{2n^2}\right), \quad = -n\left(-\frac{1}{n} - \frac{1}{2n^2} < x < -\frac{1}{n}\right),$$

$$g(x) = n\left(\frac{1}{n} < x < \frac{1}{n} + \frac{1}{2n^2}\right), \quad = -n\left(-\frac{1}{n} < x < -\frac{1}{n} + \frac{1}{2n^2}\right),$$

both functions being zero outside these intervals. Then

$$Q\int_{-1}^{1} f(x)\, dx = 0, \quad Q\int_{-1}^{1} g(x)\, dx = 0,$$

but

$$Q\int_{-1}^{1} \{f(x)+g(x)\}\, dx = \log \tfrac{1}{2}.$$

11.4. THEOREM 13. *If $g(x)$ has a Lebesgue integral over (a, b), then*

$$Q\int_a^b f(x)\,dx+\int_a^b g(x)\,dx = Q\int_a^b \{f(x)+g(x)\}\,dx,$$

provided that either Q-integral exists.

Let E_r be the set where $r \leqslant g(x) < r+1$. Then the series

$$\sum_{r=-\infty}^{\infty} |r|\, m(E_r) \tag{11.41}$$

is convergent. Also

$$\int_a^b \{[f+g]_n-[f]_n\}\,dx = \sum_{r=-\infty}^{\infty} \int_{E_r} \{[f+g]_n-[f]_n\}\,dx. \tag{11.42}$$

Suppose that $0 \leqslant r \leqslant n$, and that x is a point of E_r. Then, if $f(x) > n$,

$$[f+g]_n-[f]_n = n-n = 0.$$

If $n-r-1 < f \leqslant n$,

$$|[f+g]_n-[f]_n| \leqslant n-f < r+1.$$

If $-n < f \leqslant n-r-1$,

$$|[f+g]_n-[f]_n| = |f+g-f| < r+1.$$

If $-n-r-1 < f \leqslant n$,

$$|[f+g]_n-[f]_n| < n-(n-r-1) = r+1.$$

If $f \leqslant -n-r-1$,

$$[f+g]_n-[f]_n = -n+n = 0.$$

Also, if $r > n$,

$$|[f+g]_n-[f]_n| \leqslant 2n < 2r.$$

Hence, in any case,

$$|[f+g]_n-[f]_n| < 2r+1.$$

A similar result may be obtained if $r < 0$. Hence the series on the right of (11.42) is uniformly convergent with respect to n, by comparison with (11.41). But, for any fixed r,

$$[f+g]_n-[f]_n-g = 0$$

if $|f|<n-|r|-1$. Hence

$$\left|\int_{E_r}\{[f+g]_n-[f]_n-g\}\,dx\right|<(2|r|+1)\,mE(|f|\geqslant n-|r|-1),$$

which tends to zero when $n\to\infty$, since f is finite almost everywhere. Hence the result.

11.5. THEOREM 14. *Suppose that, as $n\to\infty$, the measure of the set where $|f(x)|>n$ is $o(1/n)$; and that $g(x)$ also has this property. Then*

$$Q\int_a^b f(x)\,dx+Q\int_a^b g(x)\,dx=Q\int_a^b\{f(x)+g(x)\}\,dx$$

if any two of the integrals exist.

We have

$$\lim_{n\to\infty}\int_a^b\{[f]_n+[g]_n-[f+g]_n\}\,dx=0.$$

For the integral is zero if $|f|<\frac{1}{2}n$ and $|g|<\frac{1}{2}n$; and the measure of the set where either of these conditions is not satisfied is $o(1/n)$, while the integrand is $O(n)$.

11.6. A slightly different definition is obtained by putting

$$[f(x)]^n=f(x) \quad \text{if} \quad |f(x)|\leqslant n,$$

$$=0 \quad \text{if} \quad |f(x)|>n,$$

and writing

$$Q'\int_a^b f(x)\,dx=\lim_{n\to\infty}\int_a^b[f(x)]^n\,dx.$$

For conjugate functions this is equivalent to the Q-definition. But this is not generally true, even if $f(x)$ is monotonic.

THEOREM 15. *For monotonic functions, Q does not necessarily imply Q', but Q' implies Q.*

(i) Let

$$f(x)=n^3\left(\frac{1}{n+1}\leqslant x<\frac{1}{n}\right), \quad =-n^3-1\left(-\frac{1}{n}<x\leqslant-\frac{1}{n+1}\right).$$

Obviously the P-integral exists, and so, by Theorem 10, the Q-integral.

But
$$\int_{-1}^{1}[f(x)]^{n^2}\,dx=\int_{-1}^{-(1/n)}f(x)\,dx+\int_{1/(n+1)}^{1}f(x)\,dx$$
$$=\frac{n^2}{n+1}+1-\frac{1}{n},$$

so that the Q'-integral does not exist.

(ii) With the notation of Theorem 1,

$$\int_{-1}^{1}[f(x)]_n\,dx=\int_0^n\{\phi(y)-\psi(y)\}\,dy,$$

$$\int_{-1}^{1}[f(x)]^n\,dx=\int_0^n\{\phi(y)-\psi(y)\}\,dy-n\{\phi(n)-\psi(n)\}.$$

The result is then a case of the following lemma* :

LEMMA. *If*
$$\int_0^x F(t)\,dt-x\,F(x)$$

tends to a limit when $x\to\infty$, *and* $F(x)=o(1)$, *then* $F(x)=o(1/x)$.

Let us suppose that F is continuous (slight additions to the argument are required if it is not). Let the limit in question be L. Then, for $x>x_0$,

$$L-\epsilon<\int_0^x F(t)\,dt-x\,F(x)<L+\epsilon.$$

Let $x_1>x_0$. Suppose that $F(x_1)>0$. Let x_2 be the least value of x greater than x_1 such that $F(x_2)=\frac{1}{2}F(x_1)$. Such a value exists, since $F\to 0$. Then

$$\tfrac{1}{2}x_1F(x_1)=x_1F(x_1)-x_2F(x_2)+(x_2-x_1)\,F(x_2)$$
$$<\left\{\int_0^{x_1}F\,dt-L+\epsilon\right\}-\left\{\int_0^{x_2}F\,dt-L-\epsilon\right\}+(x_2-x_1)\,F(x_2)$$
$$=(x_2-x_1)\,F(x_2)-\int_{x_1}^{x_2}F\,dt+2\epsilon$$
$$<2\epsilon,$$

since $F(x)>F(x_2)$ for $x_1<x<x_2$. Similarly if $F(x_1)<0$. Hence the result.

* The proof given was suggested by Pólya and Szegö's *Aufgaben und Lehrsätze*, 1 Abschn. No. 78. It was then pointed out to me by Prof. Hardy that the lemma is very easily proved by the method of his paper "On a limit theorem of Mr. Mercer", *Quart J. of Math.*, 43 (1912), 143–150, though the actual result is not given there. Mr. Vijayaraghavan has also sent me a simple proof.

THE APPROXIMATE FUNCTIONAL EQUATIONS FOR $\zeta(s)$ AND $\zeta^2(s)$

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Received 7 December, 1927.—Read 15 December, 1927.]

1. *Introduction.*

1. The first part (§ 2) of this paper is merely a supplement to two papers written in 1920 and 1921.

The zeta-function $\zeta(s) = \zeta(\sigma+it)$ is represented in the half planes $\sigma > 1$ and $\sigma < 0$ by the formulae

$$(1.1) \qquad \zeta(s) = \Sigma n^{-s}, \quad \zeta(s) = 2(2\pi)^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s)\, \Sigma n^{s-1}$$

respectively. The "approximate functional equation" is a compromise between these formulae, and represents $\zeta(s)$ as the sum of two finite series and two error terms. The representation is valid over the whole plane, but is naturally important only in the critical strip $0 \leqslant \sigma \leqslant 1$, where the formulae (1.1) fail.

The second* of the papers mentioned above contained a proof of the approximate functional equation in its most general form; in the first† we proved only an imperfect form‡ which was sufficient for our immediate application. Our first proof, however, used only the methods of "real" analysis, whereas the second was based throughout on Cauchy's theorem, and it was the first proof which seemed to us in principle the better, in spite of the imperfection of the result. We therefore now show here how, by a slight sharpening of our former analysis, we can apply the "real" method to prove the theorem in what is essentially its complete form§.

---

* Hardy and Littlewood, **2**.

† Hardy and Littlewood, **1**.

‡ For fixed σ of the open strip $0 < \sigma < 1$, and with an additional logarithm in the error terms.

§ We restrict ourselves, in order to avoid unessential complications, to the strip $-\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$. What is essential is to obtain a result holding uniformly throughout the critical strip.

In the second part (§ 3) we prove, by a method quite different from either of our earlier methods, the corresponding theorem for $\zeta^2(s)$, which is not in any way a corollary of that for $\zeta(s)$. We have had the proof for several years, and publish it now because Ingham* has used the theorem recently in his investigation of the mean value of $|\zeta(\frac{1}{2}+it)|^4$. The same method may be used to give yet another proof of the approximate functional equation for $\zeta(s)$ itself.

2. *The equation for* $\zeta(s)$.

2.1. THEOREM I. *If A is a positive constant, and*

$$-\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{3}{2}, \quad x > A, \quad y > A, \quad 2\pi xy = |t|, \tag{2.11}$$

$$\chi = \chi(s) = 2(2\pi)^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s), \tag{2.12}$$

then

$$\zeta(s) = \sum_{n \leqslant x} n^{-s} + \chi \sum_{n \leqslant y} n^{s-1} + O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}), \tag{2.13}$$

the constants implied by the O's depending only on A.

It may be verified at once that $\chi = O(|t|^{\frac{1}{2}-\sigma})$.

The proof of the theorem depends on the following lemmas.

LEMMA α. *If $\sigma > 0$, $s \neq 1$, $x > 0$, then*

$$\zeta(s) + \frac{x^{1-s}}{1-s} - \sum_{n \leqslant x}{}' n^{-s} = 2(2\pi)^{s-1} \sum_{n=1}^{\infty} n^{s-1} \int_{2n\pi x}^{\infty} u^{-s} \cos u\, du, \tag{2.14}$$

the dash implying that, when x is an integer, the term in which $n = x$ is to be halved.

There is nothing to add to our proof in **1**, Lemma 1.

2.2. LEMMA β. *Suppose that A denotes generally a positive constant, not the same at its various occurrences*†, *that*

$$\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{3}{2}, \quad t > A, \quad \xi > A, \tag{2.21}$$

and

$$I = I(\xi, s) = \int_{\xi}^{\infty} u^{-s} \cos u\, du. \tag{2.22}$$

---

* Ingham, **3**. Another proof of Ingham's result has been given recently by Titchmarsh, **6**. Both proofs are (as is to be expected) difficult.

† The inequalities satisfied by ξ and t sometimes restrict the possible values of A. Thus the A of (2.23) must be less than unity.

Then

$$I = \sin \tfrac{1}{2}s\pi\Gamma(1-s)+O\left(\frac{\xi^{1-\sigma}}{t}\right) \qquad (\xi < At < t), \tag{2.23}$$

$$I = \sin \tfrac{1}{2}s\pi\Gamma(1-s)+\frac{i\xi^{2-s}e^{i\xi}}{2t(\xi-t)}+O\left(\frac{\xi^{1-\sigma}}{t}\right)+O\left(\frac{\xi^{2-\sigma}}{(t-\xi)^3}\right) \tag{2.24}$$

$$(At < \xi < t-A),$$

$$I = \frac{i\xi^{1-s}e^{i\xi}}{2(\xi-t)}+O(\xi^{-\sigma})+O\left(\frac{t\xi^{1-\sigma}}{(\xi-t)^3}\right) \qquad (t+A < \xi < At), \tag{2.25}$$

$$I = O(\xi^{-\sigma}) \qquad (t < At < \xi), \tag{2.26}$$

and

$$I = O(\xi^{-\sigma}t^{\frac{1}{2}}) \tag{2.27}$$

in any case.

The lemma is a sharper form of Lemma 12 of our Paper **1**. The constants implied by the O's depend only on the relevant A's.

2.3. We suppose first that $t < \xi$, and write

$$I = \tfrac{1}{2}\int_\xi^\infty u^{-s}e^{iu}\,du+\tfrac{1}{2}\int_\xi^\infty u^{-s}e^{-iu}\,du = \tfrac{1}{2}I_1+\tfrac{1}{2}I_2. \tag{2.31}$$

Here
$$I_2 = \int_\xi^\infty u^{-\sigma}e^{-i(u+t\log u)}\,du = \int_\xi^\infty u^{-\sigma}e^{-iw}\,du,$$

say; and w increases with u, so that

$$I_2 = \int_{u=\xi}^\infty \frac{u^{1-\sigma}}{u+t}\,e^{-iw}\,dw.$$

The positive factor decreases as w increases*. Hence, separating the real and imaginary parts of the integral, and applying the second mean value theorem to each, we obtain

$$I_2 = O\left(\frac{\xi^{1-\sigma}}{\xi+t}\right) = O(\xi^{-\sigma}). \tag{2.32}$$

Again
$$I_1 = \int_\xi^\infty u^{-\sigma}e^{i(u-t\log u)}\,du = \int_\xi^\infty u^{-\sigma}e^{iw}\,du,$$

* Its derivative vanishes when $\sigma u = (1-\sigma)t$, which is inconsistent with $\sigma \geqslant \frac{1}{2}$ and $u > t$.

say, and here also (since $t<\xi\leqslant u$) w increases with u. Hence

$$(2.33)\quad I_1=\int_{u=\xi}^{\infty}\frac{u^{1-\sigma}}{u-t}\,e^{iw}\,dw=\frac{i\xi^{1-s}e^{i\xi}}{\xi-t}+i\int_{u=\xi}^{\infty}e^{iw}\,\frac{d}{dw}\left(\frac{u^{1-\sigma}}{u-t}\right)dw$$

$$=\frac{i\xi^{1-s}e^{i\xi}}{\xi-t}-i\sigma\int_{u=\xi}^{\infty}\frac{u^{1-\sigma}}{(u-t)^2}\,e^{iw}\,dw-it\int_{u=\xi}^{\infty}\frac{u^{1-\sigma}}{(u-t)^3}\,e^{iw}\,dw$$

$$=I_{1,1}+I_{1,2}+I_{1,3}.$$

The real part of $I_{1,3}$ is

$$t\int_{u=\xi}^{\infty}\frac{u^{1-\sigma}}{(u-t)^3}\sin w\,dw=\frac{t\xi^{1-\sigma}}{(\xi-t)^3}\int_{u=\xi}^{\xi}\sin w\,dw=O\left(\frac{t\xi^{1-\sigma}}{(\xi-t)^3}\right),$$

since

$$\frac{u^{1-\sigma}}{(u-t)^3}=u^{-2-\sigma}\left(\frac{u}{u-t}\right)^3$$

decreases as u increases; and the imaginary part may be treated similarly. Hence

$$(2.34)\qquad I_{1,3}=O\left(\frac{t\xi^{1-\sigma}}{(\xi-t)^3}\right).$$

A parallel argument gives

$$(2.35)\qquad I_{1,2}=O\left(\frac{\xi^{1-\sigma}}{(\xi-t)^2}\right).$$

If $t<At<\xi$, $I_{1,2}$ and $I_{1,3}$ are each $O(\xi^{-1-\sigma})$ and *a fortiori* $O(\xi^{-\sigma})$; and $I_{1,1}$ is obviously $O(\xi^{-\sigma})$. It follows, from (2.33), that I_1 is $O(\xi^{-\sigma})$, and then (2.26) follows from (2.31) and (2.32).

If $t+A<\xi<At$, then $t>A(\xi-t)$, and the error arising from $I_{1,2}$ may be absorbed into that arising from $I_{1,3}$. After this, (2.25) follows from (2.31), (2.32), (2.33), and (2.34).

2.4. We now suppose that $\xi<t$. If $0<\sigma<1$, we have

$$(2.41)\quad I=\Gamma(1-s)\sin\tfrac{1}{2}s\pi-\int_0^{\xi}u^{-s}\cos u\,du$$

$$=\Gamma(1-s)\sin\tfrac{1}{2}s\pi-\frac{\xi^{1-s}}{1-s}\cos\xi-\frac{1}{1-s}\int_0^{\xi}u^{1-s}\sin u\,du,$$

and the final formula (though not the proof) is obviously valid when $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{2}$. Hence

$$(2.42)\qquad I=\Gamma(1-s)\sin\tfrac{1}{2}s\pi+O\left(\frac{\xi^{1-\sigma}}{t}\right)+J,$$

where

$$(2.43)\quad J=\frac{1}{2i(s-1)}\int_0^{\xi}u^{1-s}e^{iu}\,du-\frac{1}{2i(s-1)}\int_0^{\xi}u^{1-s}e^{-iu}\,du=J_1+J_2,$$

say.

Here

$$J_2 = -\frac{1}{2i(s-1)}\int_0^\xi u^{1-\sigma}e^{-i(u+t\log u)}\,du = -\frac{1}{2i(s-1)}\int_0^\xi u^{1-\sigma}e^{-iw}\,du$$
$$= O\left(\frac{1}{t}\right)\int_{u=0}^{\xi}\frac{u^{2-\sigma}}{u+t}e^{-iw}\,dw.$$

The positive factor of the integrand increases* with u or w. Hence, separating the real and imaginary parts, and using the second mean value theorem, we obtain

$$J_2 = O\left(\frac{\xi^{2-\sigma}}{\xi+t}\right) = O\left(\frac{\xi^{1-\sigma}}{t}\right). \tag{2.44}$$

Again $$2i(s-1)J_1 = \int_0^\xi u^{1-\sigma}e^{i(u-t\log u)}\,du = \int_0^\xi u^{1-\sigma}e^{iw}\,du,$$

where now $w = u - t\log u$ decreases as u increases. Hence

$$2i(s-1)J_1 = -\int_{u=0}^{\xi}\frac{u^{2-\sigma}}{t-u}e^{iw}\,dw = \frac{i\xi^{2-s}e^{i\xi}}{t-\xi} - i\int_{u=0}^{\xi}e^{iw}\frac{d}{dw}\left(\frac{u^{2-\sigma}}{t-u}\right)dw \tag{2.45}$$
$$= \frac{i\xi^{2-s}e^{i\xi}}{t-\xi} + i(1-\sigma)\int_{u=0}^{\xi}\frac{u^{1-\sigma}}{(t-u)^2}e^{iw}\,dw + it\int_{u=0}^{\xi}\frac{u^{2-\sigma}}{(t-u)^3}e^{iw}\,dw$$
$$= J_{1,1}+J_{1,2}+J_{1,3}.$$

The positive factors of the integrands in $J_{1,3}$ and $J_{1,2}$ increase with u, and we plainly obtain

$$J_{1,3} = O\left(\frac{t\xi^{2-\sigma}}{(t-\xi)^3}\right), \tag{2.46}$$

$$J_{1,2} = O\left(\frac{\xi^{2-\sigma}}{(t-\xi)^2}\right). \tag{2.47}$$

If $\xi < At < t$, $J_{1,2}$ and $J_{1,3}$ are each $O(\xi^{2-\sigma}t^{-2})$ and *a fortiori* $O(\xi^{1-\sigma}t^{-1})$, and $J_{1,1}$ is obviously of the same form. Hence, by (2.45), J_1 is $O(\xi^{2-\sigma}t^{-2})$ and *a fortiori* is $O(\xi^{1-\sigma}t^{-1})$; and (2.23) then follows from (2.42), (2.43), and (2.44).

If $At < \xi < t-A$, we have

$$J_1 = \frac{\xi^{2-s}e^{i\xi}}{2(s-1)(t-\xi)} + O\left(\frac{1}{t}\right)(J_{1,2}+J_{1,3})$$
$$= -\frac{i\xi^{2-s}e^{i\xi}}{2t(t-\xi)} + O\left(\frac{\xi^{2-\sigma}}{t^2}\right) + O\left(\frac{\xi^{2-\sigma}}{(t-\xi)^3}\right);$$

and this equation, with (2.42), (2.43), and (2.44), gives (2.24).

* Its derivative vanishes if $(\sigma-1)u = (2-\sigma)t$, which is inconsistent with $\sigma \leqslant \frac{3}{2}$ and $u < t$.

2.5. It remains to prove (2.27). If $t<At\leqslant\xi$, (2.26) gives

$$I=O(\xi^{-\sigma})=O(\xi^{-\sigma}t^{\frac{1}{2}}).$$

If $t+t^{\frac{1}{2}}\leqslant\xi\leqslant At$, (2.25) gives

$$I=O\left(\frac{\xi^{1-\sigma}}{\xi-t}\right)+O(\xi^{-\sigma})+O\left(\frac{t\xi^{1-\sigma}}{(\xi-t)^3}\right)$$
$$=O(\xi^{1-\sigma}t^{-\frac{1}{2}})+O(\xi^{-\sigma})+O(\xi^{1-\sigma}t^{-\frac{1}{2}})=O(\xi^{-\sigma}t^{\frac{1}{2}}).$$

If $At\leqslant\xi\leqslant t-t^{\frac{1}{2}}$, (2.24) gives

$$I=O(t^{\frac{1}{2}-\sigma})+O\left(\frac{\xi^{2-\sigma}}{t(t-\xi)}\right)+O\left(\frac{\xi^{1-\sigma}}{t}\right)+O\left(\frac{\xi^{2-\sigma}}{(t-\xi)^3}\right)$$
$$=O(t^{\frac{1}{2}-\sigma})+O(\xi^{2-\sigma}t^{-\frac{3}{2}})+O(\xi^{1-\sigma}t^{-1})+O(\xi^{2-\sigma}t^{-\frac{3}{2}})=O(\xi^{-\sigma}t^{\frac{1}{2}}).$$

If $\xi\leqslant At<t$, (2.23) gives

$$I=O(t^{\frac{1}{2}-\sigma})+O(\xi^{1-\sigma}t^{-1})=O(\xi^{-\sigma}t^{\frac{1}{2}}).$$

Finally, if $t-t^{\frac{1}{2}}\leqslant\xi\leqslant t+t^{\frac{1}{2}}=\xi_1$, we have

$$I=\int_\xi^\infty=\int_\xi^{t+t^{\frac{1}{2}}}+I(\xi_1,s)=O(\xi^{-\sigma}t^{\frac{1}{2}})+O(\xi_1^{-\sigma}t^{\frac{1}{2}})=O(\xi^{-\sigma}t^{\frac{1}{2}}).$$

It follows that (2.27) holds in all cases.

This completes the proof of Lemma β. There are naturally similar results for the corresponding integral with a sine. We state as a separate lemma one result which we require.

LEMMA γ. *If the conditions* (2.21) *are satisfied, and* $t<At<\xi$, *then*

$$\int_\xi^\infty u^{-s-1}\sin u\,du=O(\xi^{-\sigma-1}). \tag{2.51}$$

The proof is substantially the same as that of (2.26).

2.6. In proving Theorem 1, we may suppose that $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{2}$. For if the theorem is proved in this case, and we change s into $1-s$, we obtain

$$\zeta(1-s)=\sum_{n\leqslant x}n^{s-1}+\chi_1\sum_{n\leqslant y}n^{-s}+O(x^{\sigma-1})+O(y^{-\sigma}|t|^{\sigma-\frac{1}{2}}).$$

where now $-\frac{1}{2}\leqslant\sigma\leqslant\frac{1}{2}$ and $\chi_1=\chi(1-s)=\chi^{-1}$. Substituting for $\zeta(1-s)$ from the functional equation, we deduce

$$\zeta(s)=\chi\sum_{n\leqslant x}n^{s-1}+\sum_{n\leqslant y}n^{-s}+O(|t|^{\frac{1}{2}-\sigma}x^{\sigma-1})+O(y^{-\sigma}),$$

which is (2.13) with x and y interchanged. We suppose then that $\sigma\geqslant\frac{1}{2}$ and, as plainly we may, that $t>0$.

We choose constants B and C so that $0<B<1<C$, $B+C=2$, and write

$$\tau = 1 \ (x \geqslant y), \quad \tau = y\sqrt{\left(\frac{2\pi}{t}\right)} \ (x \leqslant y). \tag{2.61}$$

We have then, by Lemma α,

$$\begin{aligned} \xi(s)+\frac{x^{1-s}}{1-s}-\sum_{n\leqslant x} n^{-s} &= 2(2\pi)^{s-1}(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5)\, n^{s-1}\int_{2n\pi x}^{\infty} u^{-s}\cos u\, du \\ &= S_1+S_2+S_3+S_4+S_5, \end{aligned} \tag{2.62}$$

say, where the ranges of the summations are $n<By$ in Σ_1, $By\leqslant n<y-\tau$ in Σ_2, $y-\tau\leqslant n<y+\tau$ in Σ_3, $y+\tau<n\leqslant Cy$ in Σ_4, and $Cy<n$ in Σ_5*. We dispose first of S_1 and S_5.

In S_1, $n<By$, $\xi=2\pi nx<2B\pi xy=Bt$. Hence, by (2.23),

$$S_1 = \chi \sum_{n<By} n^{s-1}+S_{1,2} = S_{1,1}+S_{1,2}, \tag{2.63}$$

where

$$S_{1,2} = O\left(\sum_{n<By} n^{\sigma-1}\frac{(nx)^{1-\sigma}}{t}\right) = O\left(\frac{yx^{1-\sigma}}{t}\right) = O(x^{-\sigma}). \tag{2.64}$$

In S_5 we write

$$\begin{aligned} 2(2\pi)^{s-1}\, n^{s-1}\int_{2n\pi x}^{\infty} u^{-s}\cos u\, du &= -\frac{x^{-s}}{n\pi}\sin 2n\pi x \\ &\quad +2s(2\pi)^{s-1}\, n^{s-1}\int_{2n\pi x}^{\infty} u^{-s-1}\sin u\, du, \end{aligned}$$

so that

$$S_5 = -\frac{x^{-s}}{\pi}\sum_{Cy<n}\frac{\sin 2n\pi x}{n}+2s(2\pi)^{s-1}\sum_{Cy<n} n^{s-1}\int_{2n\pi x}^{\infty} u^{-s-1}\sin u\, du.$$

The first term is $O(x^{-\sigma})$, since the sum of the series is bounded, and the second, by Lemma γ, is

$$O\left(t\sum_{Cy<n} n^{\sigma-1}(nx)^{-\sigma-1}\right) = O\left(tx^{-\sigma-1}\sum_{Cy<n} n^{-2}\right) = O(tx^{-\sigma-1}y^{-1}) = O(x^{-\sigma}).$$

Hence

$$S_5 = O(x^{-\sigma}). \tag{2.65}$$

---

* If $y-\tau$ is negative, there are no terms in Σ_2, and Σ_3 is to be interpreted as over $0<n<y+\tau$. We observe that $y-\tau$ can be negative only if $y<A$, when the ranges of Σ_3 and Σ_4 are bounded by A's.

2.7. We next consider S_3. Using (2.27), we obtain

$$S_3 = O\Big(\sum_{y-\tau}^{y+\tau} n^{\sigma-1}(nx)^{-\sigma}\, t^{\frac{1}{2}}\Big) = O(x^{-\sigma}y^{-1}t^{\frac{1}{2}}\tau).$$

If $x \geqslant y$, $\tau = 1$, this is $O(x^{-\sigma}y^{-1}t^{\frac{1}{2}})$ or $O(y^{\sigma-1}t^{\frac{1}{2}-\sigma})$, while if $x \leqslant y$, $\tau = Ayt^{-\frac{1}{2}}$, it is $O(x^{-\sigma})$. Hence, in any case,

$$S_3 = O(x^{-\sigma}) + O(y^{\sigma-1}t^{\frac{1}{2}-\sigma}). \tag{2.71}$$

It remains to consider S_2 and S_4. In S_4 we have $t + 2\tau\pi x < 2n\pi x \leqslant Ct$. Since $\tau \geqslant 1$, $x > A$, these are inequalities of the type which govern (2.25). Substituting from (2.25) into S_4, we obtain

$$S_4 = S_{4,1} + S_{4,2} + S_{4,3}, \tag{2.72}$$

where

$$S_{4,1} = 2(2\pi)^{s-1} \sum_{y+\tau < n \leqslant Cy} \frac{in^{s-1}(2n\pi x)^{1-s}e^{2n\pi ix}}{4\pi x(n-y)} = \frac{ix^{-s}}{2\pi} \sum_{y+\tau < n \leqslant Cy} \frac{e^{2n\pi ix}}{n-y}, \tag{2.73}$$

$$S_{4,2} = O\Big(\sum_{y+\tau < n \leqslant Cy} n^{\sigma-1}(nx)^{-\sigma}\Big) = O\Big(x^{-\sigma} \sum_{y \leqslant n \leqslant Cy} \frac{1}{n}\Big) = O(x^{-\sigma}), \tag{2.74}$$

$$S_{4,3} = O\Big(\sum_{y+\tau < n \leqslant Cy} n^{\sigma-1}\frac{t(nx)^{1-\sigma}}{x^3(n-y)^3}\Big) = O\Big(tx^{-2-\sigma} \sum_{y+\tau \leqslant n} \frac{1}{(n-y)^3}\Big). \tag{2.75}$$

If $x \geqslant y$, $\tau = 1$, the last sum is $O(tx^{-2-\sigma}) = O(x^{-\sigma})$, and if $x \leqslant y$, $\tau = Ayt^{-\frac{1}{2}}$, it is

$$O(tx^{-2-\sigma}\tau^{-2}) = O(x^{-\sigma}t^2x^{-2}y^{-2}) = O(x^{-\sigma});$$

so that in any case

$$S_4 = S_{4,1} + O(x^{-\sigma}). \tag{2.76}$$

2.8. Similarly, if we substitute from (2.24) in S_2, we obtain

$$S_2 = S_{2,1} + S_{2,2} + S_{2,3} + S_{2,4}, \tag{2.81}$$

where

$$S_{2,1} = 2(2\pi)^{s-1}\Gamma(1-s)\sin\tfrac{1}{2}s\pi \sum_{By \leqslant n < y-\tau} n^{s-1}, \tag{2.82}$$

$$S_{2,2} = 2(2\pi)^{s-1} \sum_{By \leqslant n < y-\tau} \frac{in^{s-1}(2n\pi x)^{2-s}e^{2n\pi ix}}{4\pi xt(n-y)} = \frac{ix^{1-s}}{t} \sum_{By \leqslant n < y-\tau} \frac{ne^{2n\pi ix}}{n-y}, \tag{2.83}$$

$$S_{2,3} = O\Big(\sum_{By \leqslant n < y-\tau} n^{\sigma-1}\frac{(nx)^{1-\sigma}}{t}\Big) = O\Big(\frac{x^{1-\sigma}y}{t}\Big) = O(x^{-\sigma}), \tag{2.84}$$

$$S_{2,4} = O\Big(\sum_{By \leqslant n < y-\tau} n^{\sigma-1}\frac{(nx)^{2-\sigma}}{x^3(y-n)^3}\Big) = O\Big(yx^{-\sigma-1} \sum_{n < y-\tau} \frac{1}{(y-n)^3}\Big). \tag{2.85}$$

If $x \geqslant y$, $\tau = 1$, $S_{2,4}$ is $O(yx^{-\sigma-1}) = O(x^{-\sigma})$, and if $x \leqslant y$, $\tau = Ayt^{-1}$, it is $O(yx^{-\sigma-1} \cdot y^{-2}t) = O(x^{-\sigma})$. Hence, in any case,

$$S_2 = S_{2,1} + S_{2,2} + O(x^{-\sigma}). \tag{2.86}$$

Collecting our results from (2.62), (2.63), (2.64), (2.65), (2.71), (2.76), and (2.86), we obtain

$$\zeta(s) - \sum_{n \leqslant x} n^{-s} - \chi \sum_{n \leqslant y-\tau} n^{s-1} = S_{2,2} + S_{4,1} + O(x^{-\sigma}) + O(y^{\sigma-1}t^{\frac{1}{2}-\sigma}). \tag{2.87}$$

2.9. The error involved in replacing $y-\tau$ by y on the left-hand side of (2.87) is $O(t^{\frac{1}{2}-\sigma}y^{\sigma-1}\tau)$, which is $O(t^{\frac{1}{2}-\sigma}y^{\sigma-1})$ when $x \geqslant y$, and $O(t^{-\sigma}y^{\sigma})$ or $O(x^{-\sigma})$ when $x \leqslant y$. We may therefore make this alteration.

Finally, by (2.73) and (2.83), we have

$$\begin{aligned} S_{2,2} + S_{4,1} &= \frac{ix^{1-s}}{t} \sum_{By \leqslant n < y-\tau} \frac{ne^{2n\pi ix}}{n-y} + \frac{ix^{-s}}{2\pi} \sum_{y+\tau < n \leqslant Cy} \frac{e^{2n\pi ix}}{n-y} \\ &= \frac{ix^{-s}}{2\pi}\left(\sum_{By \leqslant n < y-\tau} \frac{e^{2n\pi ix}}{y} + \sum_{By \leqslant n < y-\tau} \frac{e^{2n\pi ix}}{n-y} + \sum_{y+\tau < n \leqslant Cy} \frac{e^{2n\pi ix}}{n-y} \right). \end{aligned} \tag{2.91}$$

In the two sums last written we may replace the denominators $n-y$ by $n-[y]$, $[y]$ being the greatest integer contained in y, with a total error not exceeding

$$\sum_{n < y-\tau} O\left(\frac{1}{(n-y)^2}\right) + \sum_{y+\tau < n} O\left(\frac{1}{(n-y)^2}\right) = O(1)$$

in the sums themselves. If now we write $n = [y]+m$, the ranges of m in these sums are respectively

$$-(1-B)y \leqslant m < -\tau, \quad \tau < m \leqslant (1-B)y,$$

each modified by addition or omission of at most $O(1)$ terms at the ends, and the contributions of these terms to the sums are each $O(1)$. Hence, from (2.91),

$$S_{2,2} + S_{4,1} = O(x^{-\sigma}) + \frac{ix^{-s}}{2\pi} e^{2[y]\pi ix} \sum_{\tau < m \leqslant (1-B)y} \left(\frac{e^{-2m\pi ix}}{-m} + \frac{e^{2m\pi ix}}{m} \right).$$

The last sum is

$$2i \sum \frac{\sin 2m\pi x}{m},$$

and, since this is boundedly convergent, we obtain

$$S_{2,2} + S_{4,1} = O(x^{-\sigma}).$$

Substituting in (2.87), modified as above, we obtain (2.13).

3. *The equation for* $\zeta^2(s)$.

3.1. In what follows we restrict ourselves, for the sake of simplicity of expression, to the half-plane $t>0$. The corresponding results for negative t may be written down at once.

We write

$$X(s)=\chi^2(s)=4(2\pi)^{2s-2}\sin^2\tfrac{1}{2}s\pi\Gamma^2(1-s), \tag{3.11}$$

and define τ by

$$\log\tau=-\frac{X'(\frac{1}{2}+it)}{X(\frac{1}{2}+it)}. \tag{3.12}$$

It may be verified at once, from well known properties of the gamma-function, that $X(s)=O(t^{1-2\sigma})$, that

$$\psi''(s)=\frac{d^2}{ds^2}\{\log X(s)\}=O\left(\frac{1}{t}\right) \tag{3.13}$$

uniformly for $-A\leqslant\sigma\leqslant A$, $t\geqslant A$, that τ is real and that

$$\tau=\left(\frac{t}{2\pi}\right)^2+O(1). \tag{3.14}$$

THEOREM 2. *If*

$$-\tfrac{1}{2}\leqslant\sigma\leqslant\tfrac{3}{2},\quad x>A,\quad y>A,\quad xy=\left(\frac{t}{2\pi}\right)^2, \tag{3.15}$$

then

$$\zeta^2(s)=\sum_{n\leqslant x}\frac{d(n)}{n^s}+X(s)\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}}+O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{t}\right)^{\frac{1}{2}}\log t\right\}. \tag{3.16}$$

$d(n)$ being the number of divisors of n.

Since $\operatorname{Max}(x,y)>At$, the expression under the O is at least of the order $x^{\frac{1}{2}-\sigma}\log t$. Since $d(n)<n^{o(1)}$, the error committed by adding or omitting $O(1)$ terms at the ends of the sums in (3.16) is certainly $O(x^{\frac{1}{2}-\sigma})$ or $O(t^{1-2\sigma}y^{\sigma-\frac{1}{2}})=O(x^{\frac{1}{2}-\sigma})$ respectively; we may therefore ignore the effect of such changes. In particular, we may suppose x, y to be connected by $xy=\tau$ instead of $xy=(t/2\pi)^2$; for when y (say) is fixed the change is a change from x to x_1 in which $|x_1-x|<A$, $Ax<x_1<Ax$.

We may suppose further that $\frac{1}{2}\leqslant\sigma\leqslant\frac{3}{2}$: if Theorem 2 is proved in this case it may be extended to the strip $-\frac{1}{2}\leqslant\sigma\leqslant\frac{1}{2}$ as in § 2.6. Thus we are now supposing that

$$\tfrac{1}{2}\leqslant\sigma\leqslant\tfrac{3}{2},\quad x>A,\quad y>A,\quad xy=\tau. \tag{3.17}$$

3.2. LEMMA δ. *If the summation is defined by* $1 \leqslant m \leqslant \mu$, $1 \leqslant n \leqslant \mu$, $m \neq n$, *then*

$$\Sigma \frac{(mn)^{-\frac{1}{2}}}{|\log(n/m)|} = O(\mu \log \mu).$$

This is Lemma 5 of our paper **1**.

LEMMA ϵ. *If* $AT^{\frac{1}{2}} \leqslant H \leqslant AT$, *then*

$$\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2 dt = O(H \log T). \tag{3.21}$$

This may be deduced from more comprehensive results due to Littlewood*. We prove here what we actually require.

Taking $\sigma = \frac{1}{2}$, $x = y = z = (2\pi)^{-\frac{1}{2}} t^{\frac{1}{2}}$ in Theorem 1, we have

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<z} n^{-\frac{1}{2}-it} + O(1) \sum_{n<z} n^{-\frac{1}{2}+it} + O(t^{-\frac{1}{4}}),$$

and so

$$|\zeta(\tfrac{1}{2}+it)|^2 < A\,|\sum_{n<z} n^{-\frac{1}{2}-it}|^2 + A = A\Omega + A, \tag{3.22}$$

say. Now

$$\int_T^{T+H} \Omega dt = \int_T^{T+H} \left(\sum_{m,\, n<z} m^{-\frac{1}{2}-it} n^{-\frac{1}{2}+it} \right) dt = \sum_{m,\, n<z_1} \int_{t_{m,n}}^{T+H} m^{-\frac{1}{2}-it} n^{-\frac{1}{2}+it}\, dt,$$

where $$z_1 = \left(\frac{T+H}{2\pi}\right)^{\frac{1}{2}}, \qquad t_{m,n} = \mathrm{Max}(T,\, 2\pi m^2,\, 2\pi n^2).$$

Hence

$$\begin{aligned} \int_T^{T+H} \Omega dt &= \sum_{m=n} + \sum_{m \neq n} \qquad (3.23) \\ &= \sum_{n<z_1} \frac{T+H-t_{n,n}}{n} + \sum_{m,\, n<z_1,\ m \neq n} (mn)^{-\frac{1}{2}} \int_{t_{m,n}}^{T+H} \left(\frac{n}{m}\right)^{it} dt \\ &= O(H \log T) + O\left(\sum_{m,\, n<z_1,\ m \neq n} \frac{(mn)^{-\frac{1}{2}}}{|\log(n/m)|} \right) \\ &= O(H \log T) + O(z_1 \log z_1) = O(H \log T) + O(T^{\frac{1}{2}} \log T) \\ &= O(H \log T), \end{aligned}$$

by Lemma δ. The result of the lemma follows from (3.22) and (3.23).

* Littlewood, **4**.

3.3. LEMMA ζ. *Suppose that*

$$s=\sigma+it, \quad \tfrac{1}{2}\leqslant\sigma\leqslant\tfrac{3}{2}, \quad t>A, \quad u=\omega+iv, \tag{3.31}$$

and

$$\Phi(u)=\frac{\phi(u)}{(u-s)^2}=\frac{\tau^{u-s}X(u)-X(s)}{(u-s)^2}. \tag{3.32}$$

Then $\Phi(u)$ *is regular for* $\omega<\operatorname{Min}(\sigma, 1)$, *and also for* $\omega\leqslant\frac{1}{2}$, *and*

$$\phi(u)=O\left\{t^{1-2\sigma}\operatorname{Min}\left(1, \frac{|s-u|^2}{t}\right)\right\}, \tag{3.33}$$

when $\omega=\frac{1}{2}$, *uniformly in* s.

(i) The assertions concerning regularity are obvious except when $\sigma=\frac{1}{2}$ and $u=s$, and it is only necessary to verify that

$$\frac{d}{du}\{\tau^{u-s}X(u)\}=0$$

in this case. The function to be differentiated is

$$\{1+(u-s)\log\tau+\ldots\}\{X(s)+(u-s)X'(s)+\ldots\},$$

and the coefficient of $u-s$ vanishes in virtue of the definition of τ.

(ii) If $u=\frac{1}{2}+iv$, we have

$$X(u)=O(1), \quad \tau^{u-s}X(u)=O(t^{1-2\sigma}), \quad X(s)=O(t^{1-2\sigma}), \quad \phi(u)=O(t^{1-2\sigma}).$$

This is what we have to prove when $|s-u|^2>t$, so that we may suppose that $|s-u|^2\leqslant t$.

We write $s_1=\frac{1}{2}+it$ and

$$\phi(u)=\{\tau^{s_1-s}X(s_1)-X(s)\}+\{\tau^{u-s}X(u)-\tau^{s_1-s}X(s_1)\}=\phi_1+\phi_2. \tag{3.34}$$

Then
$$|\phi_1|=|X(s)|\left|1-\frac{\tau^{s_1-s}X(s_1)}{X(s)}\right|=O(t^{1-2\sigma})|1-e^E|,$$

where
$$\begin{aligned}E&=(s_1-s)\log\tau+\log X(s_1)-\log X(s)\\&=\psi(\tfrac{1}{2}+it)-\psi(\sigma+it)-(\tfrac{1}{2}-\sigma)\psi'(\tfrac{1}{2}+it)^*\\&=-\int_{\frac{1}{2}}^{\sigma}d\lambda\int_{\frac{1}{2}}^{\lambda}\psi''(\mu+it)\,d\mu=O\left(\frac{(\sigma-\frac{1}{2})^2}{t}\right)=O\left(\frac{|s-u|^2}{t}\right),\end{aligned}$$

by (3.13). This is $O(1)$, and so

$$\phi_1=O(t^{1-2\sigma}E)=O(t^{-2\sigma}|s-u|^2). \tag{3.35}$$

* See (3.13) for the definition of ψ.

Next, $$|\phi_2| = |\tau^{s_1-s} X(s_1)| \left|1-\frac{\tau^{\gamma-s_1} X(u)}{X(s_1)}\right| = O(t^{1-2\sigma})\,|1-e^{E'}|,$$

where
$$\begin{aligned} E' &= (u-s_1)\log\tau + \log X(u) - \log X(s_1) \\ &= \psi(\tfrac{1}{2}+iv) - \psi(\tfrac{1}{2}+it) - i(v-t)\,\psi'(\tfrac{1}{2}+it) \\ &= -\int_t^v d\lambda \int_t^\lambda \psi''(\tfrac{1}{2}+i\mu)\,d\mu = O\left(\frac{(v-t)^2}{t}\right) = O\left(\frac{|s-u|^2}{t}\right), \end{aligned}$$

as before. Hence ϕ_2 satisfies an equation similar to (3 . 35), so that (3 . 33) follows from (3 . 34).

3 . 4. Proceeding to the proof of Theorem 2, we write $f(s) = \zeta^2(s)$ and $z = \zeta + iw$. We have then (by the well known formula of Hadamard and Perron)

$$\frac{1}{2\pi i}\int_{2-\sigma} f(s+z)\frac{x^z}{z^2}\,dz = \sum_{n\leqslant x}\frac{d(n)}{n^s}\log\frac{x}{n}, \tag{3 . 41}$$

the subscript to the sign of integration connoting integration along the line $\zeta = 2-\sigma$. We choose γ so that

$$0 < \gamma < \tfrac{3}{4}, \quad \sigma - 1 < \gamma, \quad \gamma \neq \sigma, \quad \gamma \neq \sigma - \tfrac{1}{2}$$

(as is plainly possible when $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$), and we move the contour to the left to $\zeta = -\gamma$, crossing the poles at $z = 1-s$ and $z = 0$. The least possible value of $\sigma+\zeta$ exceeds $\frac{1}{2}-\frac{3}{4} = -\frac{1}{4}$, and $f(-\frac{1}{4}+iw)$ is $O(|w|^{\frac{3}{2}})$ and *a fortiori* is $o(|w|^2)$. We may therefore use Cauchy's theorem to obtain

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{-\gamma} f(s+z)\frac{x^z}{z^2}\,dz \\ &= \sum_{n\leqslant x}\frac{d(n)}{n^s}\log\frac{x}{n} - \{f(s)\log x + f'(s)\} - \frac{x^{1-s}}{(1-s)^2}\left(2C + \log x - \frac{2}{1-s}\right) \\ &= S(x,s) - F(x,s) - R(x,s) = S_x - F_x - R_x, \end{aligned} \tag{3 . 42}$$

C being Euler's constant.

If now we write $u = \omega + iv$ for $s+z$, and use the functional equation, we obtain

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{\sigma-\gamma} f(u)\frac{x^{u-s}}{(u-s)^2}\,du = \frac{1}{2\pi i}\int_{\sigma-\gamma} f(1-u)\,X(u)\left(\frac{\tau}{y}\right)^{u-s}\frac{du}{(u-s)^2} \\ &= \frac{X(s)}{2\pi i}\int_{\sigma-\gamma} f(1-u)\frac{y^{s-u}}{(u-s)^2}\,du + \frac{1}{2\pi i}\int_{\sigma-\gamma} f(1-u)\,y^{s-u}\Phi(u)\,du \\ &= X(s)\,J_1 + J_2, \end{aligned} \tag{3 . 43}$$

say; assuming that (as will appear in the sequel) these integrals are separately convergent. Comparing (3.42) and (3.43), we have

$$F_x - S_x = -X(s)\,J_1 - J_2 - R_x. \tag{3.44}$$

3.5. If $\sigma < \gamma$, the line $\omega = \sigma - \gamma$ is in the half-plane of absolute convergence of $f(1-u)$. In this case

$$J_1 = \sum_{n \leq y} \frac{d(n)}{n^{1-s}} \log \frac{y}{n} = S(y, 1-s) = S_y, \tag{3.51}$$

say. If $\gamma < \sigma$, we move the path of integration just across the pole $u = 0$ into the half-plane of absolute convergence, say to the line $\omega = -\delta$. Since $1-\sigma+\gamma > 0$, we have $f(1-u) = O(|v|)$, for large $|v|$, in the part of the plane involved, and Cauchy's theorem gives

$$J_1 = S_y - \frac{y^s}{s^2}\left(2C + \log y - \frac{2}{s}\right) = S_y - R(1-s, y) = S_y - R_y, \tag{3.52}$$

say. Thus

$$J_1 = S_y - \epsilon R_y, \tag{3.53}$$

where $\epsilon = 0$ if $\gamma > \sigma$ and $\epsilon = 1$ if $\gamma < \sigma$.

The other integral J_2 we move to the line $\omega = \frac{1}{2}$. It is easily verified that the integrand (considered as a function of u only) is $o(1)$ uniformly in the distant part of the strip concerned. In fact, if $\sigma - \gamma < \frac{1}{2}$, we have

$$f(1-u) = O(|v|^{\frac{1}{2}}), \quad \phi(u) = O(|v|^{1-2\sigma+2\gamma}),$$

$$f(1-u)\,\phi(u) = O(|v|^{\frac{3}{2}-2\sigma+2\gamma}) = O(|v|^{\frac{1}{2}+2\gamma}) = o(|v|^2);$$

while, if $\sigma - \gamma > \frac{1}{2}$*, we have

$$f(1-u) = O(|v|^{\sigma-\gamma}) = O(|v|), \quad \phi(u) = O(1), \quad f(1-u)\,\phi(u) = o(|v|^2).$$

Applying Cauchy's theorem, recalling Lemma ξ on the regularity of Φ, and noting that the pole at $u = 0$ is relevant when $\sigma < \gamma$, we obtain

$$J_2 = \frac{1}{2\pi i}\int_{\frac{1}{2}} f(1-u)\,y^{s-u}\,\Phi(u)\,du - \epsilon' X(s)\,R_y = J - \epsilon' X(s)\,R_y, \tag{3.54}$$

where $\epsilon' = 1$ if $\gamma > \sigma$ and $\epsilon' = 0$ if $\gamma < \sigma$, so that $\epsilon + \epsilon' = 1$ in either case. Collecting our results from (3.44), (3.53), and (3.54), we obtain

$$F_x - S_x + X(s)\,S_y = -R_x + X(s)\,R - J. \tag{3.55}$$

* We chose γ so that $\sigma - \gamma \neq \frac{1}{2}$.

3.6. We have

$$(3.61)\quad R_x = \frac{x^{1-s}}{(1-s)^2}\left(2C+\log x-\frac{2}{1-s}\right) = O\left(\frac{x^{1-\sigma}}{t^2}\log t\right)$$
$$= O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}\log t),$$

$$(3.62)\quad X(s)R_y = O(t^{1-2\sigma})\frac{y^s}{s^2}\left(2C+\log y-\frac{2}{s}\right)$$
$$= O\left(t^{1-2\sigma}\frac{y^\sigma}{t^2}\log t\right) = O\left(\frac{x^{-\sigma}}{t}\log t\right) = O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}\log t)$$

(in each case with a good deal to spare). We shall now establish the same upper bound for J.

Replacing the integrand in J by its modulus, and using Lemma ζ (3.33), we obtain

$$(3.63)\quad J = O\left(y^{\sigma-\frac{1}{2}}\int_{-\infty}^{\infty}|f(\tfrac{1}{2}-iv)|\frac{|\phi(u)|}{|u-s|^2}\,dv\right)$$
$$= O\left(y^{\sigma-\frac{1}{2}}t^{-2\sigma}\int_{t-t^{\frac{1}{2}}}^{t+t^{\frac{1}{2}}}|f(\tfrac{1}{2}-iv)|\,dv\right)$$
$$+O\left(y^{\sigma-\frac{1}{2}}t^{1-2\sigma}\int_{-\infty}^{t-t^{\frac{1}{2}}}+\int_{t+t^{\frac{1}{2}}}^{\infty}\frac{|f(\tfrac{1}{2}-iv)|}{|u-s|^2}\,dv\right) = O(J_a)+O(J_b),$$

say. By Lemma ϵ,

$$(3.64)\qquad J_a = O(y^{\sigma-\frac{1}{2}}t^{-2\sigma}.t^{\frac{1}{2}}\log t) = O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}\log t);$$

but J_b requires a more elaborate analysis.

Since $y^{\sigma-\frac{1}{2}}t^{1-2\sigma} = O(x^{\frac{1}{2}-\sigma})$, we have

$$(3.65)\qquad J_b = O\left\{x^{\frac{1}{2}-\sigma}\left(\int_{-\infty}^{-\frac{1}{2}t}+\int_{-\frac{1}{2}t}^{\frac{1}{2}t}+\int_{\frac{1}{2}t}^{t-t^{\frac{1}{2}}}+\int_{t+t^{\frac{1}{2}}}^{2t}+\int_{2t}^{\infty}\right)\right\}$$
$$= O(j_1+j_2+j_3+j_4+j_5),$$

say. Here

$$(3.66)\quad j_1 = O\left(x^{\frac{1}{2}-\sigma}\int_{-\infty}^{-\frac{1}{2}t}\frac{|f(\frac{1}{2}-iv)|}{v^2}\,dv\right) = O\left(x^{\frac{1}{2}-\sigma}\int_{\frac{1}{2}t}^{\infty}\frac{dv}{v^2}\right) = O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}),$$

and similarly

$$(3.67)\qquad j_5 = O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}).$$

Also

$$(3.68)\quad j_2 = O\left(\frac{x^{\frac{1}{2}-\sigma}}{t^2}\int_{-\frac{1}{2}t}^{\frac{1}{2}t}|f(\tfrac{1}{2}-iv)|\,dv\right) = O\left(\frac{x^{\frac{1}{2}-\sigma}\log t}{t}\right) = O(x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}).$$

Finally

$$j_3+j_4 = O\left(x^{\frac{1}{2}-\sigma}\sum_{-At^{\frac{1}{2}}}^{At^{\frac{1}{2}}}{}'\int_{t+mt^{\frac{1}{2}}}^{t+(m+1)t^{\frac{1}{2}}}\frac{|f(\frac{1}{2}-iv)|}{(v-t)^2}\,dv\right),$$

where the dash expresses the omission of the terms $m = 0$ and $m = -1$. Hence

$$(3.69)\quad j_3+j_4 = O\left(\frac{x^{\frac{1}{2}-\sigma}}{t}\sum_{-At^{\frac{1}{3}}}'^{At^{\frac{1}{3}}}\frac{1}{m^2}\int_{t+mt^{\frac{1}{3}}}^{t+(m+1)t^{\frac{1}{3}}}|f(\tfrac{1}{2}-iv)|\,dv\right)$$

$$= O\left(\frac{x^{\frac{1}{2}-\sigma}}{t}\,t^{\frac{1}{3}}\log t\sum_{-\infty}'^{\infty}\frac{1}{m^2}\right) = O(x^{\frac{1}{2}-\sigma}\,t^{-\frac{1}{3}}\log t).$$

3.7. It follows from (3.55) and the results of the last section that

$$(3.71)\qquad F_x - S_x + X(s)\,S_y = O(x^{\frac{1}{2}-\sigma}\,t^{-\frac{1}{2}}\log t).$$

From this we deduce (3.16) by a familiar device.

We write $x_1 = xe^{\delta}$, $y_1 = ye^{-\delta}$ (where δ will be small and positive), replace x and y by x_1 and y_1 in (3.71), and subtract the results. We have

$$(3.72)\qquad F_{x_1} - F_x = \delta f(s),$$

$$(3.73)\qquad S_{x_1} - S_x = \delta\sum_{n<x}\frac{d(n)}{n^s} + \sum_{x\leqq n<x_1}\frac{d(n)}{n^s}\log\frac{x_1}{n}$$

$$= \delta\sum_{n<x}\frac{d(n)}{n^s} + O\left(\delta x^{-\sigma}\sum_{x\leqq n<x_1}d(n)\right)$$

$$= \delta\sum_{n<x}\frac{d(n)}{n^s} + O\left[\delta x^{-\sigma}\{(x_1-x)\log x + x^{\frac{1}{2}}\}\right],$$

since

$$\sum_{n<\xi}d(n) = \xi\log\xi + (2C-1)\,\xi + O(\xi^{\frac{1}{2}});$$

and similarly

$$(3.74)\quad -X(s)(S_{y_1}-S_y) = \delta X(s)\sum_{n<y}\frac{d(n)}{n^{1-s}}$$

$$+ O\left[\delta t^{1-2\sigma}\,y^{\sigma-1}\{(y-y_1)\log y + y^{\frac{1}{2}}\}\right].$$

Since $x_1 - x = O(\delta x)$, $y_1 - y = O(\delta y)$ when δ is small, it follows that

$$(3.75)\qquad f(s) - \sum_{n<x}\frac{d(n)}{n^s} - X(s)\sum_{n<y}\frac{d(n)}{n^{1-s}} = O(M),$$

where

$$M = \mathrm{Max}\,(\delta^{-1}x^{\frac{1}{2}-\sigma}t^{-\frac{1}{2}}\log t,\ \delta x^{1-\sigma}\log x,\ x^{\frac{1}{2}-\sigma},\ \delta t^{1-2\sigma}y^{\sigma-1}\log y,\ t^{1-2\sigma}y^{\sigma-\frac{1}{2}}).$$

If $x \geqq y$, we take $\delta = (xt)^{-\frac{1}{4}}$, when each function in M is

$$O(x^{\frac{1}{2}-\sigma}\log t\,.\,x^{\frac{1}{4}}t^{-\frac{1}{4}}).$$

Similarly, if $y \geqq x$, we take $\delta = (yt)^{-\frac{1}{4}}$. In either case we obtain an error term of the type which stands in (3.16). The terms, if any, for

which $n = x$ or $n = y$ are individually of this form, and it is immaterial whether they are included or not. This completes the proof.

3·8. The most interesting case is that in which x and y are each of order t, when the error is $O(t^{\frac{1}{2}-\sigma}\log t)$. In particular, if $\sigma = \frac{1}{2}$ and $x = y$, the error is $O(\log t)$. In this case (stating the result finally for either sign of t) we obtain

$$\zeta^2(\tfrac{1}{2}+it) = \sum_{n<t/2\pi} n^{-\frac{1}{2}-it}d(n) + \left(\frac{|t|}{2\pi e}\right)^{-2it} e^{\frac{1}{4}\pi i \operatorname{sgn} t} \sum_{n<t/2\pi} n^{-\frac{1}{2}+it}d(n) + O(\log|t|).$$

This is the result used by Ingham.

References.

1. G. H, Hardy and J. E. Littlewood, "The zeros of Riemann's Zeta-function on the critical line", *Math. Zeitschrift*, 10 (1921), 283–317.
2. —— "The approximate functional equation in the theory of the Zeta-function, with applications to the divisor problems of Dirichlet and Piltz", *Proc. London Math. Soc.* (2), 21 (1923), 39–74.
3. A. E. Ingham, "Mean value theorems in the theory of the Riemann Zeta-function", *Proc. London Math. Soc.* (2), 27 (1926), 273–300.
4. J. E. Littlewood, "Researches in the theory of the Riemann ζ-function", *Proc. London Math Soc.* (2), 20 (1922), xxii–xxviii (*Records* for 10 Feb., 1921).
5. E. C. Titchmarsh, "The mean-value of the Zeta-function on the critical line", *Proc. London Math. Soc.* (2), 27 (1927), 137–150.

ON DENSITY DISTRIBUTION IN STELLAR SPACE

ECHO D. PEPPER.

[Received 25 May, 1928.—Read 14 June, 1928.]

1. The formal problem treated in this paper is the following. We have a fixed volume containing n particles. The position of any particle is independent of the position of all other particles, and the probability function for a certain particle to occupy any position in the given space, *i.e.* the probability for a particle to be in a given part of the volume, is equal to the ratio of the given part of the volume to the whole volume. We require to find the "density" distribution in the given n volume.

Since we do not have a continuous medium in our volume, the problem acquires a definite meaning only after the notion of "density" has been strictly defined. It is obvious that the notion of *density at a point* cannot have a meaning for the given problem, and we have to consider "mean densities". This can be done by dividing the whole volume into equal partial volumes and by characterizing the density by the number of particles in each of the partial volumes. If the number of partial volumes k is very small in comparison with the number of particles n, then the density will be nearly constant in all partial volumes in the sense that, in almost all volumes, the density will be nearly equal to n/k. On the other hand, if the number of partial volumes is very great in comparison with n, then in almost all partial volumes there will be either one particle or none. This density would not really characterize the distribution of the particles in the space.

We define density by dividing the given volume into n equal partial volumes, i.e. we put the number of partial volumes equal to the number of particles.

2. There are two physical problems which correspond closely to our problem.

(i) The problem of the distribution of the stars in stellar space.

(ii) The problem of the density distribution of gases.

In both of these problems (perhaps with a greater certainty in the first one) the mutual action between the "elements" (stars in the first problem, particles of gas in the second one) is weak, and we may agree that the position of an element is practically independent of the position of all other elements; and, on account of that, when we pass from the physical problem to the formal problem we may agree that the position of each element is defined "by chance".

3. We consider two questions with respect to our problem.

(i) Find the probable maximum density.

(ii) Find the probable number of volumes at which the density is equal to a given number.

At first sight one is inclined to think that the most probable distribution is a uniform one, *i.e.* one to which corresponds one particle in each of the volumes. But, in fact, this is not the case. If we enumerate the partial volumes, and denote by $k_1, k_2, \ldots, k_n$ the number of particles in each of the volumes, then the uniform distribution, *i.e.*

$$k_1 = k_2 = \ldots = k_n = 1$$

is more probable than any other "individual" non-uniform distribution. But the uniform distribution is much less probable than a distribution to which corresponds one empty partial volume, another volume with two particles, and all other volumes with one particle, because to the second distribution correspond $n(n-1)$ different "individual" systems of values of $k_1, k_2, \ldots, k_n$. This is evident by taking any one of k_i equal to zero and any other equal to two. Thus the probability that the maximum density is equal to two is greater than the probability that it is equal to one.

4. Define the number of all different distributions of n particles in n partial volumes, so that the first volume contains k_1 particles, the second k_2, and so on. It is possible to choose k_1 particles for the first volume in $\frac{n(n-1)\ldots(n-k_1+1)}{1.2\ldots k_1}$ different ways. When k_1 particles have been chosen for the first volume, it will be possible to choose, from the $n-k_1$ remaining particles, k_2 particles for the second partial volume in $\frac{(n-k_1)(n-k_1-1)\ldots(n-k_1-k_2+1)}{1.2\ldots k_2}$ different ways, and so on. From this we conclude that the number of all possible distributions of n

particles into n partial volumes, so that the first partial volume contains k_1 particles, the second k_2 particles, and so on, is equal to

$$\frac{n!}{k_1!\,k_2!\ldots k_n!}, \quad k_1+k_2+\ldots+k_n=n, \tag{1}$$

where $k_i!=1$ for $k_i=0$.

Take the product

$$n!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)$$
$$\ldots\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right),$$

the factor in brackets appearing n times. The number (1) is equal to the coefficient of the term of this product corresponding to the (k_1+1)-th term of the first factor, (k_2+1)-th term of the second factor, and so on. The number of all distributions of particles for which no partial volume contains k particles or more, is equal to the sum

$$\Sigma\frac{n!}{k_1!\,k_2!\ldots k_n!}$$

extended over all values of $k_1, k_2, \ldots, k_n$ satisfying the conditions

$$k_1+k_2+k_3+\ldots+k_n=n$$
$$k_1<k,\ k_2<k,\ \ldots,\ k_n<k.$$

It is easily seen that this number is equal to the coefficient of x in the product

$$n!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{k-1}}{(k-1)!}\right)^n.$$

The number of all possible distributions of n particles into n volumes is obviously equal to n^n, so that the probability of the above distribution is equal to the coefficient of x^n in the product

$$\frac{n!}{n^n}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^{k-1}}{(k-1)!}\right)^n,$$

and this, by Cauchy's theorem, is equal to the value of the integral

$$\frac{n!}{n^n}\,\frac{1}{2\pi i}\int_{|z|=1}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}\right)^n\frac{dz}{z^{n+1}}=P(k) \tag{2}$$

taken around the circle $|z|=1$ in the positive direction.

Our problem is to find an asymptotic value of (2) for large values of n. We consider (2) only for values of k satisfying the condition

$$n^{1-\epsilon} < k! < n^{1+\epsilon}, \tag{3}$$

where ϵ is a small fixed positive number. Divide the integral (2) into two integrals

$$\begin{aligned} J &= \frac{1}{2\pi i}\int_{|z|=1}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}\right)^n \frac{dz}{z^{n+1}} \\ &= \frac{1}{2\pi i}\int_{-n^{-\frac{1}{2}+\eta}<\phi<n^{-\frac{1}{2}+\eta}}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}\right)^n \frac{dz}{z^{n+1}} \\ &\quad + \frac{1}{2\pi i}\int_{n^{-\frac{1}{2}+\eta}<\phi<2\pi-n^{-\frac{1}{2}+\eta}}\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}\right)^n \frac{dz}{z^{n+1}} \\ &= J_1+J_2 \end{aligned} \tag{4}$$

(η being a small fixed positive number less than $\frac{1}{6}$), where the first integral is extended over the values of $z=e^{i\phi}$ for which

$$-n^{-\frac{1}{2}+\eta} < \phi < n^{-\frac{1}{2}+\eta}$$

and the second over the other values of z. We have for $|z|=1$

$$\begin{aligned} \left|\frac{z^k}{k!}+\frac{z^{k+1}}{(k+1)!}+\ldots\right| &\leqslant \frac{1}{k!}+\frac{1}{(k+1)!}+\ldots, \\ &< \frac{1}{k!}\left(1+\frac{1}{k+1}+\frac{1}{(k+1)^2}+\ldots\right) \\ &< \frac{k+1}{k}\cdot\frac{1}{k!}. \end{aligned} \tag{5}$$

From (3) and (5)

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!} = e^z+O(n^{-1+\epsilon}),$$

and

$$\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}\right)^n = e^{nz}\{1+O(n^{-1+\epsilon})\}^n = e^{nz+O(n^\epsilon)}.$$

We have

$$\begin{aligned} |J_2| &< \frac{1}{2\pi}\int_{n^{-\frac{1}{2}+\eta}}^{2\pi-n^{-\frac{1}{2}+\eta}} e^{n\cos\phi+O(n^\epsilon)}d\phi \\ &< \exp\{n\cos(n^{-\frac{1}{2}+\eta})+O(n^\epsilon)\} \\ &< \exp\{n-\tfrac{1}{2}n^{2\eta}+O(n^\epsilon)\}. \end{aligned}$$

Putting

$$\epsilon < \eta, \tag{5.1}$$

we have, for large values of n,

$$|J_2| < e^{n-\frac{1}{3}n} . \tag{6}$$

We pass now to the calculation of J_1. We have

$$J_1 = \frac{1}{2\pi}\int_{-n^{-\frac{1}{2}+\eta}}^{+n^{-\frac{1}{2}+\eta}} \left\{1+\cos\phi+\frac{\cos 2\phi}{2!}+\ldots+\frac{\cos(k-1)\phi}{(k-1)!} + i\left(\frac{\sin\phi}{1}+\frac{\sin 2\phi}{2!}+\ldots+\frac{\sin(k-1)\phi}{(k-1)!}\right)\right\} e^{-ni\phi}\,d\phi, \tag{7}$$

$$1+\cos\phi+\frac{\cos 2\phi}{2!}+\ldots+\frac{\cos(k-1)\phi}{(k-1)!} = 1+1+\frac{1}{2!}+\ldots+\frac{1}{(k-1)!} - \frac{1}{2}\phi^2\left(1^2+\frac{2^2}{2!}+\frac{3^2}{3!}+\ldots+\frac{(k-1)^2}{(k-1)!}\right)+O(\phi^4). \tag{8}$$

In the range of integration

$$\phi = O(n^{-\frac{1}{2}+\eta}), \tag{9.1}$$

$$1+\frac{1}{1}+\frac{1}{2!}+\ldots+\frac{1}{(k-1)!} = e-e\theta(k), \tag{10}$$

where

$$e\theta(k) = \frac{1}{k!}+\frac{1}{(k+1)!}+\ldots = O\left(\frac{1}{k!}\right),$$

$$\frac{1}{1}+\frac{2}{2!}+\frac{3}{3!}+\ldots+\frac{k-1}{(k-1)!} = e+O\left(\frac{1}{(k-1)!}\right), \tag{11}$$

$$\frac{1^2}{1}+\frac{2^2}{2!}+\frac{3^2}{3!}+\ldots+\frac{(k-1)^2}{(k-1)!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(k-2)!}+1+1+\frac{1}{2!}+\ldots+\frac{1}{(k-3)!}, \tag{12}$$

$$= 2e+O\left(\frac{1}{(k-2)!}\right).$$

By (3)

$$k = O(\log n),$$

consequently

$$\frac{1}{(k-1)!} = \frac{k}{k!} = O\left(\frac{\log n}{n^{1-\epsilon}}\right),$$

$$\frac{1}{(k-2)!} = \frac{k(k-1)}{k!} = O\left(\frac{(\log n)^2}{n^{1-\epsilon}}\right).$$

Because of (8), ..., (12) we have, for values of ϕ included in the range

of integration,

$$(13)\quad 1+\cos\phi+\frac{\cos 2\phi}{2!}+\ldots+\frac{\cos(k-1)\phi}{(k-1)!} +i\left(\frac{\sin\phi}{1}+\frac{\sin 2\phi}{2!}+\ldots+\frac{\sin(k-1)\phi}{(k-1)!}\right)$$
$$= e-e\theta(k)-\phi^2 e+i\phi e+O(\phi^4)+O(\phi^3) +O\left(\phi^2\frac{1}{(k-2)!}\right)+O\left(\phi\frac{1}{(k-1)!}\right).$$

We have $\phi^4 = O(n^{-2+4\eta}),\quad \phi^3 = O(n^{-\frac{3}{2}+3\eta}),$

$$\frac{\phi^2}{(k-2)!} = O(n^{-2+2\eta+(\log n)^2}),$$

$$\frac{\phi}{(k-1)!} = O(n^{-\frac{3}{2}+\eta+\epsilon\log n}).$$

and thus, by (5.1),

$$(14)\qquad O(\phi^4)+O(\phi^3)+O\left(\frac{\phi^2}{(k-2)!}\right)+O\left(\frac{\phi}{(k-1)!}\right) = O(n^{-\frac{3}{2}+3\eta}).$$

By (13), (14)

$$1+\cos\phi+\frac{\cos 2\phi}{2!}+\ldots+\frac{\cos(k-1)\phi}{(k-)!}+i\left(\frac{\sin\phi}{1}+\frac{\sin 2\phi}{2!}+\ldots+\frac{\sin(k-1)\phi}{(k-1)!}\right)$$
$$= e\left\{1-\theta(k)-\phi^2+i\phi+O(n^{-\frac{3}{2}+3\eta})\right\}.$$

We have further

$$\log\left\{1-\theta(k)-\phi^2+i\phi+O(n^{-\frac{3}{2}+3\eta})\right\} = -\theta(k)-\tfrac{1}{2}\phi^2+i\phi+O(n^{-\frac{3}{2}+3\eta}),$$
$$1-\theta(k)-\phi^2+i\phi+O(n^{-\frac{3}{2}+3\eta}) = e^{-\theta(k)}-\tfrac{1}{2}\phi^2+i\phi+O(n^{-\frac{3}{2}+3\eta}),$$

and finally

$$(15)\qquad J_1 = \frac{1}{2\pi}\int_{-n^{-\frac{1}{2}+\eta}}^{+n^{-\frac{1}{2}+\eta}} e^{n-\theta(k)n}e^{-n\phi^2/2}e^{O(n^{-\frac{1}{2}+3\eta})}d\phi$$
$$= \frac{1}{2\pi}\,e^{O(n^{-\frac{1}{2}+3\eta})}e^{n-\theta(k)n}\sqrt{\left(\frac{2}{n}\right)}\left(\int_{-\infty}^{+\infty}e^{-t^2}dt+O(e^{-n^{2\eta}})\right)$$
$$= \frac{1}{\sqrt{(2\pi n)}}\,e^{n-\theta(k)n}e^{O(n^{-\frac{1}{2}+3\eta})}.$$

Since $\theta(k)n = O(n^\epsilon)$, we conclude, from (6) and (15), that

$$(16)\qquad J \equiv J_1+J_2 = \frac{1}{\sqrt{(2\pi n)}}\,e^{n-\theta(k)n}e^{O(n^{-\frac{1}{2}+3\eta})},$$

and thus we have

$$P(k) = \frac{n!}{n^n} \cdot \frac{1}{\sqrt{(2\pi n)}}\, e^{n-\theta(k)n}\, e^{O(n^{-\frac{1}{2}+\delta_n})},$$

and, by Stirling's formula,

$$P(k) = e^{-\theta(k)n}\, e^{O(n^{-\frac{1}{2}+\delta_n})}. \tag{17}$$

This is the value of the required probability, viz. that the maximum density is less than k. Consider this expression for certain values of k. We have

$$\theta(k) = \frac{1}{e}\left(\frac{1}{k!} + \frac{1}{(k+1)!} + \ldots\right) = \frac{1}{k!\,e}\{1+O(k^{-1})\}. \tag{17.1}$$

Consider values of k satisfying the inequality

$$(\log n)^{-1\cdot 4} < \frac{n}{k!\,e} < (\log n)^{+1\cdot 4}. \tag{18}$$

We have

$$-2\log\log n < \log n - k\log k + k < 2\log\log n,$$

from which we conclude that

$$k = \{1+o(1)\}\,\frac{\log n}{\log\log n},$$

where $o(1)$ denotes a number which tends to zero as n tends to infinity. Then we see that the inequality (18) is satisfied by two or by three consecutive values of k, and for the largest of them

$$(\log n)^{-1\cdot 4} < \frac{n}{k!\,e} < (\log n)^{-0\cdot 4}, \tag{19}$$

and for the smallest

$$(\log n)^{0\cdot 4} < \frac{n}{k!\,e} < (\log n)^{1\cdot 4}. \tag{20}$$

If k' and k'' are respectively the smallest and largest values of k, we deduce from (17.1) that, when $k = k'$, $\theta(k)n$ has a large value, and when $k = k''$, $\theta(k)n$ has a small value; and this shows that $P(k')$ is small and $P(k'')$ is near to 1.

Thus the probability that all $k_1, k_2, \ldots, k_n$ are less than k'' is near to 1, and the probability that all of them are less than k' is near to zero, *i.e. it is almost certain that all* $k_1, k_2, \ldots, k_n$ *are less than* k'', *and that some of them are not less than* k'.

Now the numbers k' and k'' are either two consecutive numbers or their difference is 2. Thus we can state nearly with certainty that *in the first case the greatest density is* k' *and in the second case it is either* k' *or*

$k'+1$. The value k' may be defined as the number for which $n/k!e$ passes from large values to small values.

It is remarkable that in a problem which seems to deal with quite indefinite matter we have arrived at so definite a result.

5. We pass now to the second problem of § 3. Let $P(l, m)$ denote the probability that m out of n partial volumes contain l particles, and none of the others contains l particles. Consider the product

$$\frac{n!}{n^n}\Pi_1\Pi_2\dots\Pi_n, \tag{21}$$

where
$$\Pi_i = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots \quad (i = 1, 2, \dots, n).$$

We take $n!/n$ and one term from each Π_i and then form their product and consider (21) as the sum of all such products. If we take the term of power k_1 from Π_1, of power k_2 from Π_2, and so on, then the coefficient of this term will give the probability that the first volume contains k_1 particles, the second contains k_2 particles, and so on. The probability that $k_{i_1} = k_{i_2} = \dots = k_{i_m} = l$ and that all other $k_{j_1}, k_{j_2}, \dots, k_{j_{n-m}}$ have definite values different from l, is given by the coefficient of the term of (21) corresponding to taking the term of power l from $\Pi_{i_1}, \Pi_{i_2}, \dots, \Pi_{i_m}$, and of power k_{j_1} from Π_{j_1}, of power k_{j_2} from Π_{j_2}, and so on. Now the probability that $k_{i_1} = k_{i_2} = \dots = k_{i_m} = l$ and that all other $k_{j_1}, k_{j_2}, \dots, k_{j_{n-m}}$ have arbitrary values different from l, is equal to the sum of the coefficients of the terms formed in the above manner. We can obtain all these terms by taking always $x^l/l!$ from $\Pi_{i_1}, \Pi_{i_2}, \dots, \Pi_{i_m}$ and any term different from $x^l/l!$ from each of $\Pi_{j_1}, \Pi_{j_2}, \dots, \Pi_{j_{n-m}}$, *i.e.* any term of the series

$$1+\frac{x}{1!}+\dots+\frac{x^{l-1}}{(l-1)!}+\frac{x^{l+1}}{(l+1)!}+\frac{x^{l+2}}{(l+2)!}+\dots = e^x-\frac{x^l}{l!}.$$

Thus the above probability is equal to the coefficient of x^n in the product

$$\frac{n!}{n^n}\left(\frac{x^l}{l!}\right)^m\left(e^x-\frac{x^l}{l!}\right)^{n-m}. \tag{22}$$

Form this probability corresponding to each system $\Pi_{i_1}, \Pi_{i_2}, \dots, \Pi_{i_m}$, and take the sum of all these probabilities, then we shall have the value of $P(l, m)$. But (22) shows that the probability is the same for every system $\Pi_{i_1}, \Pi_{i_2}, \dots, \Pi_{i_m}$. Since the number of all possible systems $\Pi_{i_1}, \Pi_{i_2}, \dots, \Pi_{i_m}$ is obviously $\frac{n!}{m!\,(n-m)!}$, we have $P(l, m)$ equal to the

coefficient of x^n in the product

$$\frac{n!}{m!\,(n-m)!}\,\frac{n!}{n^n}\left(\frac{x^l}{l!}\right)^m\left(e^x-\frac{x^l}{l!}\right)^{n-m}. \tag{23}$$

By Cauchy's theorem

$$P(l,m)=\frac{n!}{m!\,(n-m)!}\,\frac{n!}{n^n}\,\frac{1}{(l!)^m}\,\frac{1}{2\pi i}\int_{|z|=1}\left(e^z-\frac{z^l}{l!}\right)^{n-m}z^{-n-1+ml}\,dz, \tag{23.1}$$

the integral being taken around the circle $|z|=1$ in the positive direction. Write

$$J=\frac{1}{2\pi i}\int_{|z|=1}\left(e^z-\frac{z^l}{l!}\right)^{n-m}z^{-n-1+ml}\,dz \tag{24}$$

$$=\frac{1}{2\pi i}\int_{-n^{-\frac{1}{2}+\eta}<\phi<n^{-\frac{1}{2}+\eta}}+\frac{1}{2\pi i}\int_{n^{-\frac{1}{2}+\eta}<\phi<2\pi-n^{-\frac{1}{2}+\eta}},$$

in which J_1 is extended over those values of $z=e^{i\phi}$ which satisfy the condition

$$-n^{-\frac{1}{2}+\eta}<\phi<n^{-\frac{1}{2}+\eta}$$

(η being a fixed positive number less than $\frac{1}{6}$), and J_2 is extended over the other values of z.

We shall first compute $|J_2|$. In our problem we have to consider l as constant and n as a large number.

Consider

$$\left|1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^{l-1}}{(l-1)!}+\frac{z^{l+1}}{(l+1)!}+\ldots\right|=\left|e^z-\frac{z^l}{l!}\right| \tag{25}$$

for $$n^{-\frac{1}{2}+\eta}<\phi<2\pi-n^{-\frac{1}{2}+\eta}.$$

Series (25) contains at least one of the two pairs of terms

$$1+z,\quad \frac{z^2}{2!}+\frac{z^3}{3!}, \tag{26}$$

for, if l is different from 0 to 1, the series contains the first pair; if l is equal to one of these numbers, the series contains the second pair of terms. It is easily seen that the modulus of each of these pairs is a decreasing function of ϕ in the interval $0\leqq\phi\leqq\pi$, and thus, in the range of integration of J_2, we have

$$|1+z|<|1+z|_{\phi=n^{-\frac{1}{2}+\eta}},$$

$$\left|\frac{z^2}{2!}+\frac{z^3}{3!}\right|=\tfrac{1}{2}\left|1+\frac{z}{3}\right|<\tfrac{1}{2}\left|1+\frac{z}{3}\right|_{\phi=n^{-\frac{1}{2}+\eta}},$$

$$|1+z|^2_{\phi=n^{-\frac{1}{2}+\eta}}=(1+\cos n^{-\frac{1}{2}+\eta})^2+\sin^2 n^{-\frac{1}{2}+\eta}$$

$$=2+2\cos n^{-\frac{1}{2}+\eta}<4-\tfrac{1}{2}n^{-1+2\eta},$$

since $\cos\epsilon < 1-\frac{1}{4}\epsilon^2$ for small ϵ. Hence

$$|1+z|_{\phi=n^{-\frac{1}{2}+\eta}} < 2-\tfrac{1}{8}n^{-1+2\eta}.$$

In the same way

$$\left|1+\frac{z}{3}\right|_{\phi=n^{-\frac{1}{2}+\eta}} > \tfrac{4}{3}-\tfrac{1}{16}n^{-1+2\eta},$$

and thus, in the whole range of integration of J_2, we have

$$|1+z| < |1|+|z|-\tfrac{1}{16}n^{-1+2\eta},$$

$$\left|\frac{z^2}{2!}+\frac{z^3}{3!}\right| < \left|\frac{z^2}{2!}\right|+\left|\frac{z^3}{3!}\right|-\tfrac{1}{16}n^{-1+2\eta}.$$

We have shown that the series (25) contains at least one of the pairs (26), say, the first one. We have

$$\begin{aligned}(27)\quad &\left|1+\frac{z}{1!}+\ldots+\frac{z^{l+1}}{(l+1)!}+\frac{z^{l+2}}{(l+2)!}+\ldots\right| \\ &\leqslant |1+z|+\ldots+\left|\frac{z^{l+1}}{(l+1)!}\right|+\left|\frac{z^{l+2}}{(l+2)!}\right|+\ldots \\ &< 1+|z|+\ldots+\left|\frac{z^{l+1}}{(l+1)!}\right|+\left|\frac{z^{l+2}}{(l+2)!}\right|+\ldots-\tfrac{1}{16}n^{-1+2\eta} \\ &< 1+1+\ldots+\frac{1}{(l+1)!}+\frac{1}{(l+2)!}+\ldots-\tfrac{1}{16}n^{-1+2\eta} \\ &< e-\frac{1}{l!}-\tfrac{1}{16}n^{-1+2\eta},\end{aligned}$$

from which we conclude that

$$\begin{aligned}(28)\quad |J_2| &< \left(e-\frac{1}{l!}-\tfrac{1}{16}n^{-1+2\eta}\right)^{n-m} \\ &< \left(e-\frac{1}{l!}\right)^{n-m}\exp\left(-\frac{(n-m)n^{-1+2\eta}}{16(e-1/l!)}\right).\end{aligned}$$

Pass now to the calculation of J_1. In the range of integration of J_1 we have

$$(29)\qquad \phi = O(n^{-\frac{1}{2}+\eta}).$$

We now make some transformations in J_1, where

$$J_1 = \frac{1}{2\pi i}\int_{-n^{-\frac{1}{2}+\eta}}^{+n^{-\frac{1}{2}+\eta}}\left(e^z-\frac{z^l}{l!}\right)^{n-m} z^{-n-1+ml}\,dz.$$

We have

$$z = e^{i\phi} = 1+i\phi-\tfrac{1}{2}\phi^2+O(\phi^3),$$

$$e^z-\frac{z^l}{l!} = e^{1+i\phi-\frac{1}{2}\phi^2}-\frac{e^{li\phi}}{l!}+O(\phi^3) = e(1+i\phi-\phi^2)-\frac{1}{l!}(1+li\phi-\tfrac{1}{2}l^2\phi^2)+O(\phi^3)$$

$$= \left(e-\frac{1}{l!}\right)+\left\{i\left(e-\frac{l}{l!}\right)\phi-\left(e-\tfrac{1}{2}\frac{l^2}{l!}\right)\phi^2+O(\phi^3)\right\}.$$

We have, further,

$$\log\left(e^z-\frac{z^l}{l!}\right)=\log\left(e-\frac{1}{l!}\right)+\left\{i\,\frac{(e-l/l!)\phi}{e-1/l!}-\frac{(e-\frac{1}{2}l^2/l!)\phi^2}{e-1/l!}\right.$$
$$\left.+\frac{(e-l/l!)^2\phi^2}{2(e-1/l!)^2}\right\}+O(\phi^3)$$
$$=\log\left(e-\frac{1}{l!}\right)+\left\{i\,\frac{(e-l/l!)\phi}{e-1/l!}\right.$$
$$\left.-\frac{\{e^2-(2e-2el+el^2)/l!\}\phi^2}{2(e-1/l!)^2}\right\}+O(\phi^3).$$

Now we have

$$\left(e^z-\frac{z^l}{l!}\right)^{n-m}z^{-n+ml}$$
$$=\left(e-\frac{1}{l!}\right)^{n-m}\exp\left[i\,\frac{(n-m)(e-l/l!)-(n-ml)(e-1/l!)}{e-1/l!)^2}\phi\right.$$
$$\left.-\frac{(n-m)\{e^2-(2e-2el+el^2)/l!\}}{2(e-1/l!)^2}\phi^2+O(\phi^3)\right].$$

Thus we have, to a sufficient approximation,

$$\Re(J_1)=\frac{1}{2\pi}\left(e-\frac{1}{l!}\right)^{n-m}\int_{-n^{-\frac{1}{2}+\eta}}^{+n^{-\frac{1}{2}+\eta}}e^{-a\phi^2}\cos b\phi\,d\phi, \tag{30}$$

where $\Re(J_1)$ is the real part of J_1, and

$$a=\frac{\{e^2-(2e-2el+l^2e)/l!\}(n-m)}{2(e-1/l!)^2}$$
$$b=\frac{(l-1)(me-n/l!)}{e-1/l!},$$

Since

$$\int_{\phi_0}^{\infty}e^{-a\phi^2}d\phi<\frac{e^{-a\phi_0^2}}{2a\phi_0},$$

if we assume that

$$n-m>\beta n, \tag{30.1}$$

where β is a fixed positive number ($\beta>0$), we see that

$$\int_{-n^{-\frac{1}{2}+\eta}}^{+n^{-\frac{1}{2}+\eta}}e^{-a\phi^2}\cos b\phi\,d\phi=\int_{-\infty}^{\infty}e^{-a\phi^2}\cos b\phi\,d\phi+O(e^{-an^{-1+2\eta}})$$
$$\sqrt{\pi}\,.\,e^{-\frac{1}{4}b^2/a}+O(e^{-an^{-1+2\eta}}). \tag{31}$$

Let a be a positive number such that

$$an^{-1+2\eta}>an^{2\eta} \tag{32}$$

and

$$\frac{(n-m)n^{-1+2\eta}}{16(e-1/l!)}>an^{2\eta}. \tag{33}$$

We have, from (30), (31), and (32),

$$\mathfrak{R}(J_1) = \frac{1}{2\sqrt{\pi}} \left(e - \frac{1}{l!}\right)^{n-m} \left\{e^{-\frac{1}{4}h^2/\alpha} + O(e^{-\alpha n^{2\epsilon}})\right\}.$$

By (28) and (33),

$$\mathfrak{R}(J_1+J_2) = \frac{1}{2\sqrt{\pi}} \left(e - \frac{1}{l!}\right)^{n-m} \left\{e^{-\frac{1}{4}h^2/\alpha} + O(e^{-\alpha n^{2\epsilon}})\right\}.$$

Since $J_1+J_2 = J$, J being a real number, we have

$$J = \frac{1}{2\sqrt{\pi}} \left(e - \frac{1}{l!}\right)^{n-m} \left\{e^{-\frac{1}{4}h^2/\alpha} + O(e^{-\alpha n^{2\epsilon}})\right\}. \tag{34}$$

From (23 . 1), we have

$$P(l, m) = \frac{1}{2\sqrt{\pi}} \frac{n!}{n^n} \left\{\frac{n!}{m!\,(n-m)!} \left(\frac{1}{l!}\right)^m \left(e - \frac{1}{l!}\right)^{n-m}\right\} \times \left\{e^{-\frac{1}{4}h^2/\alpha} + O(e^{-\alpha n^{2\epsilon}})\right\}. \tag{35}$$

Formula (35) has been deduced with the assumption (30 . 1). If (30 . 1) is not satisfied, then it is easily seen that the following formula will hold :

$$P(l, m) = \frac{1}{2\sqrt{\pi}} \frac{n!}{n^n} \left\{\frac{n!}{m!\,(n-m)!} \left(\frac{1}{l!}\right)^m \left(e - \frac{1}{l!}\right)^{n-m}\right\} \left\{e^{-\frac{1}{4}h^2/\alpha} + O(1)\right\}. \tag{36}$$

Obviously (36) holds also for values of m satisfying (30 . 1). Formula (36) may be replaced by

$$P(l, m) = \frac{1}{2\sqrt{\pi}} \frac{n!\,e^n}{n^n} \left\{\frac{n!}{m!\,(n-m)!} \left(\frac{1}{e\,.\,l!}\right)^m \left(1 - \frac{1}{e\,.\,l!}\right)^{n-m}\right\} O(1) \tag{36 . 1}$$

$$= \sqrt{n} \left\{\frac{n!}{m!\,(n-m)!} \left(\frac{1}{e\,.\,l!}\right)^m \left(1 - \frac{1}{e\,.\,l!}\right)^{n-m}\right\} O(1).$$

Take

$$m_1 = \frac{e\,.\,l_1}{n} - z\sqrt{\left[2n \frac{1}{e\,.\,l!} \left(1 - \frac{1}{e\,.\,l!}\right)\right]}$$

and

$$m_2 = \frac{n}{e\,.\,l!} + z\sqrt{\left[2n \frac{1}{e\,.\,l!} \left(1 - \frac{1}{e\,.\,l!}\right)\right]},$$

where $z = n^\epsilon$ $(0 < \epsilon < \frac{1}{6})$.

By a well known deduction from the binomial theorem, we have

$$\sum_{m < m_1} \frac{n!}{m!\,(n-m)!} \left(\frac{1}{e\,.\,l!}\right)^m \left(1 - \frac{1}{e\,.\,l!}\right)^{n-m} = O(e^{-n^{2\epsilon}}) \tag{37}$$

and

$$\sum_{m > m_2} \frac{n!}{m!\,(n-m)!} \left(\frac{1}{e\,.\,l!}\right)^m \left(1 - \frac{1}{e\,.\,l!}\right)^{n-m} = O(e^{-n^{2\epsilon}}). \tag{38}$$

From (36 . 1), (37), (38),

$$\sum_{m<m_1} P(l, m) + \sum_{m>m_2} P(l, m) = O(\sqrt{n} \, . \, e^{-n^{2\epsilon}}). \tag{39}$$

Obviously

$$\sum_{m=1}^{m \cdot \lceil n/l \rceil} P(l, m) = 1, \tag{40}$$

because the summation is extended over all possible values of m. From (39) and (40),

$$\sum_{m_1}^{m_2} P(l, m) = 1 - O(\sqrt{n} \, . \, e^{-n^{2\epsilon}}). \tag{41}$$

Equation (41) expresses our final result, which may be interpreted as follows :

We can state almost with certainty that the number of partial volumes with density l may be represented by

$$\frac{n}{e \, . \, l!} + O(n^{\frac{1}{2}+\epsilon}).$$

Thus we have arrived at the following very elegant result :

The number of partial volumes with density 0, 1, 2, 3, ... *is given approximately by the numbers*

$$\frac{n}{e}, \frac{n}{e \, . \, 1}, \frac{n}{e \, . \, 2!}, \frac{n}{e \, . \, 3!}, \cdots$$

and the maximal density is given by the value of l for which

$$\frac{n}{e \, . \, l!}$$

passes from large values to small values.

THE BEHAVIOUR OF CERTAIN SERIES ASSOCIATED WITH LIMITING CASES OF ELLIPTIC THETA-FUNCTIONS: II

By R. COOPER.

[Received 14 February, 1928.—Read 16 February, 1928.]

1. In a previous paper* I have investigated some properties of the six functions defined at least for $\sigma > 1$ by the series

$$F_1(s, \theta) = \sum_1' e^{\pi i \theta n^2} n^{-s}; \qquad F_2(s, \theta) = \sum_1^\infty (-1)^n e^{\pi i \theta n^2} n^{-s};$$

$$F_3(s, \theta) = \sum_1^\infty e^{-\pi i \theta n^2} n^{-s}; \qquad F_4(s, \theta) = \sum_1^\infty (-1)^n e^{-\pi i \theta n^2} n^{-s};$$

$$F_5(s, \theta) = \sum_0' e^{\pi i \theta (n+\frac{1}{2})^2} (n+\tfrac{1}{2})^{-s}; \quad F_6(s, \theta) = \sum_0^\infty e^{-\pi i \theta (n+\frac{1}{2})^2} (n+\tfrac{1}{2})^{-s};$$

where $0 < \theta < 1$ and $s = \sigma + it$. Transformation formulae connecting $F_\alpha(s, \theta)$ with $F_\beta(s, \theta_1)$, where $\theta_1 = \theta^{-1} - [\theta^{-1}]$†, were found and were applied when θ was rational‡ or a quadratic irrational§ to examine the analytic character of the functions. In the latter case it was shown that the functions are meromorphic with a double infinity of possible simple poles whose position depends on both the non-recurring and the recurring part of the simple continued fraction for θ, being in this last respect unlike

$$\Sigma' \{n\theta\}^{-s}, \quad (\{n\theta\} = n\theta - [n\theta] - \tfrac{1}{2}),$$

the position of whose poles depends only on the recurring part. The object of this paper is to examine the behaviour of $F_\alpha(s, \theta)$ when θ is a non-quadratic irrational. It will appear that in certain cases F_α has

* 1.

† $[x]$ means the greatest integer not less than x.

‡ 1, § 6, (425).

§ 1, Theorem 6, (418).

a singular line parallel to the imaginary axis and is regular in the half-plane to the right of this line*. Irrationals are divided into classes as follows. Let

$$\theta = \frac{1}{a_1+}\,\frac{1}{a_2+}\ldots; \quad \theta^{-1} = a_1+\theta_1, \quad \theta_1^{-1} = a_2+\theta_2, \quad \ldots.$$

Then θ is said to be of class λ if λ is the lower bound of the numbers μ (if such exist) for which

$$(\theta\theta_1 \ldots \theta_{n-1})^{\mu}\,\theta_n^{-1} \to 0$$

as $n \to \infty$ †. If no such numbers μ exist, the class λ is ∞. Hardy and Littlewood have shown‡ that, if λ is not zero or infinity, $\Sigma\{n\theta\}x^{-s}$ has $\sigma = \lambda/(1+\lambda)$ as a singular line, on the right of which it is regular. The behaviour of the functions $F_a(s, \theta)$ is similar but more complicated, and it will appear later that a finer classification of irrationals than the one above is necessary. Also the term in the transformation formula, on which the work depends, corresponds to a term of no importance in the transformation formula for $\Sigma\{n\theta\}n^{-s}$.

2. The transformation formula for $F_a(s, \theta)$ may be written

$$(2.1) \quad F_a(s, \theta) = I_a(s, \theta)+e^{\frac{1}{2}\delta\pi i}\,\theta^{s-\frac{1}{2}}\,F_{a_1}(s, \theta_1)+\theta^{s+\frac{1}{2}}R_0(s, -1)+I'_a+I''_a,$$

where we retain the notation of the previous paper§; δ is one of 0, 1, 2, ..., 7‖. The suffixes are attached to preserve the identities of the functions. The expansion has not been carried to as many terms as in the preceding paper, and I', I'' are the terms corresponding to (2.134) and (2.136) there. We have then

$$I_1(s, \theta) = -\int_{L(\frac{1}{2})} \frac{e^{\pi i\theta z^2}\,z^{-s}}{e^{2\pi iz}-1}\,dz, \quad I_3(s, \theta) = -2\int_{L'(\frac{1}{2})} \frac{e^{-\pi i\theta z^2}\,z^{-s}}{e^{2\pi iz}-1}\,dz, \quad \text{etc.}$$

Repeated applications of (2.1) give

$$F_a(s, \theta) = \sum_{n=0}^{m} e^{\frac{1}{2}\delta\pi i}(\theta\theta_1 \ldots \theta_{n-1})^{s-\frac{1}{2}}\{I_{a_n}(s, \theta_n)+I'_{a_n}+I''_{a_n}\}$$
$$+e^{\frac{1}{2}\delta\pi i}(\theta\theta_1 \ldots \theta_{m-1})^{s-\frac{1}{2}}F_{a_{m+1}}(s, \theta_{m+1})+\sum_{n=0}^{m}(\theta\theta_1 \ldots \theta_n)^{s-\frac{1}{2}}\theta_n\,R_n(s, -1),$$

* The precise statement is in the theorem of § 6.

† This definition is given in **2**. The class so defined differs by unity from that used in **3**.

‡ **2**.

§ **1**.

‖ δ does not always represent the same number.

where $\theta_0 = \theta$, $a_0 = a$ and an empty bracket ($n = 0$) denotes 1. For $\sigma > 2$, the functions $|F_a(s, \theta)|$ have a finite upper bound independent of θ. Letting $m \to \infty$, we find, for $\sigma > 2$,

$$F_a(s, \theta) = \sum_0^{\infty} e^{\frac{1}{4}\delta\pi i}(\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} \{I_{a_n}(s, \theta_n) + I'_{a_n} + I''_{a_n} + \theta_n^{s+\frac{1}{2}} R_n(s, -1)\},$$

and so, by analytic continuation,

$$(2.2) \quad F_a(s, \theta) = \sum_0^{\infty} e^{\frac{1}{4}\delta\pi i}(\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} I_{a_n}(s, \theta_n) + \sum_0^{\infty} e^{\frac{1}{4}\delta\pi i}(\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} \{I'_{a_n} + I''_{a_n} + \theta_n^{s+\frac{1}{2}} R_n(s, -1)\},$$

throughout any region in which $F_a(s, \theta)$ and the sum functions of the series exist and are regular, provided that this region is a connected domain overlapping $\sigma > 2$.

It is sufficient to consider those terms I'_{a_n}, I''_{a_n} which arise from F_1. We have

$$|I'_1(s, \theta)| \leqslant \theta^{\sigma}(2/\theta)^{\frac{1}{2}} \sum_1^{\infty} e^{-\frac{1}{8}\pi\nu^2/\theta} \leqslant 2^{\frac{1}{2}}\theta^{\sigma-\frac{1}{2}} \sum_1^{\infty} e^{-\frac{1}{8}\pi\nu^2},$$

$$|I''_1(s, \theta)| \leqslant A(s)\,\theta^{\sigma-\frac{1}{2}} 2^{\frac{1}{2}} \sum_1^{\infty} e^{-\frac{1}{8}\pi\nu^2/\theta}*.$$

The functions $R_n(s, -1)$ are bounded in any half-plane $\sigma \geqslant -1+\epsilon (\epsilon > 0)$ uniformly with respect to θ. Hence the second series in (2.2) represents a function regular for $\sigma > \frac{1}{2}$, and so

$$(2.3) \qquad F_a(s, \theta) = \sum_0^{\infty} e^{\frac{1}{4}\delta\pi i}(\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} I_{a_n}(s, \theta_n) + R(s, \tfrac{1}{2}),$$

subject to the same conditions as (2.2). It remains to consider the functions $I_{a_n}(s, \theta_n)$.

3. We have

$$(3.1) \qquad I_1(s, \theta) = \int_{L(0)} \frac{(z+\frac{1}{2})^{-s} \exp\{\pi i\theta(z+\frac{1}{2})^2\}}{e^{2\pi iz}+1}\, dz$$

$$= \tfrac{1}{2} \int_{L(\frac{1}{2})} z^{-s} e^{\pi i\theta z^2}\, dz - \tfrac{1}{2} i \int_{L(0)} (z+\tfrac{1}{2})^{-s} \tan \pi z \exp\{\pi i\theta(z+\tfrac{1}{2})^2\}\, dz.$$

* $A(s)$ denotes here and elsewhere a positive constant, not always the same, depending on the bounded closed region in which s varies.

Now, for real values of x,

$$\tan(x+ix) = \pm i + O(e^{-2|x|})$$

according as $x \to \pm\infty$. Therefore

$$\left| \int_{a\sqrt{i}}^{\infty\sqrt{i}} (\tan \pi z - i)(z+\tfrac{1}{2})^{-s} \exp\{\pi i\theta(z+\tfrac{1}{2})^2\}\, dz \right|$$

$$= O(e^{-2\pi a}) \int_{|z|=a}^{\infty} |(z+\tfrac{1}{2})^{-s} \exp\{\pi i\theta(z+\tfrac{1}{2})^2\}|\, |dz|,$$

and

$$\left| \int_{-a\sqrt{i}}^{-\infty\sqrt{i}} (\tan \pi z + i)(z+\tfrac{1}{2})^{-s} \exp\{\pi i\theta(z+\tfrac{1}{2})^2\}\, dz \right|$$

$$= O(e^{-2\pi a}) \int_{|z|=a}^{\infty} |(\tfrac{1}{2}-z)^{-s} \exp\{\pi i\theta(\tfrac{1}{2}-z)^2\}|\, |dz|,$$

where the integrals on the left-hand sides are taken along portions of $L(0)$, and a is a number greater than 1, which will be defined more precisely later by making it depend on θ in such a way as to make certain error terms as small as possible. Applying these results to (3.1) we find, for $\sigma < 1$,

$$(3.2) \quad \left| I_1(s,\theta) - \int_{a\sqrt{i}}^{\infty\sqrt{i}} z^{-s} e^{\pi i\theta z^2}\, dz \right|$$

$$\leqslant A(s)\, a^{1-\sigma} + O(e^{-2\pi a}) \int_a^{\infty} \{\phi(y) + \phi(-y)\}\, dy,$$

where $$\phi(y) = |(\tfrac{1}{2} + yi^{\frac{1}{2}})^{-s} \exp\{\pi i\theta(\tfrac{1}{2} + yi^{\frac{1}{2}})^2\}|.$$

When $y > 0$, writing η for $2^{-\frac{3}{2}}$, we have

$$\phi(\pm y) \leqslant A(s)(y \pm \eta)^{-\sigma} e^{-\pi\theta(y\pm\eta)^2}.$$

Therefore, if $0 < \sigma < 1$,

$$\int_a^{\infty} \phi(y)\, dy \leqslant A(s) \int_{a+\eta}^{\infty} e^{-\pi\theta y^2} y^{-\sigma}\, dy = A(s)\left\{\int_{a+\eta}^{b} + \int_b^{\infty}\right\}$$

$$\leqslant A(s)\, b^{1-\sigma} + A(s)\, b^{-\sigma} \int_b^{\infty} e^{-\pi\theta y^2}\, dy$$

$$\leqslant A(s)\, b^{1-\sigma} + A(s)\, b^{-\sigma} \theta^{-\frac{1}{2}}.$$

Taking $b = (a+\eta)\theta^{-\frac{1}{2}}$, we find

$$(3.3) \quad \int_a^{\infty} \phi(y)\, dy \leqslant \{A(s)(a+\eta)^{1-\sigma} + A(s)(a+\eta)^{-\sigma}\}\, \theta^{\frac{1}{2}(\sigma-1)}$$

$$\leqslant A(s)(a+\eta)^{1-\sigma}\, \theta^{\frac{1}{2}(\sigma-1)} \leqslant A(s)\, a^{1-\sigma}\, \theta^{\frac{1}{2}(\sigma-1)},$$

with a similar result for the other part of the integral on the right-hand side of (3.2).

Next, for $0<\sigma<1$,

$$\int_{a\sqrt{i}}^{\infty\sqrt{i}} e^{\pi i\theta z^2} z^{-s}\,dz = \int_0^{\infty\sqrt{i}} e^{\pi i\theta z^2} z^{-s}\,dz + O\{A(s)\,a^{1-\sigma}\}; \tag{3.4}$$

and

$$\int_0^{\infty\sqrt{i}} e^{\pi i\theta z^2} z^{-s}\,dz = \theta^{\frac{1}{2}(s-1)} \int_0^{\infty\sqrt{i}} e^{\pi i z^2} z^{-s}\,dz = \theta^{\frac{1}{2}(s-1)} J(s), \quad \text{say,} \tag{3.5}$$

where the integrals are taken along parts of $L(0)$. $J(s)$ is regular at least in the strip $0<\sigma<1$.

Combining (3.2) ... (3.5) we find, for $0<\sigma<1$,

$$|I_1(s,\theta) - \theta^{\frac{1}{2}(s-1)} J(s)| < A(s)\,a^{1-\sigma} + A(s)\,e^{-2\pi a} a^{1-\sigma}\,\theta^{\frac{1}{2}(\sigma-1)}.$$

Now fix a by taking it so that

$$2\pi a = 2\pi - \log\theta.$$

Then

$$|I_1(s,\theta) - \theta^{\frac{1}{2}(s-1)} J(s)| < A(s)\left(1-\frac{1}{2\pi}\log\theta\right)^{1-\sigma}\{1+e^{-2\pi}\theta^{\frac{1}{2}(\sigma+1)}\} < A(s)\left(\log\frac{2}{\theta}\right)^{1-\sigma}. \tag{3.6}$$

We also need an expression for $I_1(s,\theta)$ which is valid in a domain containing points on both sides of $\sigma=1$. Now $(e^{2\pi iz}-1)^{-1}$ is bounded on $L(\frac{1}{2})$ and tends exponentially to zero on the part of $L(\frac{1}{2})$ below the real axis. Let $c = 2^{-\frac{3}{2}}(7^{\frac{1}{2}}-1)$. Then

$$\left|\int_{L(\frac{1}{2})} \frac{e^{\pi i\theta z^2} z^{-s}}{e^{2\pi iz}-1}\,dz\right| = \left|\int_{\frac{1}{2}-\infty\sqrt{i}}^{\frac{1}{2}+c\sqrt{i}} + \int_{\frac{1}{2}+c\sqrt{i}}^{\frac{1}{2}+\infty\sqrt{i}}\right| \leqslant A(s) + C\int_{|z^{-1}-c}^{\infty} |e^{\pi i\theta(z+\frac{1}{2})^2}(z+\tfrac{1}{2})^{-s}|\,|dz|,$$

where C is a constant. The last integral is

$$\int_c^\infty \left|\frac{\exp\{\pi i\theta(\frac{1}{2}+xi^{\frac{1}{2}})^2\}}{(\frac{1}{2}+xi^{\frac{1}{2}})^s}\right| dx \leqslant e^{\frac{1}{4}\pi|t|}\int_c^\infty \frac{\exp\{-\pi\theta(x^2+2^{-\frac{1}{2}}x)\}}{(x^2+2^{-\frac{1}{2}}x+\frac{1}{4})^{\sigma/2}}\,dx \leqslant e^{\frac{1}{4}\pi|t|+\frac{1}{8}\pi\theta}\int_c^\infty \frac{\exp\{-\pi\theta(x+2^{-\frac{3}{2}})^2\}}{(x^2+2^{-\frac{1}{2}}x+\frac{1}{4})^{\sigma_1/2}}\,dx,$$

for $\sigma \geqslant \sigma_1$, since $x^2+2^{-\frac{1}{2}}x+\frac{1}{4} \geqslant 1$ for $x \geqslant c$. This expression does not exceed

$$e^{\frac{1}{4}\pi|t|+\frac{1}{8}\pi}\int_c^\infty \frac{\exp\{-\pi\theta(x+2^{-\frac{3}{2}})^2\}}{(x+2^{-\frac{3}{2}})^{\sigma_1}}\,dx = e^{\frac{1}{4}\pi|t|+\frac{1}{8}\pi}\int_{c+2^{-\frac{3}{2}}}^\infty e^{-\pi\theta x^2} x^{-\sigma_1}\,dx.$$

Applying to this integral the process which we applied to the integral on the right-hand side of (3 . 2), we find, if $0 < \sigma_1 < 1$,

$$(3 . 7) \qquad |I_1(s, \theta)| < A(s) + e^{\frac{1}{2}\pi|t|+\frac{1}{2}\pi} \frac{C\sigma_1}{1-\sigma_1} (c+2^{-\frac{1}{2}})^{1-\sigma_1} \theta^{\frac{1}{2}(\sigma_1-1)}$$

$$< A(s) + \frac{A(s)}{1-\sigma_1} \theta^{\frac{1}{2}(\sigma_1-1)} \quad (\sigma \geqslant \sigma_1).$$

The same arguments applied to $I_5(s, \theta)$ show that it also satisfies (3 . 6) and (3 . 7).

4. Applying the same arguments to $I_3(s, \theta)$ and $I_6(s, \theta)$, we find that they satisfy (3 . 7). For $0 < \sigma < 1$, however, the integral in (3 . 4) is replaced by

$$2 \int_{\alpha i \sqrt{i}}^{\infty i \sqrt{i}} e^{-\pi i \theta z^2} z^{-s} \, dz$$

taken along part of $L'(0)$, so that corresponding to (3 . 6) we find, for $\beta = 3$ or 6,

$$(4 . 1) \qquad |I_\beta(s, \theta) - 2i^{1-s} \theta^{\frac{1}{2}(s-1)} J(s)| < A(s) \left(\log \frac{2}{\theta}\right)^{1-\sigma}.$$

In the integrands of $I_2(s, \theta)$, $I_4(s, \theta)$ there is a factor $e^{\pi i z}/(e^{2\pi i z}-1)$ which tends exponentially to zero as $z \to \infty$ in each direction along the path of integration. Therefore

$$(4 . 2) \qquad |I_2(s, \theta)| < A(s), \qquad |I_4(s, \theta)| < A(s).$$

5. Applying the results (4 . 2) to (2 . 3), we see that the part of the series for which $a_n = 2$ or 4 converges uniformly for $\sigma \geqslant \frac{1}{2}+\epsilon$ ($\epsilon > 0$), and so

$$(5 . 1) \qquad F_a(s, \theta) = \sum_{n=n_j} e^{\frac{1}{4}\delta\pi i} (\theta\theta_1 \ldots \theta_{n-1})^{s-\frac{1}{2}} I_{a_n}(s, \theta_n) + R(s, \tfrac{1}{2}),$$

where n_j denotes the sequence of integers n for which $a_n = 1$, 3, 5, or 6.

Now suppose that θ is an irrational of finite class λ. Then the sum of the first n coefficients in the Dirichlet's series, by which $F_a(s, \theta)$ was originally defined, is $o(n^{k+\epsilon})$ for any positive ϵ where $k = (1+\lambda)/(2+\lambda)$*. Therefore the Dirichlet's series converges and $F_a(s, \theta)$ is regular in the half-plane $\sigma > (1+\lambda)/(2+\lambda)$.

* 3, Theorem A. See the foot-note on the definition of λ in § 1.

Also, when λ is finite,

$$\sum_0^\infty (\theta\theta_1 \dots \theta_{n-1})^{\sigma-\frac{1}{2}} \left(\log \frac{2}{\theta_n}\right)^{1-\sigma}$$

is convergent for $\sigma > \frac{1}{2}$ and uniformly convergent for $\sigma \geqslant \frac{1}{2}+\epsilon\,(\epsilon > 0)$. Further, the series

$$\sum_0^\infty a_n (\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} \theta_n^{\frac{1}{2}(s-1)}, \tag{5.2}$$

where $|a_n|$ lies between two non-zero constants, converges absolutely or diverges according as $\sigma - \frac{1}{2} + \frac{1}{2}\lambda(\sigma-1)$ is positive or negative. Its axis of convergence therefore is

$$\sigma = \frac{1+\lambda}{2+\lambda}.$$

Combining these results with (3.6), (3.7), and (4.1) we see that the series in (5.1) converges uniformly for

$$\sigma \geqslant \frac{1+\lambda}{2+\lambda} + \epsilon$$

for any positive ϵ*, and that

$$\begin{aligned} F_a(s, \theta) = {} & J(s)\,\Sigma_1\, e^{\frac{1}{2}\delta\pi i} (\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} \theta_n^{\frac{1}{2}(s-1)} \\ & + 2i^{1-s} J(s)\,\Sigma_2\, e^{\frac{1}{2}\delta\pi i} (\theta\theta_1 \dots \theta_{n-1})^{s-\frac{1}{2}} \theta_n^{\frac{1}{2}(s-1)} \\ & + R(s, \tfrac{1}{2}) \quad \left(\frac{1+\lambda}{2+\lambda} < \sigma < 1\right), \end{aligned} \tag{5.3}$$

where Σ_1 is taken over those values of n for which a_n is 1 or 5, and Σ_2 over those for which a_n is 3 or 6. Table I gives the values of a_{n+1} in terms of those of a_n. It summarizes parts of Theorems 1–4 of the previous paper†.

TABLE I.

a_n	1	2	3	4	5	6
a_{n+1}, $[\theta_n^{-1}]$ even	3	6	1	5	4	2
a_{n+1}, $[\theta_n^{-1}]$ odd	4	6	2	5	3	1

* Hence it follows independently of the argument from 3 that $F_a(s, \theta)$ is regular for $\sigma > (1+\lambda)/(2+\lambda)$.

† 1.

Now divide the irrationals of class λ (finite) into sub-classes $(\lambda, a, \lambda_1, \lambda_2)$ with respect to the function F_a by defining λ_1 as the lower bound of the numbers μ for which

$$(\theta\theta_1 \dots \theta_{n-1})^\mu \theta_n^{-1} \to 0$$

as $n \to \infty$ through those values of n for which a_n is 1 or 5; the sequence for which a_n is 3 or 6 defines λ_2 similarly. Obviously $\lambda \geqslant \text{Max}(\lambda_1, \lambda_2)$. If one of the sequences defining λ_1 and λ_2 reduces to a finite number of terms, we leave a blank in the corresponding place in the symbol $(\lambda, a, \lambda_1, \lambda_2)$; we see, from Table I, that this cannot happen to *both* sequences.

6. We can now prove the theorem which gives our principal result.

THEOREM 1. *If θ is an irrational of finite class λ and of sub-class $(\lambda, a, \lambda_1, \lambda_2)$ and $\lambda_1 \neq \lambda_2$, the function $F_a(s, \theta)$ has $\sigma = (1+\Lambda)/(2+\Lambda)$, where $\Lambda = \text{Max}(\lambda_1, \lambda_2)$, for a singular line, when $\Lambda \neq 0$, and is regular everywhere to the right of this line.*

Even if $\lambda_1 = \lambda_2$ or $\Lambda = 0$, $F_a(s, \theta)$ is regular in the half-plane $\sigma > (1+\Lambda)/(2+\Lambda)$.

To establish the theorem it is sufficient to prove that the series Σ_1 in (5 . 3) has $\sigma = (1+\lambda_1)/(2+\lambda_1)$ for a singular line if $\lambda_1 > 0$, since, using the argument which we applied to (5 . 2), we see that Σ_1 converges, and so represents a function regular in the half-plane to the right of this line, and we have already shown that F_a is regular for $\sigma > (1+\lambda)/(2+\lambda)$.

Choose a number μ such that $0 < \mu < \lambda_1$, and divide the terms of Σ_1 into two sets $n = n'$ and $n = n''$, writing $n = n'$ if

$$\theta_n < (\theta\theta_1 \dots \theta_{n-1})^\mu$$

and $n = n''$ otherwise. By the definition of λ_1, the sequence $n = n'$ is infinite. The portion of Σ_1 corresponding to $n = n''$ is absolutely convergent for $\sigma > (1+\mu)/(2+\mu)$, which is a greater half-plane than $\sigma > (1+\lambda_1)/(2+\lambda_1)$.

Let $n_1, n_2, \dots, n_k, \dots$ be the values of n' and write

$$l_k = -\log(\theta\theta_1\theta_2 \dots \theta_{n_k-1}\theta_{n_k}^{\frac{1}{2}}).$$

Then the portion of Σ_1 corresponding to $n = n'$ is

$$\sum_{k=1}^{\infty} e^{\frac{1}{2}\delta\pi i}(\theta\theta_1 \dots \theta_{n_k})^{\frac{1}{2}} e^{-l_k s}, \tag{6 . 1}$$

and

$$l_{k+1}-l_k = \log(\theta_{n_k}\theta_{n_k+1}\dots\theta_{n_{k+1}-1})^{-1}+\tfrac{1}{2}\log\theta^{-1}_{n_{k+1}}-\tfrac{1}{2}\log\theta^{-1}_{n_k}$$

$$\geqslant \tfrac{1}{2}\log\theta^{-1}_{n_{k+1}} > \tfrac{1}{2}\mu\log(\theta\theta_1\dots\theta_{n_{k+1}-1})^{-1}$$

$$\to\infty \quad \text{as} \quad k\to\infty.$$

Therefore, by a theorem given by Wennberg*, the axis of convergence $\sigma = (1+\lambda_1)/(2+\lambda_1)$ of the series (6.1) is a singular line for its sum function. It must also be a singular line for Σ_1, and this completes the proof of the theorem.

7. *Illustrations.* We now show how to construct irrationals to illustrate some of the possibilities given by the theorem for the behaviour of $F_\alpha(s, \theta)$. It will appear that, for a given θ, the six functions F_α need not all have the same singular line, and that we may be able to establish the existence of a singular line for some but not for all of the functions.

Using properties of irrationals given by Hardy and Littlewood† we see that, if p_n/q_n is the n-th convergent of the simple continued fraction for θ, so that $p_1 = 1$, $q_1 = a_1$, etc.

$$q_{n+1} > Aq_n^{\mu+1} \quad \text{implies} \quad \theta_n^{-1} > 2^{-1-\mu}A(\theta\theta_1\dots\theta_{n-1})^{-\mu}$$

and

$$q_{n+1} < Bq_n^{\mu+1} \quad \text{implies} \quad \theta_n^{-1} < 2B(\theta\theta_1\dots\theta_{n-1})^{-\mu},$$

and that corresponding implications hold in the reverse direction.

Since $a_{n+1}q_n < q_{n+1} < 2a_{n+1}q_n$, λ will certainly be the lower bound of the numbers μ for which

$$(\theta\theta_1\dots\theta_{n_j-1})^{\mu}\theta^{-1}_{n_j}\to 0 \quad \text{as} \quad j\to\infty,$$

where the integers n_j satisfy $n_j < n_{j+1}$, if positive (non-zero) constants K_1, K_2 exist such that, for $n = n_j$,

$$K_1 q_n^{\lambda} < a_{n+1} < K_2 q_n^{\lambda} \qquad (\lambda \neq 0)$$

or

$$K_1 \log q_n < a_{n+1} < K_2 \log q_n \quad (\lambda = 0).$$

Tables II, III give the values of a_n in the series (2.3) for certain types of θ, making in these cases the right-hand side of (5.3) and the sequences used to define λ_1, λ_2 precise. They are derived from Table I.

* **5** (3–7).

† **4**, §§ 1.2, 1.3.

TABLE II.

Partial quotients a_n of θ all even.

α	1	2	3	4	5	6
$a_1 = a_{2n+1}$	3	6	1	5	4	2
$a_1 = a_{2n+2}$	1	2	3	4	5	6

TABLE III.

Partial quotients a_{2n} odd and a_{2n+1} even.

α	1	2	3	4	5	6
$a_1 = a_{4n+1}$	3	6	1	5	4	2
$a_2 = a_{4n+2}$	2	1	4	3	5	6
$a_3 = a_{4n+3}$	6	3	5	1	4	2
$a_4 = a_{4n+4}$	1	2	3	4	5	6

We shall write (x) in the equations which we give to define a_n to denote whichever of x and $x+1$ gives the right parity to the partial quotient.

Example 1. Take for θ an irrational with every a_n even. The first few a_n's (say $n < 2n_0$) can be taken arbitrarily. For $n \geqslant n_0$ define step by step

$$a_{2n} = (q_{2n-1}^2), \qquad a_{2n+1} = (q_{2n}).$$

Then θ is of class 2 and of sub-classes (2, 1, 1, 2), (2, 2, —, 2), (2, 3, 2, 1), (2, 4, 2, —), (2, 5, 1, —), and (2, 6, —, 1), so that $\sigma = \frac{3}{4}$ is a singular line for F_1, F_2, F_3, F_4, and $\sigma = \frac{7}{8}$ for F_5, F_6. The functions are regular to the right of these lines.

Example 2. Take for θ every a_n even. Take the first few a_n's arbitrarily, and subsequently take

$$a_{2n} = ([\log q_{2n-1}]), \qquad a_{2n+1} = (q_{2n}) ;$$

and then θ is of class 1 and of sub-classes (1, 1, 1, 0), (1, 2, —, 0), (1, 3, 0, 1), (1, 4, 0, —), (1, 5, 1, —), and (1, 6, —, 1). F_1, F_3, F_5, F_6 have $\sigma = \frac{2}{3}$ as a singular line and are regular for $\sigma > \frac{7}{8}$. The theorem gives no information about F_2, F_4, except that they are regular for $\sigma > \frac{1}{2}$.

Example 3. Take for θ partial quotients such that a_{2n} is odd and a_{2n+1} is even, and, for all large values of n,

$$a_{4n+1} = (q_{4n}), \quad a_{2n+2} = (q_{2n+1}^2), \quad a_{4n+3} = (q_{4n+2}^2) ;$$

and then θ is of class 2 and of sub-classes (2, 1, 1, 2), (2, 2, 2, 2), (2, 3, 2, 1), (2, 4, 2, 2), (2, 5, 2, —), and (2, 6, —, 2), so that each

function is regular for $\sigma > \frac{3}{4}$, and $\sigma = \frac{3}{4}$ is a regular line for F_1, F_3, F_5, F_6. Nothing more can be deduced from the theorem about F_2, F_4 in this case.

8. We can get a result corresponding to Theorem 1 for some irrationals of infinite class. We may say they are of only logarithmically infinite class. We say that an irrational θ of infinite class is of logarithmic class L_λ if λ is the lower bound of numbers μ, if such exist, for which

$$(\theta\theta_1 \dots \theta_{n-1})^\mu \log \theta_n \to 0 \quad \text{as} \quad n \to \infty .$$

For such an irrational the series

$$\sum_1^\infty (\theta\theta_1 \dots \theta_{n-1})^{\sigma-\frac{1}{2}} \left(\log \frac{2}{\theta_n}\right)^{1-\sigma}$$

converges or diverges according as σ is greater or less than $(1+\lambda)/(2+\lambda)$.

If we define λ_1 and λ_2 exactly as in § 5, we have the following theorem. Its proof now differs only trivially from the arguments above*.

THEOREM 2. *If θ is of sub-class $(L_\lambda, \alpha, \lambda_1, \lambda_2)$, $\lambda_1 \neq \lambda_2$ and $\Lambda = \operatorname{Max}(\lambda_1, \lambda_2) > \lambda \geqslant 0$, the function $F_\alpha(s, \theta)$ has $\sigma = (1+\Lambda)/(2+\Lambda)$ as a singular line to the right of which it is regular.*

References.

1. E. Cooper, "The behaviour of certain series associated with limiting cases of elliptic theta-functions", *Proc. London Math. Soc.* (2), 27 (1928), 410–426.
2. G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: the analytical properties of certain Dirichlet's series associated with the distribution of numbers to modulus unity", *Trans. Camb. Phil. Soc.*, 22 (1923), 519–534.
3. G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: a further note on the trigonometrical series associated with elliptic theta-functions", *Proc. Camb. Phil. Soc.*, 21 (1923), 1–5.
4. G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: the lattice points of a right-angled triangle", *Hamburg. Math. Abh.*, 1 (1922), 213–249.
5. S. Wennberg, "Zur Theorie der Dirichlet'schen Reihen", *Inaugural Dissertation* (Upsala, 1920).

* The argument in § 4 depending on a result from 3 is not essential, as was pointed out in the first footnote on page 117.

ON THE REPRESENTATION OF A NUMBER AS A SUM OF SQUARES AND PRIMES

By G. K. STANLEY.

[Received 3 January, 1928.—Read 12 January, 1928.]

Introduction.

1.1. It has been shown by Hardy and Littlewood* that if a certain hypothesis is true, then every large odd number n is the sum of three odd primes, and they give an asymptotic formula for the number of such representations. Their hypothesis, which they call Hypothesis R, is a generalization of Riemann's hypothesis concerning the zeros of his zeta-function. In his dissertation† Kloosterman used the same method to find a general asymptotic formula for the number of representations of a number in the form $ax_1^2+bx_2^2+\ldots+kx_r^2$, $r \geqslant 5$. Later he gave an asymptotic formula for the number of representations when $r = 4$‡. In this problem, naturally, Hypothesis R is not involved.

My object here is to find in what circumstances every large number is the sum of r squares and s primes, and to find an asymptotic formula for the number of such representations. It is to be observed that if it has been established that every large number can be expressed as the sum of r_1 squares and s_1 primes, and an asymptotic formula has been found for the number of representations, then the corresponding theorem that every large number can be represented as the sum of r squares and s primes, where $r \geqslant r_1$, $s \geqslant s_1$, is trivial, but the finding of an asymptotic formula embodies a new theorem. Also, if we have made a proof for $r = r_1$, $s = s_1$, then the proof for $r \geqslant r_1$, $s \geqslant s_1$, though not strictly in-

* G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum' (III): On the expression of a number as the sum of primes", *Acta Math.*, 44 (1922), 1–70. This memoir will hereafter be referred to as P.N. 3.

† H. D. Kloosterman, *Over het splitsen van geheele positieve getallen in een som van quadraaten* (1924). This will hereafter be referred to as K.

‡ H. D. Kloosterman, "On the expression of a number in the form $ax^2+by^2+cz^2+dt^2$", *Acta Math.*, 49 (1926), 407–464. See also "Asymptotische Formeln für die Fourier-koeffizienten ganzer Modulformen λ", *Abhandlungen aus dem mathematischer Seminar der Hamburgischen Universität*, 5 (1927), 337–352.

cluded in the first proof, is, in fact, easier. It is therefore only necessary, *e.g.* when $s = 2$, to go into details for $r = 1$, the proof for $r = 2$ being simpler. Certain parts of the analysis apply with equal ease to all values of r and s considered, and in these cases I take general values.

Hardy and Littlewood* give formulae for such representation when $r = 4$, $s = 1$, and when $r = 2$, $s = 1$, and they say that the first of these should be capable of rigorous proof. In Part 1, I consider this representation, a proof being included in Theorem A, in which I prove a formula for the cases

$$s \geqslant 3; \quad s = 2, \ r \geqslant 3; \quad s = 1, \ r \geqslant 4.$$

Here the proof follows closely the lines of proof in P.N. 3 or in K. The analysis of the general case fails for other values of r and s, but I give a special proof for the case $r = 3$, $s = 1$ in Theorem B ($r = 2$, $s = 1$ cannot be proved along these lines). There remains the consideration of the cases when $s = 2$, $r \leqslant 2$. I give the proof for the more difficult case $r = 1$, $s = 2$ in Theorem C. Hardy and Littlewood also say, in another memoir†, that a use of this method will show that almost all numbers are the sums of two squares and a prime. A proof of this statement is given in Part 2, where I show that the number of numbers less than n which cannot be so represented is $O(n^{\frac{1}{2}+\epsilon})$, when n is large.

1 . 2. The fundamental hypothesis which I shall assume throughout is

HYPOTHESIS $R^*_\ddagger$. *Every zero of every Dirichlet's function*

$$L(s) = \Sigma \frac{\chi(m)}{m^s},$$

where $\chi(m)$ is a character to modulus q, has a real part less than or equal to $\frac{1}{2}$ for all q and all χ.

All the theorems of this paper are subject to this hypothesis.

Also, throughout the paper, the O's refer to the limit process $n \to \infty$, A denotes an absolute constant, though not necessarily the same constant, while B denotes a constant which depends only on r and s. I assume that r and s are never zero, for in that case the problem becomes

---

* P.N. 3, Conjecture J, 50.

† G. H. Hardy and J. E. Littlewood, "Some problems of 'Partitio Numerorum' (V): A further contribution to the study of Goldbach's problem", *Proc. London Math. Soc.* (2), 22 (1924), 46–56. See, in particular, 56. This memoir will be referred to as P.N. 5.

‡ Hypothesis R, which Hardy and Littlewood used in P.N. 3, is slightly less drastic than this.

that of P.N. 3 or of K, and I use the following notation: $\mathfrak{p}$ is a prime divisor of n, ϖ is a prime, and p an integer. If $q = 1$, $p = 0$; otherwise

$$0 < p < q, \quad (p, q) = 1.$$

$\mu(n)$ is Mobius' function, and $\phi(n)$ the number of numbers less than and prime to n. The functions which are fundamental in all the analysis are

$$f(x) = \sum_{\varpi} \log \varpi \,.\, x^{\varpi}, \quad \mathfrak{S}(x) = \sum_{m=-\infty}^{\infty} x^{m^2}.$$

By $N_{r,s}(n)$ we denote the number of representations of n as the sum of r squares and s primes, attention being paid to the order of the squares among themselves and of the primes among themselves, while those representations which differ only in the signs of the squares are counted as distinct. By $\nu_{r,s}(n)$ we denote the sum

$$\nu_{r,s}(n) = \sum_{m_1^2+m_2^2+\ldots+m_r^2+\varpi_1+\varpi_2+\ldots+\varpi_s=n} \log \varpi_1 \log \varpi_2 \ldots \log \varpi_s,$$

so that we have

$$\sum_{n=2}^{\infty} \nu_{r,s}(n) x^n = \{\mathfrak{S}(x)\}^r \{f(x)\}^s,$$

and, if $n \geqslant 2$,

$$\nu_{r,s}(n) = \frac{1}{2\pi i} \int \{\mathfrak{S}(x)\}^r \{f(x)\}^s \frac{dx}{x^{n+1}},$$

the path of integration being the circle

$$|x| = e^{-(1/n)}$$

which we shall call Γ. We use the Farey dissection of order $N = [\sqrt{n}]$, and call the typical arc thus obtained $\xi_{p,q}$. There is in this problem, as in P.N. 3 or in K, no distinction between "major" and "minor" arcs. We have also the *singular series*

$$\mathfrak{S}_{r,s}(n) = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\phi(q)}\right)^s \frac{a_q}{q^r},$$

where

$$a_q = \sum_{p} (S_{p,q})^r e^{-2np\pi i/q}.$$

2.1. We need the following preliminary lemmas.

LEMMA 1. *On $\xi_{p,q}$, where $x = e^{(2p\pi i/q)+\beta+i\alpha}$,*

$$f(x) = \psi(x) + \Psi(x), \tag{2.11}$$

where

$$\psi(x) = \frac{\mu(q)}{\phi(q)} \left(\frac{1}{n} - i\alpha\right)^{-1} = O\{nq^{-1}(\log q)^4 (1+n^2\alpha^2)^{-\frac{1}{2}}\}, \tag{2.121}$$

$$\Psi(x) = O(n^{\frac{3}{4}+\epsilon}). \tag{2.122}$$

See P.N. 3, p. 19, equations (2 . 821) and (2 . 822), p. 21, equation (2 . 91), and also p. 23, equation (3 . 124), putting $\Theta = \frac{1}{2}$.

2 . 2. LEMMA 2. *If x lies on Γ*

$$\int_0^{2\pi} |f(x)|^2 \, d\alpha = O(n \log n). \tag{2 . 21}$$

See P.N. 3, p. 24, § 3 . 14.

2 . 3. LEMMA 3. *We have*

$$\Sigma \int_{\xi_{p,q}} |\psi|^2 \, d\alpha = O\{n(\log n)^A\}. \tag{2 . 31}$$

From Lemma 1

$$\Sigma \int_{\xi} |\psi|^2 \, d\alpha = O\left(n^2 \sum_{p,q} q^{-2} (\log q)^A \int_{\xi} (1+n^2\alpha^2)^{-1} \, d\alpha\right)$$

$$= O\left(n(\log n)^A \sum_{p,q} q^{-2}\right) = O\{n(\log n)^A\},$$

since

$$\sum_{p,q} q^{-2} < \sum_{q} q^{-1} = O(\log n).$$

2 . 4. LEMMA 4. *We have*

$$\Sigma \int_{\xi} |\Psi|^2 \, d\alpha = O\{n(\log n)^A\}. \tag{2 . 41}$$

For

$$|\Psi|^2 \leqslant A(|f|^2 + |\psi|^2).$$

Hence

$$\int_{\Gamma} |\Psi|^2 \, d\alpha = O(n \log n) + O\{n(\log n)^A\} = O\{n(\log n)^A\}.$$

2 . 5. LEMMA 5. *On $\xi_{p,q}$*

$$\vartheta(x) = \theta(x) + \Theta(x), \tag{2 . 51}$$

where

$$\theta(x) = \pi^{\frac{1}{2}} q^{-1} S_{p,q} \left(\frac{1}{n} - i\alpha\right)^{-\frac{1}{2}} \tag{2 . 521}$$

and

$$\Theta(x) = O(n^{\frac{1}{4}+\epsilon}). \tag{2 . 522}$$

See K., p. 21, equations (17 . 1) and (17 . 2), and p. 24, equation (19 . 5).

2 . 6. LEMMA 6. *We have*

$$\theta(x) = O\{q^{-\frac{1}{2}} n^{\frac{1}{2}} (1+n^2\alpha^2)^{-\frac{1}{4}}\}. \tag{2 . 61}$$

For $$S_{p,q} = O(q^{\frac{1}{2}}),$$

and the result follows at once from (2 . 521).

2 . 7. LEMMA 7. *We have*

$$\phi(x) > Aq(\log q)^{-A}. \tag{2 . 71}$$

This is Lemma 10 of P.N. 3. A proof of rather more than this is given by Landau*.

PART I. *Results for all large n.*

Summation of the singular series.

3 . 11. LEMMA 8. *If* $(q, q') = 1$, *then*

$$A_{qq'} = A_q A_{q'}, \tag{3 . 112}$$

where $$A_q = \left(\frac{\mu(q)}{\phi(q)}\right)^s q^{-r} a_q, \quad a_q = \Sigma (S_{p,q})^r e^{-2np\pi i/q}.$$

It is well known that, if $(q, q') = 1$, then

$$\mu(qq') = \mu(q)\mu(q'), \quad \phi(qq') = \phi(q)\phi(q').$$

Now, if $P = pq' + p'q$,

$$a_{qq'} = \sum_P (S_{P, qq'})^r \exp\left(-\frac{2nP\pi i}{qq'}\right)$$

$$= \sum_P (S_{Pq, q'})^r (S_{Pq', q})^r \exp\left(-\frac{2nP\pi i}{qq'}\right).$$

But

$$S_{Pq, q'} = \sum_{j=1}^{q'} \exp\left(\frac{2j^2 Pq\pi i}{q'}\right) = \sum_{j=1}^{q'} \exp\left(\frac{2(jq)^2 p'\pi i}{q'}\right) = S_{p', q'},$$

and in the same way $$S_{Pq', q} = S_{p, q}.$$

Hence it follows that

$$a_{qq'} = \sum_{p, p'} (S_{p,q})^r (S_{p',q'})^r \exp\left(-\frac{2nP\pi i}{qq'}\right)$$

$$= \sum_p (S_{p,q})^r \exp\left(-\frac{2np\pi i}{q}\right) \sum_{p'} (S_{p',q'})^r \exp\left(-\frac{2np'\pi i}{q'}\right) = a_q a_{q'}.$$

Thus we have $$A_{qq'} = A_q A_{q'}.$$

3 . 12. LEMMA 9. *If* $r + 2s > 4$, *the singular series is absolutely convergent.*

* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (1909), 217.

We have $$\mathfrak{S}_{r,s}(n) = \sum_{q=1}^{\infty} A_q$$

and $$|A_q| = \left|\frac{1}{\phi(q)}\right|^s |q^{-r}| . \left|\sum_p \exp\left(-\frac{2np\pi i}{q}\right)(S_{p,q})^r\right|$$

$$\leqslant B\left|\frac{1}{\phi(q)}\right|^s |q^{-r}| . |\phi(q)| . |q^{r/2}| \leqslant Bq^{-(r/2)-s+1}(\log q)^A$$

by (2 . 71) combined with the result that $S_{p,q} = O(q^{\frac{1}{2}})$. Hence the singular series is absolutely convergent if the series

$$\Sigma q^{-(r/2)-s+1}(\log q)^A$$

is convergent, which is obviously true when $r+2s > 4$.

This is a very crude argument for the convergence of the singular series, but it gives all that we need in Part I.

3 . 13. LEMMA 10. *If* $r+2s > 4$,

$$\mathfrak{S}_{r,s} = \prod_{\varpi} \chi_{\varpi},$$

where $$\chi_{\varpi} = 1 + A_{\varpi}.$$

This follows at once from the last two lemmas, since, by virtue of the factor $\mu(q)$,

$$A_{\varpi^2} = A_{\varpi^3} = A_{\varpi^4} = \ldots = 0.$$

3 . 2. LEMMA 11. *If* $r+2s > 4$, *and*

$$\mathfrak{S}_{r,s} = \sum_{q=1}^{\infty}\left[\frac{\mu(q)}{\phi(q)}\right]^s \frac{a_q}{q^r};$$

then

$$(3.211)\quad \mathfrak{S}_{r,s} = \prod_{\varpi=3}^{\infty}\left[1-\frac{(-1)^s}{\varpi^{\frac{1}{2}r}(\varpi-1)^s}\right]\prod_{\mathfrak{p}}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s}\right]$$

$$[r \equiv 0 \pmod 4],$$

$$(3.212)\quad = \prod_{\varpi=3}^{\infty}\left[1+\left(\frac{n}{\varpi}\right)\frac{(-1)^s}{\varpi^{\frac{1}{2}(r-1)}(\varpi-1)^s}\right]\prod_{\mathfrak{p}}\left[\frac{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s}{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s+(n/\mathfrak{p})(-1)^s}\right]$$

$$[r \equiv 1 \pmod 4],$$

$$(3.213)\quad = \prod_{\varpi=3}^{\infty}\left[1-\left(-\frac{1}{\varpi}\right)\frac{(-1)^s}{\varpi^{\frac{1}{2}r}(\varpi-1)^s}\right]$$

$$\times \prod_{\mathfrak{p}\equiv 1 \pmod 4}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s}\right]$$

$$\times \prod_{\mathfrak{p}\equiv 3 \pmod 4}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s}\right] [r \equiv 2 \pmod 4],$$

$$(3.214)\quad = \prod_{\varpi=3}^{\infty}\left[1+\left(\frac{-n}{\varpi}\right)\frac{(-1)^s}{\varpi^{\frac{1}{2}(r-1)}(\varpi-1)^s}\right]$$

$$\times \prod_{\mathfrak{p}}\left[\frac{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s}{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s+\{(-n)/\mathfrak{p}\}(-1)^s}\right] [r \equiv 3 \pmod 4].$$

The values of a_q/q^r have been given in Hardy's paper "On the representation of a number as the sum of any number of squares, and, in particular, of five"*. I quote these values as far as they are needed.

When $\varpi = 2$,

$$a_\varpi = 0, \quad A_\varpi = 0. \tag{3.221}$$

When ϖ is odd $\mu(\varpi) = -1$, $\phi(\varpi) = \varpi - 1$. If, in addition, $\varpi \mid n$,

$$\varpi^{-r} a_\varpi = (\varpi-1)\varpi^{-\frac{1}{2}r}, \quad 0, \quad \text{or} \quad (-1)^{\frac{1}{2}(\varpi-1)}(\varpi-1)\varpi^{-\frac{1}{2}r}, \tag{3.222}$$

according as r is evenly even, odd, or oddly even. If, on the other hand, $\varpi \nmid n$

$$\begin{cases} \varpi^{-r} a_\varpi = -\varpi^{-\frac{1}{2}r}, & \left(\frac{n}{\varpi}\right)\varpi^{-\frac{1}{2}(r-1)}, \quad -(-1)^{\frac{1}{2}(\varpi-1)}\varpi^{-\frac{1}{2}r}, \\ \text{or} & (-1)^{\frac{1}{2}(\varpi-1)}\left(\frac{n}{\varpi}\right)\varpi^{-\frac{1}{2}(r-1)}, \end{cases} \tag{3.223}$$

according as r is convergent to 0, 1, 2, or 3 to modulus 4. Hence, if ϖ is odd and $\varpi \mid n$,

$$\chi_\varpi = 1 + \frac{(-1)^s}{(\varpi-1)^{s-1}\varpi^{\frac{1}{2}r}}, \quad 1 \quad \text{or} \quad 1 + \frac{(-1)^{\frac{1}{2}(\varpi-1)+s}}{(\varpi-1)^{s-1}\varpi^{\frac{1}{2}r}}, \tag{3.231}$$

according as r is evenly even, odd, or oddly even. On the other hand, if $\varpi \nmid n$

$$\begin{cases} \chi_\varpi = 1 - \frac{(-1)^s}{(\varpi-1)^s\varpi^{\frac{1}{2}r}}, & 1 + \left(\frac{n}{\varpi}\right)\frac{(-1)^s}{(\varpi-1)^s\varpi^{\frac{1}{2}(r-1)}}, \quad 1 - \frac{(-1)^{\frac{1}{2}(\varpi-1)+s}}{(\varpi-1)^s\varpi^{\frac{1}{2}r}}, \\ \text{or} & 1 + \left(\frac{n}{\varpi}\right)\frac{(-1)^{\frac{1}{2}(\varpi-1)+s}}{(\varpi-1)^s\varpi^{\frac{1}{2}(r-1)}}; \end{cases} \tag{3.232}$$

according as r is congruent to 0, 1, 2, or 3 to modulus 4.

Thus we have, when $r \equiv 0 \pmod 4$,

$$\mathfrak{S}_{r,s} = \prod_{\varpi \mid n}\left(1 + \frac{(-1)^s}{\varpi^{\frac{1}{2}r}(\varpi-1)^{s-1}}\right)\prod_{\varpi \nmid n}\left(1 - \frac{(-1)^s}{\varpi^{\frac{1}{2}r}(\varpi-1)^s}\right),$$

which gives (3.211). When $r \equiv 1 \pmod 4$ we have

$$\mathfrak{S}_{r,s} = \prod_{\varpi \nmid n}\left\{1 + \left(\frac{n}{\varpi}\right)\frac{(-1)^s}{(\varpi-1)^s\varpi^{\frac{1}{2}(r-1)}}\right\},$$

* *Trans. Amer. Math. Soc.*, 17 (1920), 255–284. The a_q/q^r in question is his A_p.

which gives (3 . 212). When $r \equiv 2 \pmod 4$ we have

$$\mathfrak{S}_{r,s} = \prod_{p \equiv 1} \left(1 + \frac{(-1)^s}{(p-1)^{s-1} p^{\frac{1}{2}r}}\right) \prod_{p \equiv 3} \left(1 - \frac{(-1)^s}{(p-1)^{s-1} p^{\frac{1}{2}r}}\right)$$

$$\times \prod_{\substack{\varpi \equiv n \\ \varpi \equiv 1}} \left(1 - \frac{(-1)^s}{(\varpi-1)^s \varpi^{\frac{1}{2}r}}\right) \prod_{\substack{\varpi \equiv n \\ \varpi \equiv 3}} \left(1 + \frac{(-1)^s}{(\varpi-1)^s \varpi^{\frac{1}{2}r}}\right),$$

the congruences being to modulus 4. This gives

$$\mathfrak{S}_{s,r} = \prod_{\varpi \equiv 3}^{\infty} \left\{1 - \left(\frac{-1}{\varpi}\right) \frac{(-1)^s}{(\varpi-1)^s \varpi^{\frac{1}{2}r}}\right\}$$

$$\prod_{p \equiv 1} \left(\frac{p^{\frac{1}{2}r}(p-1)^s + (-1)^s(p-1)}{p^{\frac{1}{2}r}(p-1)^s - (-1)^s}\right) \prod_{p \equiv 3} \left(\frac{p^{\frac{1}{2}r}(p-1)^s - (-1)^s(p-1)}{p^{\frac{1}{2}r}(p-1)^s + (-1)^s}\right)$$

which is (3 . 213). When $r \equiv 3 \pmod 4$

$$\mathfrak{S}_{r,s} = \prod_{\varpi \equiv n} \left\{1 + \left(\frac{-n}{\varpi}\right) \frac{(-1)^s}{(\varpi-1)^s \varpi^{\frac{1}{2}(r-1)}}\right\}$$

which gives (3 . 214).

The main theorems.

4 . 1. THEOREM A. *If $s \geqslant 3$; $s = 2$, $r \geqslant 3$; or if $s = 1$, $r \geqslant 4$, then*

$$\nu_{r,s}(n) \sim \frac{\pi^{\frac{1}{2}r}}{\Gamma(\frac{1}{2}r+s)} n^{\frac{1}{2}r+s-1} \mathfrak{S}_{r,s}(n). \tag{4 . 11}$$

For
$$\nu(n) = \frac{1}{2\pi i} \int \Im^r f^s \frac{dx}{x^{n+1}}$$

and
$$\Im^r f^s = (\theta + \Theta)^r (\psi + \Psi)^s.$$

We consider first the term involving $\theta^r \psi^s$. From (2 . 121) and (2 . 521) this gives

$$\pi^{\frac{1}{2}r} \sum_q \left(\frac{\mu(q)}{\phi(q)}\right)^s \frac{1}{q^r} \sum_p (S_{p,q})^r \frac{1}{2\pi i} \int_\xi x^{-n-1} \left(\frac{1}{\mu} - i\alpha\right)^{-\frac{1}{2}r-s} dx. \tag{4 . 12}$$

But now
$$\left(\frac{1}{n} - i\alpha\right)^{-\frac{1}{2}r-s} = \frac{1}{\Gamma(\frac{1}{2}r+s)} F(X) + O(1),$$

where
$$F(X) = \Sigma m^{\frac{1}{2}r+s-1} X^{m*}, \quad X = e^{-(1/n)+i\alpha}.$$

* See K., 25.

Hence, if we replace $(n^{-1}-i\alpha)^{-\frac{1}{2}r-s}$ by $F(X)/\Gamma(\frac{1}{2}r+s)$ we have an error of

$$O\left\{\sum_q \left(\frac{\mu(q)}{\phi(q)}\right)^s q^{-r} \sum_p (S_{p,q})^r \int \frac{dx}{x^{n+1}}\right\} = O\left\{\sum_q \left(\frac{\mu(q)}{\phi(q)}\right)^s q^{-r} \sum_p q^{\frac{1}{2}r} \frac{1}{qN}\right\},$$

since $S_{p,q} = O(q^{\frac{1}{2}})$ and the arc of integration is $O(1/qN)$. Hence the error is

$$O\left(\sum_q q^{-s}(\log q)^A q^{-\frac{1}{2}r} n^{-\frac{1}{2}}\right) = O(q^{-\frac{1}{2}r-s+1+\epsilon} n^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}}),$$

since $\frac{1}{2}r+s-1$ is always positive.

If we now replace

$$\frac{1}{2\pi i}\int_{\xi_{p,q}} x^{-n-1} F(X)\, dx$$

by the integral over the whole circle, we have an error of

$$O\left(\int_{\kappa/qN}^{\infty} \frac{n^{\frac{1}{2}r+s}\, d\alpha}{(1+n^2\alpha^2)^{\frac{1}{4}r+\frac{1}{2}s}}\right) = O\left(\int_{\kappa n/qN}^{\infty} \frac{n^{\frac{1}{2}r+s-1}\, dt}{(1+t^2)^{\frac{1}{4}r+\frac{1}{2}s}}\right) = O\left\{n^{\frac{1}{2}r+s-1}\left(\frac{q}{\sqrt{n}}\right)^{\frac{1}{2}r+s-1}\right\},$$

since $$F(X) = O\left(\frac{1}{n^2}+\alpha^2\right)^{-\frac{1}{4}r-\frac{1}{2}s}.$$

Thus the total error in (4.12) by these transformations is

$$O\left\{\sum_q \left(\frac{\mu(q)}{\phi(q)}\right)^s q^{-r} \sum_p q^{\frac{1}{2}r} n^{\frac{1}{2}r+s-1}\left(\frac{q}{\sqrt{n}}\right)^{\frac{1}{2}r+s-1}\right\} = O(n^{\frac{1}{2}r+\frac{1}{2}s+\epsilon}),$$

since $\phi(q) > q(\log q)^{-A}$ and q runs between 0 and $\sqrt{n}$. And, since

$$\frac{1}{2\pi i}\int_\Gamma x^{-n-1} F(X)\, dx = n^{\frac{1}{2}r+s-1}\exp\left(-\frac{2p\pi i n}{q}\right),$$

the term (4.12) can be written

$$\frac{\pi^{\frac{1}{2}r}}{\Gamma(\frac{1}{2}r+s)}\, n^{\frac{1}{2}r+s-1} \sum_{q=1}^{N} A_q + O(n^{\frac{1}{2}r+\frac{1}{2}s+\epsilon}). \tag{4.13}$$

Finally the error involved in replacing the finite series by the infinite series

$$\sum_{q=1}^{\infty} A_q$$

is

$$O\left\{n^{\frac{1}{2}r+s-1}\sum_{q>N}\left(\frac{\mu(q)}{\phi(q)}\right)^s q^{-r}\sum_p (S_{p,q})^r \exp\left(-\frac{2p\pi i n}{q}\right)\right\}$$

$$= O\left(n^{\frac{1}{2}r+s-1}\sum_{q>N} q^{-s+1}(\log q)^A q^{-r+\frac{1}{2}r}\right) = O(n^{\frac{1}{2}r+\frac{1}{2}s+\epsilon}).$$

Thus in (4.13) we can replace the finite series by the infinite series.

To complete the proof of (4 . 11) we must study the term

$$\sum_{a,b} \int_{\mathfrak{r}} B(a,b)\, \theta^r \Theta^{r-a} \psi^b \Psi^{s-b} \frac{dx}{x^{n+1}},$$

where $0 \leqslant a \leqslant r-1$, $0 \leqslant b \leqslant s-1$, both a and b being integers. Then, from (2 . 61), (2 . 522), (2 . 121), and (2 . 122) we have

$$\int_{\mathfrak{r}} |\theta^r \Theta^{r-a} \psi^b \Psi^{s-b}| \frac{dx}{x^{n+1}} = O\left(n^{\frac{1}{4}(r+a+3s+b)-1+\epsilon} \sum_{p,q} q^{-\frac{1}{2}a-b} \int (1+n^2\alpha^2)^{-\frac{1}{4}a-\frac{1}{2}b} d\alpha\right)$$

$$= O(n^{\frac{1}{4}(r+a+3s+b)-1+\epsilon}),$$

provided that $a+2b \geqslant 4$. For the series

$$\sum_{p,q} q^{-\frac{1}{2}a-b} < \sum_{q} q^{-\frac{1}{2}a-b+1},$$

which is convergent when $\frac{1}{2}a+b > 2$, and is $O(\log n)$ when $\frac{1}{2}a+b = 2$. Also $\frac{1}{4}(r+a+3s+b)-1 < \frac{1}{2}r+s-1$ for all values of r and s. If, however, $a+2b < 4$ we must consider the value of the error term more carefully. When $a+2b = 3$ this error term becomes

$$O(n^{\frac{1}{4}(r+a+3s+b)-\frac{3}{4}+\epsilon})$$

for

$$\sum_{p,q} q^{-\frac{3}{2}} = O(n^{\frac{1}{4}+\epsilon}).$$

Now $\frac{1}{4}(r+a+3s+b)-\frac{3}{4} < \frac{1}{2}r+s-1$ provided that $a+b < r+s-1$, which is always true. Next suppose that $a+2b = 2$. Then $a = 2$, $b = 0$, or $a = 0$, $b = 1$, and the error term becomes

$$O(n^{\frac{1}{4}(r+a+3s+b)-\frac{1}{2}+\epsilon}).$$

Now $\frac{1}{4}(r+a+3s+b)-\frac{1}{2} < \frac{1}{2}r+s-1$, except when $a+b \geqslant r+s-2$. But this is only possible if $r = 3$, $s = 1$, or if $r = 1$, $s = 2$, both of which cases have been excluded by our hypothesis. If $a+2b = 1$, $a = 1$, $b = 0$, and we write the error term in the form

$$O\left(\int |\theta \Theta^{r-1} \Psi^s| d\alpha\right) = O\left(\max(|\theta \cdot \Theta^{r-1} \Psi^{s-2}|) \int_0^{2\pi} |\Psi^2| d\alpha\right) = O(n^{\frac{1}{4}(r+3s)+\epsilon}).$$

And $\frac{1}{4}(r+3s) < \frac{1}{2}r+s-1$, except when $r+s < 4$. But as $a = 1$, $b = 0$, it follows that $r \geqslant 2$ and so $s \leqslant 2$, which is again excluded by the hypothesis. Finally, suppose that $a+2b = 0$, *i.e.* $a = b = 0$. Then the error term is

$$O\left(\int_{\mathfrak{r}} |\Theta^r \Psi^s| d\alpha\right) = O\left(\max(\Theta^r \Psi^{s-2}) \int_0^{2\pi} |\Psi^2| d\alpha\right) = O(n^{\frac{1}{4}(r+3s)-\frac{1}{2}+\epsilon}),$$

and $\frac{1}{4}(r+3s)-\frac{1}{2} < \frac{1}{2}r+s$ provided that $r+s > 2$. This completes the proof of the theorem.

4.2. THEOREM B. *If $r=3$, $s=1$, then*

$$\nu_{3,1}(n) = \tfrac{4}{3}\pi n^2 \mathfrak{S}_{3,1}(n) + O(n^{\frac{11}{8}+\epsilon}).$$

We have

$$\nu(n) = \frac{1}{2\pi i}\int_\Gamma \mathfrak{I}^3 f \frac{dx}{x^{n+1}} = \frac{1}{2\pi i}\int \{\theta^3\psi + \mathfrak{I}^3\Psi + \psi(\mathfrak{I}^3-\theta^3)\}\frac{dx}{x^{n+1}}.$$

As in Theorem A, we can show that

$$\frac{1}{2\pi i}\int \theta^3\psi \frac{dx}{x^{n+1}} = \tfrac{4}{3}\pi n^2 \mathfrak{S}_{3,1}(n) + O(n^{\frac{7}{4}+\epsilon}). \tag{4.21}$$

We then have

$$\int |\mathfrak{I}^3\Psi|\,d\alpha \leqslant A\Big[\int |\theta^3\Psi|\,d\alpha + \int |\Theta^3\Psi|\,d\alpha\Big]. \tag{4.22}$$

Now, if $q < n^{\frac{1}{2}}$, we have, from (2.122) and (2.61),

$$\int |\theta^3\Psi|\,d\alpha = O\Big(n^{\frac{3}{4}+\epsilon}\sum_{p,q} q^{-\frac{3}{2}}\int (1+n^2\alpha^2)^{-\frac{3}{4}}\,d\theta\Big)$$

$$= O\Big(n^{\frac{3}{4}+\epsilon}\sum_q q^{-\frac{1}{2}}\Big) = O(n^{\frac{11}{8}+\epsilon}).$$

If, on the other hand, $n^{\frac{1}{2}} > q > n^{\frac{1}{3}}$, we have, from (2.41),

$$\int |\theta^3\Psi|\,d\alpha \leqslant \Big\{\int |\theta|^6\,d\alpha \int |\Psi|^2\,d\alpha\Big\}^{\frac{1}{2}} = O\Big\{n^{\frac{1}{2}+\epsilon}\Big(\int |\theta|^6\,d\alpha\Big)^{\frac{1}{2}}\Big\}.$$

But, from (2.61),

$$\int |\theta|^6\,d\alpha = O\Big(n^3\sum_{p,\,q>n^{\frac{1}{3}}} q^{-3}\int_\xi (1+n^2\alpha^2)^{-\frac{3}{2}}\,d\alpha\Big) = O\Big(n^2\sum_{q>n^{\frac{1}{3}}} q^{-2}\Big)$$

$$= O(n^{\frac{1}{3}}).$$

Hence, when $n^{\frac{1}{3}} > q > 0$,

$$\int |\theta^3\Psi|\,d\alpha = O(n^{\frac{11}{8}+\epsilon}). \tag{4.221}$$

For the other part of (4.22) we have, from (2.31) and (2.522),

$$\int |\Theta^3\Psi|\,d\alpha \geqslant \Big\{\int |\Psi|^2\,d\alpha \int |\Theta|^6\,d\alpha\Big\}^{\frac{1}{2}} = O(n^{\frac{3}{2}+\epsilon}). \tag{4.222}$$

We still have to consider

$$\int |\psi(\mathfrak{I}^3-\theta^3)|\,d\alpha \leqslant A\Big[\int |\psi\theta^2\Theta|\,d\alpha + \int |\psi\Theta^3|\,d\alpha\Big]. \tag{4.23}$$

The first part of this gives, by (2.121), (2.522), and (2.61),

$$\int |\psi\theta^2\Theta|\,d\alpha = O\Big(n^{\frac{3}{4}+\epsilon}\sum_{p,q} q^{-2}\int_\xi (1+n^2\alpha^2)^{-1}\,d\alpha\Big) = O(n^{\frac{3}{4}+\epsilon}). \tag{4.231}$$

The second part gives

$$\int |\psi\Theta^3|\,d\alpha = O\left(n^{\frac{3}{4}+\epsilon}\sum_{p,q} q^{-1}\int (1+n^2\alpha^2)^{-\frac{1}{2}}\,d\alpha\right) \tag{4.232}$$

$$= O\left(n^{\frac{3}{4}+\epsilon}\sum_{q} 1\right) = O(n^{\frac{3}{4}+\epsilon}).$$

Hence, from (4.21), (4.22), (4.221), (4.222), (4.231), and (4.232), we have the theorem.

4.3. THEOREM C. *If* $r=1$, $s=2$, *then*

$$\nu_{1,2}(n) = \tfrac{4}{3}n^2\mathfrak{S}_{1,2}(n)+O(n^{\frac{11}{8}+\epsilon}). \tag{4.31}$$

We have

$$\nu(n) = \frac{1}{2\pi i}\int \Im f^2\,\frac{dx}{x^{n+1}} = \frac{1}{2\pi i}\int [\theta\psi^2+f^2\Theta+\theta(f^2-\psi^2)]\,\frac{dx}{x^{n+1}}. \tag{4.32}$$

As in Theorem A, we can easily show that

$$\frac{1}{2\pi i}\int \theta\psi^2\,\frac{dx}{x^{n+1}} = \tfrac{4}{3}n^2\mathfrak{S}(n)+O(n^{\frac{3}{4}+\epsilon}). \tag{4.33}$$

We then have, from (2.21) and (2.522),

$$\int |f^2\Theta|\,d\alpha = O\left(n^{\frac{1}{4}+\epsilon}\int_0^{2\pi}|f|^2\,d\alpha\right) = O(n^{\frac{3}{4}+\epsilon}). \tag{4.34}$$

The remaining part of this integral can be written

$$\int \theta(2\psi\Psi+\Psi^2)\,d\alpha \tag{4.35}$$

and, when $q<n^{\frac{1}{4}}$, the first part of this gives us, by (2.121), (2.122), and (2.61),

$$\int |\theta\psi\Psi|\,d\theta = O\left(n^{\frac{3}{4}+\epsilon}\sum_{p,q} q^{-\frac{3}{2}}\int (1+n^2\alpha^2)^{-\frac{3}{4}}\,d\alpha\right)$$

$$= O\left(n^{\frac{3}{4}+\epsilon}\sum_{q<n^{\frac{1}{4}}} q^{-\frac{1}{2}}\right) = O(n^{\frac{11}{8}+\epsilon}).$$

When $q>n^{\frac{1}{4}}$, we have

$$\int |\theta\psi\Psi|\,d\alpha = O\left\{\int |\theta^2\psi^2|\,d\alpha\int |\Psi|^2\,d\alpha\right\}^{\frac{1}{2}}$$

$$= O\left(n^{\frac{1}{2}+\epsilon}\left[n^3\sum_{p,q} q^{-3}\int_{\mathfrak{F}_{p,q}} (1+n^2\alpha^2)^{-\frac{3}{2}}\,d\alpha\right]^{\frac{1}{2}}\right)$$

$$= O\left(n^{\frac{1}{2}+\epsilon}\left[n^2\sum_{q>n^{\frac{1}{4}}} q^{-2}\right]^{\frac{1}{2}}\right) = O(n^{\frac{11}{8}+\epsilon}).$$

Thus we have, for all values of $q < n^{\frac{1}{3}}$,

$$\int |\theta\psi\Psi| \, d\alpha = O(n^{\frac{1}{6}+\epsilon}).$$

We still have to consider the second part of the integral (4.35). Again we have to divide up the range of values of q. When $q < n^{\frac{1}{3}}$, we use

$$\int |\theta\Psi^2| \, d\alpha \leqslant \Big[\int |\theta|^2 \, d\alpha \int |\Psi|^4 \, d\alpha\Big]^{\frac{1}{2}}.$$

Then, by (2.61),

$$\int |\theta|^2 \, d\alpha = O\Big(n \sum_{p,q} q^{-1} \int (1+n^2\alpha^2)^{-\frac{1}{2}} \, d\alpha\Big) = O\Big(n^{\epsilon} \sum_{q<n^{\frac{1}{3}}} 1\Big) = O(n^{\frac{1}{3}+\epsilon});$$

and, by (2.122) and (2.31),

$$\int |\Psi|^4 \, d\alpha = O\Big(n^{\frac{1}{2}+\epsilon} \int |\Psi|^2 \, d\alpha\Big) = O(n^{\frac{1}{2}+\epsilon}).$$

From these it follows that, when $q < n^{\frac{1}{3}}$,

$$\int |\theta\Psi^2| \, d\alpha = O(n^{\frac{1}{6}+\epsilon}).$$

When $q > n^{\frac{1}{3}}$, we write the integral in the form

$$\int_{\Gamma} |\theta\Psi^2| \, d\alpha \leqslant \operatorname{Max} \theta \int_{\Gamma} |\Psi|^2 \, d\alpha.$$

Now, from (2.61), we have

$$\max \theta = O\Big[\Big(\frac{n}{q}\Big)^{\frac{1}{2}}\Big] = O(n^{\frac{1}{3}})$$

and so we have, for all values of $q > n^{\frac{1}{3}}$,

$$\int_{\Gamma} |\theta\Psi^2| \, d\alpha = O(n^{\frac{1}{6}+\epsilon}).$$

Collecting these results we have the theorem stated.

Proof of the final formulae.

5.1. LEMMA 12. *If* $r+2s > 4$, *then*

$$\nu_{r,s}(n) > Bn^{\frac{1}{2}r+s-1}$$

for $n \geqslant n_0(r, s)$.

If $r+2s > 4$, it follows, from Theorems A, B, and C, that

$$\nu_{r,s}(n) \sim \frac{\pi^{\frac{1}{2}r}}{\Gamma(\frac{1}{2}r+s)} n^{\frac{1}{2}r+s-1} \mathfrak{S}_{r,s}(n),$$

and the sum of the singular series has been found in Lemma 11.

If s is even,

$$\prod_{\mathfrak{p}}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s}\right]>1.$$

If s is odd

$$\prod_{\mathfrak{p}}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s}\right]>\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-\mathfrak{p}}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s}\right)$$

$$>\prod_{\varpi}{}'\left(1-\frac{\varpi}{\varpi^{\frac{1}{2}r}(\varpi-1)^s}\right)=B.$$

This, taken in conjunction with (3.211), proves the lemma when $r\equiv 0 \pmod 4$. Also when s is even and $(n/\mathfrak{p})=-1$, or when s is odd and $(n/\mathfrak{p})=+1$,

$$\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s}{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s+(n/\mathfrak{p})(-1)^s}\right)>1.$$

For the other factors

$$\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s}{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s+1}\right)>\prod_{\mathfrak{p}}\left(\frac{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s-1}{\mathfrak{p}^{\frac{1}{2}(r-1)}(\mathfrak{p}-1)^s}\right)>\prod_{\varpi}\left(1-\frac{1}{\varpi^{\frac{1}{2}(r-1)}(\varpi-1)^s}\right)=B.$$

Hence, taking account of (3.212), we have a proof of the lemma when $r\equiv 1 \pmod 4$. From (3.213) we see that, when $r\equiv 2 \pmod 4$, we have still to consider

$$\prod_{\mathfrak{p}\ \equiv 3 \pmod 4}\left[\frac{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s-(-1)^s(\mathfrak{p}-1)}{\mathfrak{p}^{\frac{1}{2}r}(\mathfrak{p}-1)^s+(-1)^s}\right].$$

But this is exactly like the expression considered when $r\equiv 0 \pmod 4$, except that the two cases "s is odd" and "s is even" must be interchanged. Finally we see, from (3.214), that the proof for the case $r\equiv 3 \pmod 4$ is exactly like that for $r\equiv 1 \pmod 4$ if we replace $(n/\mathfrak{p})$ by $(-n/\mathfrak{p})$ whenever it occurs.

5.2. THEOREM 11. *If $r+2s>4$, then*

$$N_{r,s}(n)\sim\frac{\nu_{r,s}(n)}{(\log n)^s}. \tag{5.21}$$

We observe first that

$$N_{r,s}(n)=\sum_{m_1^2+m_2^2+\ldots+m_r^2+\varpi_1+\varpi_2+\ldots+\varpi_s=n}1\leqslant\sum_{m_1^2+m_2^2+\ldots+m_r^2+n_1+n_2+\ldots+n_s=n}1$$

$$\leqslant B\sum_{m_1}\sum_{m_2}\ldots\sum_{m_r}\sum_{n_1+n_2+\ldots+n_s=n-m_1^2-\ldots-m_r^2}1$$

where $m_k \leqslant \sqrt[r]{n}$, $(k = 1, 2, \ldots, r)$. Then

$$N_{r,s}(n) \leqslant B \sum_{m_1} \sum_{m_2} \ldots \sum_{m_r} (n - m_1^2 - m_2^2 - \ldots - m_r^2)^{s-1}$$

$$\leqslant B \sum_{m_1} \sum_{m_2} \ldots \sum_{m_{r-1}} n^{s-1}(n - m_1^2 - m_2^2 - \ldots - m_{r-1}^2)^{s-1} \leqslant \ldots$$

$$\leqslant B n^{s-1+\frac{1}{2}r}$$

and

$$\nu_{r,s}(n) = \sum_{m_1^2+\ldots+m_r^2+\varpi_1+\ldots+\varpi_s=n} \log \varpi_1 \ldots \log \varpi_s \tag{5.211}$$

$$\leqslant (\log n)^s N_{r,s}(n) < B n^{\frac{1}{2}r+s-1}(\log n)^s.$$

Now suppose that

$$\nu = \nu' + \nu'', \quad N = N' + N'', \tag{5.212}$$

where ν' and N' include all those terms of the summations for which

$$\varpi_j \geqslant n^{1-\delta} \quad (0 < \delta < 1, \quad j = 1, 2, \ldots, s).$$

Then plainly

$$\nu'_{r,s}(n) \geqslant (1-\delta)^s (\log n)^s N'_{r,s}(n). \tag{5.213}$$

Also

$$N''_{r,s}(n) \leqslant s \sum_{\varpi_j < n^{1-\delta}} \left(\sum_{m_1^2+\ldots+m_r^2+\varpi_1+\ldots+\varpi_{s-1}=n} 1 \right)$$

$$< B \sum_{\varpi_s < n^{1-\delta}} N_{r,s-1}(n - \varpi_s) < B(n^{1-\delta} \cdot n^{\frac{1}{2}r+s-2}) < B n^{\frac{1}{2}r+s-1-\delta},$$

$$\nu''_{r,s}(n) \leqslant (\log n)^s N''_{r,s}(n) < B n^{\frac{1}{2}r+s-1-\delta}(\log n)^s.$$

But $\nu_{r,s}(n) > B n^{\frac{1}{2}r+s-1}$ for $n > n_0(r, s)$, by Lemma 12; and so

$$(\log n)^s N''_{r,s}(n) = O\{\nu_{r,s}(n)\}, \quad \nu''_{r,s}(n) = O\{\nu_{r,s}(n)\}, \tag{5.214}$$

for every positive δ.

From (5.211), (5.212), (5.213), and (5.214) we deduce

$$(1-\delta)^s (\log n)^s (N_{r,s} - N''_{r,s}) \leqslant \nu_{r,s} - \nu''_{r,s} \leqslant (\log n)^s N_{r,s},$$

$$(1-\delta)^s (\log n)^s N_{r,s} \leqslant \nu_{r,s} + O(\nu_{r,s}) \leqslant (\log n)^s N_{r,s},$$

$$(1-\delta)^s \leqslant \underline{\lim} \frac{\nu_{r,s}}{(\log n)^s N_{r,s}}, \quad \overline{\lim} \frac{\nu_{r,s}}{(\log n)^s N_{r,s}} \leqslant 1.$$

Since δ is arbitrary, this proves (5.21).

5.3. THEOREM E. *Every large number n is the sum of two squares and two primes. The asymptotic formula for the number of representa-*

tions is

$$(5.31)\quad N_{2,2}(n) \sim \tfrac{1}{2}\pi C_{2,2}\frac{n^2}{(\log n)^2}\prod_{\mathfrak{p}\equiv 1 \pmod 4}\left(\frac{(\mathfrak{p}^2-\mathfrak{p}+1)(\mathfrak{p}-1)}{(\mathfrak{p}^3-2\mathfrak{p}^2+\mathfrak{p}-1)}\right)\times\prod_{\mathfrak{p}\equiv 3 \pmod 4}\left(\frac{(\mathfrak{p}^2-\mathfrak{p}-1)(\mathfrak{p}-1)}{(\mathfrak{p}^3-2\mathfrak{p}^2+\mathfrak{p}+1)}\right),$$

where $\mathfrak{p}$ *is an odd prime divisor of* n, *and*

$$(5.32)\quad C_{2,2} = \prod_{\varpi=3}^{\infty}\left\{1-\left(\frac{-1}{\varpi}\right)\frac{1}{\varpi(\varpi-1^2)}\right\}.$$

This follows at once from Theorem C, Lemma 11, equation (3.213), and Theorem D.

THEOREM F. *Every large number* n *is the sum of three squares and a prime. The asymptotic formula for the number of representations is*

$$(5.33)\quad N_{3,1}(n) \sim \tfrac{4}{3}\pi\frac{n^2}{\log n}C_{3,1}\Pi\left(\frac{(\mathfrak{p}-1)\mathfrak{p}}{\mathfrak{p}(\mathfrak{p}-1)-\{(-n)/\mathfrak{p}\}}\right),$$

where $\mathfrak{p}$ *is an odd prime divisor of* n, *and*

$$C_{3,1} = \prod_{\varpi=3}^{\infty}\left\{1-\left(\frac{-n}{\varpi}\right)\frac{1}{\varpi(\varpi-1)}\right\}.$$

This is a corollary of Theorems B and D and of equation (3.214) in Lemma 11.

THEOREM G. *Every large number* n *is the sum of one square and two primes. The asymptotic formula for the number of representations is*

$$N_{1,2}(n) \sim \tfrac{4}{3}\frac{n}{(\log n)^2}C_{1,2}\Pi\left(\frac{(\mathfrak{p}-1)^2}{(\mathfrak{p}-1)^2+(n/\mathfrak{p})}\right),$$

where $\mathfrak{p}$ *is an odd prime divisor of* n, *and*

$$C_{1,2} = \sum_{\varpi=3}^{\infty}\left\{1+\left(\frac{n}{\varpi}\right)\frac{1}{(\varpi-1)^2}\right\}.$$

This follows immediately from Theorems C and D, and equation (3.212) of Lemma 11.

PART II. *Results for almost all* n.

We must now define various new functions. As, in this part of the paper, we are concerned throughout with two squares and one prime,

we shall drop the suffixes in $\nu(n)$, $\mathfrak{S}(n)$, etc. We write

$$F(x) = \pi\Sigma\Omega(m)x^m = \pi\Sigma m\mathfrak{S}(m)x^m,$$

$$C_q(-m) = \sum_{r}' \exp(-2pm\pi i/q),$$

$$F_{p,q} = \pi\frac{\mu(q)}{\phi(q)}\left(\frac{S_{p,q}}{q}\right)^2 \sum_m m(xe^{-2p\pi i/q})^m = \pi\frac{\mu(q)}{\phi(q)}\left(\frac{S_{p,q}}{q}\right)^2 g(xe^{-2p\pi i/q}),$$

where

$$g(z) = \sum_m mz^m = \frac{z}{(1-z)^2},$$

so that

$$F = F(x) = \pi\sum_m mx^m \sum_q A_q(m) = \pi\sum_m mx^m \sum_q \frac{\mu(q)}{\phi(q)}\frac{1}{q^2}\sum_p (S_{p,q})^2 e^{-2pm\pi i/q}$$

$$= \pi\sum_{q,p}\frac{\mu(q)}{\phi(q)}\left(\frac{S_{p,q}}{q}\right)^2 \sum_m m(xe^{-2p\pi i/q})^m = \sum_{q,p} F_{p,q}.$$

Finally we write

$$\mathfrak{S}(m) = \mathfrak{S}_1(m) + \mathfrak{S}_2(m),$$

$$F = \pi\Sigma m\mathfrak{S}(m)x^m = \pi\Sigma m\mathfrak{S}_1(m)x^m + \pi\Sigma m\mathfrak{S}_2(m)x^m.$$

where the suffix 1 limits q to the range $q \leqslant \lambda$, and the suffix 2 to the range $q > \lambda$. Then

$$F_1 = \sum_{q\leqslant\lambda,\ p} F_{p,q}, \quad F_2 = \sum_{q>\lambda,\ p} F_{p,q}.$$

6.2. We shall need the following lemmas, in addition to those at the beginning of the paper.

LEMMA 13. *We have*

$$\Sigma\int_\xi |f\mathfrak{S}^2 - \psi\theta^2|^2\,da = O(n^{3+\epsilon})$$

for every positive ϵ.

For

$$|f\mathfrak{S}^2 - \psi\theta^2|^2 < A|\Psi^2\mathfrak{S}^4| + A|\psi(\mathfrak{S}^2-\theta^2)|^2$$

and

$$|\mathfrak{S}^4| < A|\theta|^4 + A|\Theta|^4.$$

Now, from (2.122) and (2.61),

$$\Sigma\int_\xi |\Psi^2\theta^4|\,da = O\left(n^{\frac{3}{2}+\epsilon}\cdot n^2\sum_{p,q} q^{-2}\int(1+n^2a^2)^{-1}\,da\right) = O(n^{3+\epsilon}).$$

Also, from (2.122) and (2.522),

$$\Sigma\int_\xi |\Psi^2\Theta^4|\,da = O(n^{3+\epsilon}).$$

We have, moreover,

$$|\psi^2(\Sigma^2-\theta^2)^2| < A|\psi^2\theta^2\Theta^2| + A|\psi^2\Theta^4|,$$

and from (2.121), (2.522), and (2.61),

$$\Sigma\int_{\xi}|\psi^2\theta^2\Theta^2|\,d\alpha = O\left(n^{\frac{1}{2}+\epsilon}\sum_{p,q} q^{-3}\int(1+n^2\alpha^2)^{-\frac{3}{2}}d\alpha\right) = O(n^{\frac{1}{2}+\epsilon}),$$

$$\Sigma\int_{\xi}|\psi^2\Theta^4|\,d\alpha = O\left(n^{1+\epsilon}\sum_{n,q} q^{-2}\int(1+n^2\alpha^2)^{-1}d\alpha\right) = O(n^{2+\epsilon}).$$

These results give the lemma.

6.3. LEMMA 14. *If* $q>1$, $|\mu(q)|>0$ *and* $q_1 q_2 = q$, *where* $(q_2, n)=1$, *then*

$$|A_q(n)| < \frac{A(\log q)^A}{q_1 q_2^2}.$$

For
$$A_q(n) = \frac{1}{\phi(q)}\frac{a_q}{q^2},$$

and
$$|a_q| = |\Sigma(S_{p,q})^2 e^{-2np\pi i/q}| \leqslant q|\Sigma e^{-2np\pi i/q}| = q|c_q(-n)|,$$

since
$$(S_{p,q})^2 = 0 \quad (q=2), = (-1)^{\frac{1}{2}q-1}\cdot q \quad (q \text{ odd}).$$

Hence
$$A_q(n) < \frac{A|c_q(-n)|}{q\phi(q)} < \frac{A(\log q)^A}{q^2}|c_q(-n)|.$$

But, if $q = \Pi\varpi_s$, where ϖ_s denotes a prime,

$$|c_q(-n)| = \prod_{\varpi_s/n}(\varpi_s - 1) < \prod_{\varpi_s/n}\varpi_s = q_1,$$

which gives the result of the lemma.

LEMMA 15. *If* $\lambda>1$, $n>1$, *then*

$$|\mathfrak{S}(n)| < A(\log n)^A d(n)\frac{(\log\lambda)^A}{\lambda^2}.$$

where $d(n)$ *is the number of divisors of* n.

As $|A_q(n)|$ satisfies the same inequality here as it does in P.N. 5, the proof of this lemma is identical with the corresponding lemma* in that paper.

* P.N. 5, Lemma 6, p. 50.

6.4. LEMMA 16. *If* $x = |x|e^{i\omega}$, *then*

$$\int_0^{2\pi} |F_2(x)|^2 d\omega = O\left(n^3 (\log n)^4 \frac{(\log \lambda)^A}{\lambda^2}\right).$$

This follows from P.N. 5, Lemma 7, p. 50, for the same reason.

LEMMA 17. *We have*

$$\Sigma \int_\xi |F_{p,q} - \theta^2 \psi|^2 d\alpha = O(1).$$

This follows immediately, for, from (2.121) and (2.521),

$$\begin{aligned} F_{p,q} - \theta^2\psi &= \pi \frac{\mu(q)}{\phi(q)} \left(\frac{S_{p,q}}{q}\right)^2 \left\{ g(xe^{-2p\pi i/q}) - \frac{1}{\{(1/n) - i\alpha)^2} \right\} \\ &= \pi \frac{\mu(q)}{\phi(q)} \left(\frac{S_{p,q}}{q}\right)^2 \left\{ \frac{xe^{-2p\pi i/q}}{(1 - xe^{-2p\pi i/q})^2} - \frac{1}{\{\log (1/xe^{-2p\pi i/q})\}^2} \right\} \\ &= O(1). \end{aligned}$$

LEMMA 18. *If* ξ_2 *denotes a typical Farey arc for which* $q > \lambda$, *then*

$$\Sigma \int_{\xi_2} |\theta^2 \psi|^2 d\alpha = O\left(n^3 \frac{(\log \lambda)^A}{\lambda^2}\right).$$

We have, from (2.121) and (2.61),

$$\int_{\xi_2} |\theta^2 \psi|^2 d\alpha = O\left(n^4 q^{-4} (\log q)^A \int \frac{d\alpha}{(1 + n^2\alpha^2)^2}\right) = O\{n^3 q^{-4} (\log q)^A\}.$$

Hence

$$\Sigma \int_{\xi_2} |\theta^2 \psi|^2 d\alpha = O\left(n^3 \sum_{q > \lambda} \frac{(\log q)^A}{q^3}\right) = O\left(n^3 \frac{(\log \lambda)^A}{\lambda^2}\right).$$

6.5. LEMMA 19. *If* x *lies on* $\xi_{p,q}$ *and* $\xi_{P,Q}$ *is a Farey arc other than* $\xi_{p,q}$, *then*

$$|F_{P,Q}| < \frac{An(\log Q)^A}{(pQ - Pq)^2}.$$

We have $$x = |x| e^{2p\pi i/q} \cdot e^{-i\alpha} = |x| e^{2P\pi i/Q} \cdot e^{-i\omega},$$

where $$\omega = 2\pi \left(\frac{P}{Q} - \frac{p}{q}\right) + \alpha,$$

and

$$\begin{aligned} |F_{P,Q}| &= \pi \left|\frac{\mu(Q)}{\phi(Q)}\right| \cdot \left|\frac{S_{P,Q}}{Q}\right|^2 |g(xe^{-2P\pi i/Q})| \\ &= \pi \left|\frac{\mu(Q)}{\phi(Q)}\right| \cdot \left|\frac{S_{P,Q}}{Q}\right|^2 |g(|x| e^{-i\omega})| < \frac{A(\log Q)^A}{Q^2 |\omega|^2}. \end{aligned}$$

Suppose first that these arcs are not adjacent in the Farey dissection Then

$$\left|2\pi\left(\frac{p}{q}-\frac{P}{Q}\right)\right| \geqslant \frac{4\pi}{qQ} \geqslant 2\,\frac{2\pi}{qN} \geqslant 2\,|\alpha|,$$

and so

$$|\omega| > A\left|\frac{p}{q}-\frac{P}{Q}\right| = A\,\frac{|pQ-qP|}{qQ},$$

$$F_{P,Q}| < \frac{A(\log Q)^A}{Q^2}\,\frac{q^2Q^2}{(pQ-qP)^2} < \frac{An(\log Q)^A}{(pQ-qP)^2},$$

since $q^2 \leqslant N^2 \leqslant n$. This argument fails when the arcs are adjacent, as in that case $|pQ-qP|=1$. Since x is outside the arc $\xi_{P,Q}$, $|\omega| > A/QN$, and so

$$|F_{P,Q}| < A\,\frac{(\log Q)^A}{Q^2}\,Q^2N^2 < An(\log Q)^A.$$

Hence the lemma is true in either case.

6.6. LEMMA 20. *If*

$$G = \overline{\Sigma}|F_{P,Q}|$$

where the symbol $\overline{\Sigma}$ *signifies summation over those pairs of values* (P, Q) *which are distinct from* (p, q) *for which* $Q \leqslant \lambda$, *then*

$$\Sigma \int_{\xi_{p,q}} G^2\,d\alpha = O\{n^3(\log n)^A\lambda^{-4}\}.$$

As in Lemmas 15 and 16, the proof may be taken direct from P.N. 5, Lemma 11, p. 52.

6.7. LEMMA 21. *The singular series*

$$\mathfrak{S}(n) = \sum_q \frac{u(q)}{\phi(q)} \sum_p \left(\frac{S_{p,q}}{q}\right)^2 e^{-2n\pi i/q}$$

is absolutely convergent.

For

$$|A_q| \leqslant \frac{A\,|c_q(-n)|}{q\phi(q)},$$

as in Lemma 14. Hence

$$|A_q| < Aq^{-2}(\log q)^A \sum_{\delta|q,\ \delta|n} \delta,$$

and so

$$\sum_q |A_q| < A\sum_q \frac{(\log q)^A}{q^2} \sum_{\delta|q,\ \delta|n} \delta \leqslant A\sum_q q^{-2+\epsilon}.$$

as $$\sum_{\delta/n} \delta = O(n^{\epsilon}),$$

which shows that the singular series is absolutely convergent.

It follows that the results of Lemma 10, 11, and 12 are true in this case.

7.1. THEOREM H. *For every positive* ϵ

$$\sum_{1}^{n} \{\nu(m)-\Omega(m)\}^2 = O(n^{\frac{5}{3}+\epsilon}).$$

It is sufficient to prove* that

$$\Sigma \{\nu(m)-\Omega(m)\}^2 |x|^{2m} = O(n^{\frac{2}{3}+\epsilon}),$$

when $|x| = e^{-1/n}$, or that

$$\int_0^{2\pi} |f\mathfrak{S}^2-F|^2 d\omega = O(n^{\frac{5}{3}+\epsilon}),$$

when $$x = |x| e^{i\omega} = e^{-(1/n)+i\omega}.$$

Now

$$\int_0^{2\pi} |f\mathfrak{S}^2-F|^2 d\omega = \Sigma \int_{\xi} |f\mathfrak{S}^2-F|^2 d\alpha$$

$$< A\Sigma \int_{\xi} |f\mathfrak{S}^2-\psi\theta^2|^2 d\alpha + A\Sigma \int_{\xi} |\psi\theta^2-F|^2 d\alpha,$$

and $$\Sigma \int_{\xi} |f\mathfrak{S}^2-\psi\theta^2|^2 d\alpha = O(n^{\frac{5}{3}+\epsilon})$$

by Lemma 13. It is, therefore, sufficient to prove that

$$\Sigma \int_{\xi} |\psi\theta^2-F|^2 d\alpha = O(n^{\frac{5}{3}+\epsilon}). \tag{7.11}$$

7.2. Since $F = F_1+F_2$ we have

$$\Sigma \int_{\xi} |\psi\theta^2-F|^2 d\alpha < A\Sigma \int_{\xi} |\psi\theta^2-F_1|^2 d\alpha + A\Sigma \int_{\xi} |F_2|^2 d\alpha \tag{7.21}$$

and

$$\Sigma \int_{\xi} |F_2|^2 d\alpha < An^3(\log n)^A \frac{(\log\lambda)^A}{\lambda^2}, \tag{7.22}$$

* Cf. P.N. 5, p. 54.

by Lemma 16. Next

$$F_1 = \sum_{P,\,Q \leqq \lambda} F_{P,\,Q} = F_{p,\,q} + \bar{\Sigma} F_{P,\,Q},$$

$$\text{or} \quad F_1 = \bar{\Sigma} F_{P,\,Q},$$

according as $q \leqq \lambda$ or $q > \lambda$, the range of summation indicated by $\bar{\Sigma}$ being defined as in Lemma 20. If $q \leqq \lambda$, we have

$$|\psi\theta^2 - F_1|^2 < A\,|\psi\theta^2 - F_{p,\,q}|^2 + A\,|\bar{\Sigma} F_{P,\,Q}|^2 < A\,|\psi\theta^2 - F_{p,\,q}|^2 + AG^2.$$

If, on the other hand, $q > \lambda$, we have

$$|\psi\theta^2 - F_1|^2 < A\,|\psi\theta^2|^2 + A\,|F_1|^2 < A\,|\psi\theta^2|^2 + A\,|\bar{\Sigma} F_{P,\,Q}|^2$$

$$< A\,|\psi\theta^2|^2 + AG^2.$$

Hence

$$(7.23) \quad \Sigma \int_\xi |\psi\theta^2 - F_1|^2\,d\alpha < A \sum_{q \leqq \lambda} \int_\xi |\psi\theta^2 - F_{p,\,q}|^2\,d\alpha$$

$$+ A \sum_{q > \lambda} \int_\xi |\psi\theta^2|^2\,d\alpha + A\Sigma \int_\xi G^2\,d\alpha$$

$$< A + A\,\frac{n^3(\log\lambda)^A}{\lambda^2} + An^{\frac{3}{2}}(\log n)^A \lambda^4,$$

by Lemmas 17, 18, and 20. From (7.21), (7.22), and (7.23) we now have

$$\Sigma \int_\xi |\psi\theta^2 - F|^2\,d\alpha < An^3(\log n)^A\,\frac{(\log\lambda)^A}{\lambda^2} + An^{\frac{3}{2}}(\log n)^A \lambda^4.$$

Taking $\lambda = [n^{\frac{1}{4}}]$, we obtain (7.11), and this completes the proof of the theorem.

7.3. LEMMA 22. *We have*

$$\mathfrak{S}(n) = C \prod_{\mathfrak{p} \equiv 1 \pmod 4} \left(\frac{(\mathfrak{p}-1)^2}{\mathfrak{p}^2 - \mathfrak{p} + 1}\right) \prod_{\mathfrak{p} \equiv 3 \pmod 4} \left(\frac{\mathfrak{p}^2 - 1}{\mathfrak{p}^2 - \mathfrak{p} - 1}\right) > A.$$

As was noted at the end of Lemma 21, the results of Lemmas 11 and 12 hold in this case. Hence, putting $r = 2$, $s = 1$, and using equation (3.213) we have the result.

THEOREM I. *The number of numbers less than n which cannot be represented as the sum of two squares and a prime is $O(n^{\frac{1}{2}+\epsilon})$.*

We have $$\Omega(m) = m\mathfrak{S}(m) > Am$$

from Lemma 22, and thus

$$\{\nu(m)-\Omega(m)\}^2 = \{\Omega(m)\}^2 > Am^2$$

for every number m for which the representation as two squares and a prime is impossible. The number of such numbers between $\frac{1}{2}n$ and n (inclusive of the limits) is therefore less than

$$B(\epsilon)n^{\frac{1}{2}+\epsilon},$$

where $B(\epsilon)$ is a function of ϵ only, and the number of such numbers less than n is less than

$$B(\epsilon)n^{\frac{1}{2}+\epsilon}\left(1+\left(\tfrac{1}{2}\right)^{\frac{1}{2}+\epsilon}+\left(\tfrac{1}{3}\right)^{\frac{1}{2}+\epsilon}+\ldots\right) < B(\epsilon)n^{\frac{1}{2}+\epsilon}.$$

LINEAR SYSTEMS OF ALGEBRAIC CURVES ON A PLANE AND ON A CONE

By W. V. D. HODGE.

[Received 5 March, 1928.—Read 8 March, 1928.]

LINEAR systems of plane algebraic curves of genus $\pi = 0$, 1, 2 have been investigated by Guccia*, Martinetti†, and others. Plane linear systems of general genus π have also been considered by Jung‡ in two papers. The general results of the theory are clearly applicable to any rational surface and are of interest as illustrating how the results of the modern theory of algebraic surfaces may be obtained by elementary methods for the simplest kind of algebraic surface. The investigation on these lines can be extended to ruled surfaces, and, indeed, to any surface on which we can find a set of systems of curves in terms of which any system of curves on the surface may be expressed symbolically§.

The present paper is in two sections : the first is devoted to researches on linear systems of plane algebraic curves, and the second to linear systems of curves on a cone. The results obtained are therefore applicable to any surface which can be birationally transformed into a ruled surface (rational or irrational).

In the first section we obtain some general theorems on plane linear systems of curves, and then attempt to solve the problem of determining all the birationally independent linear systems of curves of given genus on a plane. In this we confine ourselves to systems in which the base points are all ordinary singularities.

* *Rend. di Palermo*, 1 (1887), 338–349.

† *Lombardo Ist. Rend.* (2), 20 (1887), 264–269.
Ann. di Mat. (2), 15 (1888), 277–312; 16 (1889), 291–327.

§ See Picard and Simart, *Fonctions algébriques de deux variables*, 2 (1906), 498.

In the second section we obtain by elementary methods a number of theorems which have been obtained by Severi* for a class of surfaces of which ruled surfaces form a special case, and then consider the problem of the enumeration of systems of curves of given genus on a ruled surface.

I. *Linear systems of curves in a plane.*

1. We say that linear systems of plane algebraic curves which can be obtained from one another by means of birational transformations of the plane belong to the same *family*. We assume throughout that the general curve of a linear system is irreducible, and that the systems which we speak of are complete, being subject to no restrictions except those imposed by the base points. We show that in each family there is one system which cannot by any Cremona transformation have its order reduced, and that, as regards the number and multiplicity of its base points, this system is unique. We call such a system a *minimal* system, and in general we represent a family by its minimal system.

In accordance with the usual practice we distinguish two methods of description of a linear system—the virtual and the effective. In the virtual description a linear system is defined as the system formed by the curves of order n which satisfy the conditions of having the points $a_1, a_2, \ldots, a_t$ as base points of multiplicities $r_1, r_2, \ldots, r_t$. The grade D is defined as

$$D = n^2 - \sum_1^t r_i^2,$$

and the genus as

$$\pi = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2}\sum_1^t r_i(r_i-1).$$

The dimension is given by

$$k = D - \pi + 1.$$

On the other hand, the effective description is as follows. $a_1, a_2, \ldots, a_t$ are now the actual base points of the system, and $r_1, r_2, \ldots, r_t$ are the multiplicities of the curves at these points. The grade of the system is then the number of variable intersections of two curves of the system, and hence is

$$D = n^2 - \sum_1^t r_i^2.$$

* *Mem. della R. Acc. di Torino* (2), 54 (1904), 1-49.

and the genus is

$$\pi = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2}\sum_1^t r_i(r_i-1).$$

The dimension k is the number of independent conditions that the curves of the system may be made to satisfy. Since the characteristic series of a complete system is complete, we have

$$k-1 = D-\pi+\tau,$$

where τ is the index of speciality of the characteristic series. Hence

$$k = D-\pi+\tau+1,$$

and τ is called the *superabundance* of the system. If $\tau = 0$, the system is said to be *regular*, and the conditions imposed by the base points are independent. Thus $\tau = 0$ is the necessary and sufficient condition that the virtual and effective descriptions of a system should be identical.

The terms defined are clearly invariants of the system under birational transformations of the plane. What follows may be applied equally well to the virtual or effective descriptions of a system.

2. A *fundamental curve* of a linear system is a curve which meets the general curve of the system only in base points. If a fundamental curve is reducible, each of its components is a fundamental curve, so we shall suppose all fundamental curves to be irreducible. Then the order of a fundamental curve is less than the order of the system. For if we make a curve of the system pass through one point of the fundamental curve it contains the fundamental curve as part, and so the latter must be a component of a reducible curve of the system, or must itself be a curve of the system, which is then a pencil. But in the latter case we do not regard the curves of the pencil as proper fundamental curves.

There are only a finite number of fundamental curves for any linear system. For, otherwise, there would be a fundamental curve through any point of the plane, and hence any curve of the system would be reducible. We define the *excess* ϵ of a linear system as the number of base points less the number of fundamental curves. We prove that ϵ is an invariant of the systems of a family; indeed, if the system be represented projectively by the hyperplane sections of a rational surface, $\omega = 10-\epsilon$ is the relative invariant of this surface which is closely related to the linear genus of the class of rational surfaces.

As any Cremona transformation can be made up of a series of quadratic transformations with distinct fundamental points, it is sufficient to prove

the invariance of ϵ for such a transformation. Let A, B, C be the fundamental points, and let these be base points of the system of multiplicity r, s, t. Any base point or fundamental curve in the first plane, other than A, B, C, or BC, CA, AB, remains such for the transformed system. The condition that BC should be a fundamental curve of the system in the first plane is

$$s+t+X=n,$$

where X is the sum of the multiplicities of the α other base points on BC. When this relation holds there are α base points at A', the point of the second plane corresponding to BC; but, if BC is not a fundamental curve of the first system, there are $\alpha+1$ base points at A'. Hence, corresponding to the disappearance of the fundamental curve BC of the first plane, there is the disappearance of a base point. The same argument holds for the fundamental curves of the second plane. Hence it follows that the excess must be invariant.

3. The *adjoint system* of index i of a linear system of curves of order n is defined as the linear system of curves of order $n-3i$, whose virtual base points are the base points of the system of multiplicity r greater than i. In these points the adjoint system has base points of multiplicity $r-i$. Adjoint systems are known to be regular, and it can be shown that they are invariant under Cremona transformations of the plane. Indeed, if we apply the transformation of the previous paragraph, the first adjoint system of the given system becomes a system which, when taken with the fundamental curves of the second plane which arise from the points A, B, C which are not base points of the system, satisfies all the conditions of adjointness of the transformed system. A similar argument applies to adjoint systems of higher index.

4. Let $|F|$ be a linear system of curves of order n having base points $a_1, a_2, \ldots, a_t$ of multiplicities $r_1, r_2, \ldots, r_t$, where we may suppose that

$$r_1 \geqslant r_2 \geqslant \ldots \geqslant r_t.$$

If the base points are ordinary singularities, and are of general position, in the sense that the conditions that any curve of order less than n should have the given multiplicities at these points are independent, the necessary and sufficient condition that the system should be minimal is

$$r_1+r_2+r_3 \leqslant n.$$

For if

$$r_1+r_2+r_3 > n,$$

a quadratic transformation with a_1, a_2, a_3 as fundamental points transforms the system into one of order

$$2n-r_1-r_2-r_3 < n.$$

This proves that the condition is necessary. To prove it sufficient, consider any Cremona transformation

$$T_m = \begin{pmatrix} v_1^{s_1}, & v_2^{s_2}, & \ldots, & v_\theta^{s_\theta}, & \ldots, & v_\phi^{s_\phi} \\ w_1^{\sigma_1}, & w_2^{\sigma_2}, & \ldots & \ldots & \ldots, & w_\phi^{\sigma_\phi} \end{pmatrix},$$

where m is the order of the transformation, v_1, v_2, ..., v_ϕ are the fundamental points of the first plane of multiplicities s_1, s_2, ..., s_ϕ, of which v_1, v_2, ..., v_θ coincide with a_1', a_2', ..., a_θ', θ base points of $|F|$, and w and σ similarly refer to the second plane. The transformed system will be of order

$$N = mn - \sum_1^\theta s_k r_k'.$$

From the theory of Cremona transformations, we have

$$\sum_1^\phi s_k = 3(m-1) \quad (s_k \leqslant m-1),$$

and therefore

$$\sum_1^\theta s_k r_k' \leqslant (m-1)(r_1+r_2+r_3).$$

Hence

$$N \geqslant n+(m-1)(n-r_1-r_2-r_3)$$

$$\geqslant n$$

if our condition is satisfied. If equality holds, we have either $m = 1$, in which case the transformation is a homography, or $m \geqslant 2$, in which case we require

$$n = r_1+r_2+r_3,$$

$$\sum_1^\theta s_k r_k' = (m-1)(r_1+r_2+r_3),$$

$$\theta = \phi.$$

If $m = 2$, the transformation is quadratic with base points a_1, a_2, a_3.

If $m > 2$, we have either

(1) $$r_1 = r_2 = \ldots = r_\phi \quad (a_k = a_k'),$$

or

(2) $$s_1 = m-1, \quad s_2 = s_3 = \ldots s_\phi = 1,$$

$$r_1' = r_1, \quad r_2 = r_3 = \ldots = r_\phi \quad (a_k = a_k').$$

None of these transformations alters the description of the system and hence the sufficiency of our condition is established, together with the uniqueness of the minimal system. It will be noticed that if a system is minimal all its adjoint systems are minimal.

5. We now consider the connection between the number of fundamental curves of a system and the condition that the system should be minimal. The following is an extension of a theorem due to Caporali* : *If a linear system of curves with base points of general position, in the sense of the previous paragraph, possesses a fundamental curve* Γ, *this curve is rational and the system is not minimal.* There are two exceptions :

(1) If Γ is a straight line joining two points of multiplicity r and s, such that $r+s=n$, and the system has no other base points, the system is minimal by the theorem just proved.

(2) If the system consists of a pencil of elliptic curves of order $3m$ ($m>1$) with 9 m-ple points, Γ is an elliptic cubic and the system is minimal.

Suppose that Γ is a curve of order ν which has in $a_1', a_2', \ldots, a_\phi'$ points of multiplicities $s_1, s_2, \ldots, s_\phi$. Then

$$\sum_1^\phi s_k r_k' = n\nu, \qquad (a)$$

$$\sum_1^\phi s_k(s_k+1) = \nu(\nu+3),$$

since the base points are of general position. Now, if $k \geqslant 2$, we can find a curve of the system to touch a line not tangent to Γ at a point of Γ. This curve breaks up into Γ and another curve Γ' of order $n-\nu$ which meets Γ in at least one point outside the base points. Hence

$$\nu(n-\nu) - \sum_1^\phi s_k(r_k'-s_k) \geqslant 1,$$

and from this we deduce that

$$\sum_1^\phi s_k(s_k-1) \geqslant (\nu-1)(\nu-2).$$

But Γ is irreducible, hence equality only can hold and Γ is rational.

* Caporali, *Mem. di Geom.* (1888), 171.

Further, we have $\sum_1^\phi s_k = 3\nu - 1.$

If we write sr as the sum of s terms each equal to r, the relation (a) becomes

$$\epsilon_1 + \epsilon_2 + \ldots + \epsilon_\nu = n\nu,$$

where $$\epsilon_j = r_\alpha + r_\beta + r_\gamma \leqslant r_1 + r_2 + r_3 \quad (j \neq \nu)$$

$$\epsilon_\nu = r_\xi + r_\eta.$$

Hence, if $\nu > 1$, $$\nu(r_1 + r_2 + r_3) > n\nu,$$

and the system is not minimal. If $\nu = 1$, $\epsilon_\nu = n$, and the system is, or is not, minimal, according as there is not, or is, another base point, and the exceptional case (1) may arise.

If $k = 1$, we cannot prove that Γ' meets Γ. If it does, the preceding argument holds. If not, we have

$$\nu(n - \nu) = \sum_1^\phi s_k(r_k' - s_k),$$

and hence Γ is elliptic, and

$$\sum_1^\phi s_k = 3\nu.$$

Proceeding, as before, we now have

$$\epsilon_1 + \epsilon_2 + \ldots + \epsilon_\nu = n\nu,$$

where $$\epsilon_i = r_\alpha + r_\beta + r_\gamma \leqslant r_1 + r_2 + r_3 \quad (j \leqslant \nu),$$

and hence $$r_1 + r_2 + r_3 \geqslant n.$$

Equality only holds when

$$r_1' = r_2' = \ldots = r_\phi' = r_1 = r_2 = r_3 = r,$$

and $n = 3r$. The equation

$$\Sigma r^2 = n^2 - D$$

now becomes $$\phi r^2 + \Sigma' r^2 = 9r^2 - D,$$

where Σ' denotes summation over base points of the system which do not lie on Γ. Hence $\phi \leqslant 9$. But Γ is elliptic, so we require $\phi = 9$, and the exceptional case (2) arises.

If the base points are of general position, the adjoint system of index i contains a fundamental curve as a fixed component counted i times. We may leave out the case in which the system is a pencil of

elliptic curves, for which the adjoint systems are unique curves. For rational fundamental curves we have

$$3\nu-1 = \sum_1^{\phi} s_k.$$

But the fundamental curve meets the first adjoint system in

$$\sum_1^{\phi} s_k(r_k'-1) = n\nu - \sum_1^{\phi} s_k$$
$$= (n-3)\nu+1$$

points, and hence is a fixed component. The fundamental curve meets the residual system in

$$\Sigma s_k(r_k'-s_k-1) = (n-3)\nu+1-\Sigma s_k^2$$
$$= (n-3-\nu)\nu$$

points, which shows that Γ is reckoned simply as a component of the adjoint system. Again, if $|F_i|$ denotes adjoint system of index i,

$$|F_i| = |F+i(F_1-F)|,$$

and hence $|F_i|$ contains the fundamental curve counted i times. Conversely, a fixed component of the first adjoint system must be a fundamental curve, since otherwise, on a general curve of $|F|$, the canonical series would have fixed points. Hence it follows that, if $|F|$ is a minimal system, the only fixed component of any adjoint system is a straight line [except in the exceptional case (2)].

6. For convenience in what follows, we here quote the results of Guccia* for linear systems of genus $\pi = 0, 1$. Guccia has shown that any complete linear system of rational curves with ordinary base points is transformable into one of the following systems :

(1) A pencil of lines.

(2) The system formed by all the lines of a plane.

(3) The system formed by all the conics of a plane.

(4) A system of curves of order n with an $(n-1)$-fold base point.

(5) The system (4) with a simple base point added.

* *Loc. cit.*

A complete linear system of elliptic curves in which the base points are all ordinary multiple points can be transformed to one of the following systems :

(1) The system of cubics with r simple base points ($r = 1, \ldots, 8$)

(2) The system of quartics with two double base points.

(3) The system of curves of order $3m$ with 9 m-ple base points*.

It will also be useful to notice that, if $|F|$ is a minimal system of genus π greater than 2, whose adjoint system $|F_1|$ is formed by $\pi-1$ curves of a pencil, this pencil is formed by straight lines. Suppose, first, that $|F|$ has a fundamental curve. It has then two base points of multiplicities r and s ($r \geqslant s$). $|F_1|$ consists of the line joining the base points and the system of curves of order $n-4$ with an $(r-2)$-fold point and an $(s-2)$-fold point. This is only reducible if $r = n-2$, in which case it consists of $n-4$ lines of a pencil. Now suppose that $|F|$ has no fundamental curve. Then, if P is a base point of the pencil $|C|$ from which $|F_1|$ is formed, not at a base point of $|F|$, the curves of $|F|$ which pass through P must consist of $k-1$ curves of $|C|$. Thus $|C|$ must be a pencil of cubics. But there is no reducible system of curves formed by r ($r > 1$) cubic curves of a pencil which form an adjoint system†. Thus $|C|$ has no base points outside the base points of $|F|$. Hence, if the curves of $|C|$ are of genus p and order m, and have s-fold points in the r-fold base points of $|F|$,

$$\Sigma s^2 = m^2, \quad \Sigma s = 3m+2p-2.$$

But $$(\pi-1)s = r-1, \quad (\pi-1)m = n-3,$$

hence

$$\Sigma(r-1)^2 = (n-3)^2, \quad \Sigma(r-1) = 3(n-3)+2(p-1)(\pi-1),$$

and therefore $$\Sigma r(r-1) = n(n-3)+2(p-1)(\pi-1),$$

i.e. $$2p(\pi-1) = 0,$$

and since $\pi > 2$, $p = 0$. Thus $|C|$ is a pencil of lines.

Finally, we notice that if, for a given system of dimension $k > 0$, the adjoint system of index i does not exist, no adjoint system of index greater than i exists. This is clearly true for systems of genus $\pi = 0$, 1,

* Halphen, *Bull. Soc. Math. de France*, 10 (1882), 162–172.

† Enriques, *Teoria geom. d. equazioni*, 3 (1924), 197.

and for systems whose adjoint systems are formed by lines of a pencil. Suppose, if possible, that, for a given system $|F|$, the adjoint systems $|F_i|$ of index $i < i_0$ exist, but that $|F_{i_0}|$ does not exist, and that $|F_{(i_0+l)}|$ does exist for some $l > 0$. Then, from our knowledge of the case $\pi = 0$, it follows that $|F_{(i_0-1)}|$ is a unique rational curve, and hence $|F_{(i_0-2)}|$ is a system of elliptic curves. From our knowledge of the case $\pi = 1$, we see that $|F_{(i_0-2)}|$ is a unique curve. Proceeding in this way we see that $|F|$ is a unique curve. Thus our theorem is proved. We observe that, if $\pi > 1$, we do not require the restriction $k > 0$.

7. In enumerating the birationally independent systems of curves of genus π, we may suppose that $\pi > 1$, as the cases $\pi = 0$, $\pi = 1$ have already been dealt with.

If π can be written in the form

$$\pi = \tfrac{1}{2}(n-1)(n-2),$$

where n is a positive integer, there exists of a system of curves of genus π without base points. For this system

$$D = n^2, \quad k = \tfrac{1}{2}n(n+3), \quad \epsilon = 0.$$

If the minimal system of curves of genus π is of order n and has just one base point of multiplicity $r = n-i$, we have

$$(n-i)(n-i-1) = (n-1)(n-2)-2\pi,$$

i.e.

$$\pi = (i-1)\,r+\tfrac{1}{2}(i-1)(i-2),$$

where $i \geqslant 2$, since, if $i = 1$, $\pi = 0$.

This system has only a finite number of solutions for r and i in positive integers. Hence there are only a finite number of minimal systems of curves of genus π with one base point. Also

$$\begin{aligned} D &= n^2-r^2 \\ &= 2\pi+2i+\frac{2\pi}{i-1}. \end{aligned}$$

Since D is an integer, we must have

$$i = 2\pi+1 \quad \text{or} \quad i \leqslant \pi+1.$$

The first does not lead to a possible solution of our equation, and, in the second, equality is possible only when $\pi > 1$, if $\pi = 3$. But when $r = 0$ and the system has not a base point. Hence, if the system has just one base point of multiplicity $n-i$,

$$2 \leqslant i \leqslant \pi.$$

Regarding D as a function of i, we see that D is greatest when $i=2$, when $D=4\pi+4$. The system consists of curves of order $\pi+2$ with a π-fold point. This value for the grade is, moreover, greater than the grade of the system of genus $\pi(\pi>1)$ without base points, if such exists, except in the case $\pi=3$, when the system of quartics is of grade $4\pi+4=16$.

Now consider minimal systems of curves of genus π with two base points of multiplicities r and s. We may suppose that $r\geqslant s\geqslant 2$ as the case in which $s=1$ may be deduced from the foregoing by the imposition of a simple base point, which does not alter the genus. Let the order of the curves be

$$n=r+s+i \quad (i\geqslant 0).$$

Then $$\pi=rs+(i-1)(r+s)+\tfrac{1}{2}(i-1)(i-2).$$

Since $$\tfrac{1}{2}(i-1)(i-2)\geqslant 0 \quad \text{and} \quad r\geqslant s\geqslant 2,$$

we have $$0\leqslant i\leqslant \tfrac{1}{4}\pi.$$

Again, it is clear that, given π, there are only a finite number of solutions of this equation for i, r, s in positive integers, and hence there are only a finite number of minimal systems of curves of genus π with two base points. Now for such a system

$$D=2(\pi+r+s+i-1)+i$$

$$=2\left(\pi+\frac{\pi+\tfrac{1}{2}i(i-1)}{s+i-1}+s\right)+i=2\left(\pi+\frac{\pi+\tfrac{1}{2}s(s-1)}{s+i-1}+i\right)+s.$$

Taking the second form, regard D as a function of s for a fixed value of i. We allow s to vary from $s=2$ to the value corresponding to $r=2$. By symmetry D has the same values at the two extremities of the range, and from the nature of the function we see that D has there its greatest values. Thus the greatest value of D, for a given value of i, is given by $s=2$. Similarly, taking the third form, we examine the variation of D with i, as s remains constant. Here $0\leqslant i\leqslant \tfrac{1}{2}\pi$, and, by a method similar to that already used, we can show that D has its greatest value for $i=0$. Hence, for a minimal system of curves of genus π with two base points, the grade is greatest when the system consists of curves of order $\pi+3$, with a $(\pi+1)$-fold point and a double point. Then $D=4\pi+4$. We shall see immediately that, if a minimal system has more than two base points, its grade is less than $4\pi+4$. Therefore we have proved that, if $\pi>1$, the grade of a linear system of curves of genus π never exceeds $4\pi+4$, and that this limit is reached by the systems formed by curves of

order $\pi+2$ with a π-fold base point, and by curves of order $\pi+3$ with a $(\pi+1)$-fold point and a double point as base points. In the case $\pi = 3$, this limit is also reached by the system of plane quartics*. All the systems mentioned are minimal.

Other systems of genus π may be deduced from the foregoing by the imposition of simple base points. We may therefore confine ourselves in the next paragraph to systems for which $r_3 \geqslant 2$.

8. To consider the enumeration of minimal systems of curves of genus π, with more than two base points, we recall the equations

$$\Sigma r_i^2 = n^2 - D, \tag{1}$$

$$\sum_i r_i(r_i-1) = (n-1)(n-2) - 2\pi, \tag{2}$$

$$\sum_i r_i = 3n - D + 2\pi - 2. \tag{3}$$

We suppose that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_t$,

and, since the system is minimal,

$$r_1 + r_2 + r_3 \leqslant n. \tag{4}$$

We suppose that $r_3 \geqslant 2$. Then

$$r_3 \sum_4^t r_i \geqslant \sum_4^t r_i^2;$$

hence $$r_3(3n - D + 2\pi - 2) - r_3(r_1 + r_2) \geqslant n^2 - r_1^2 - r_2^2 - D,$$

i.e. $$n(n-3r_3) + (r_3-1)(D-2\pi+2) - r_1(r_1-r_3) - r_2(r_2-r_3) \leqslant 2\pi - 2, \tag{5}$$

and from this, taken with (4), we deduce

$$2(r_1 r_2 - r_3^2) + (r_3-1)(D-2\pi+2) \leqslant 2\pi - 2. \tag{6}$$

Now $$r_1 r_2 \geqslant r_3^2, \quad r_3 \geqslant 2;$$

hence for all systems of the type considered

$$D \leqslant 4\pi - 4.$$

We notice that equality only holds for sextics with t double points $(3 \leqslant t \leqslant 9)$.

Any system of genus π must be such that the multiplicities of its base points satisfy equation (3). Now, if the right-hand side of this equation is fixed, we see at once that there can only be a finite number of values

* This theorem is due to Castelnuovo, *Ann. di Mat.* (2), 18 (1890), 119–128.

$r_1, r_2, \ldots, r_t$, which satisfy this equation. If π is given, there are only a finite number of possible values of D, and if we can show that there are only a finite number of values of n for which the system may be minimal, we shall then be able to find all the possible values of $n, D, r_1, r_2, \ldots, r_t$ which satisfy (3), and also (2) and (4), and therefore all the minimal systems of genus π, and show that there are only a finite number of them. This paragraph is therefore devoted to finding a limit for n.

For this purpose suppose that the t base points consist of α_1 simple base points, α_2 double base points, etc. Then equations (1) and (3) become

$$\Sigma r^2 \alpha_r = n^2 - D,$$

$$\Sigma r \alpha_r = 3n - D + 2\pi - 2.$$

Consider the linear series cut out on a general curve of the system by the adjoint curves of index i (supposed existing). This may not, of course, be a complete series. The order of the series is

$$n(n-3i) - \sum_{r=1}^{+1} r(r-i)\alpha_r \leqslant D - i(D-2\pi+2).$$

Hence, if $D > 2\pi-2$, there are, at most, $D/(D-2\pi+2) \leqslant 2\pi-1$ adjoint systems. If i_0 be the index of the last adjoint system which exists, and I_0 the order of the series cut by this system on a general curve of the given system, the adjoint system is of dimension less than I_0. Now this adjoint system is minimal, hence it consists of a system of rational curves of order less than $\frac{1}{2}(I_0+1)$, or of reducible curves formed by lines of a pencil, with, possibly, a fixed line added, counted, at most, i_0 times. In any case, the adjoint curves are of order I_0+i_0 at most, and therefore the original system is, at most, of order

$$4i_0 + I_0,$$

to which a limit is simply determined in terms of π.

Now consider the case in which $D = 2\pi-2$. Equation (6) then becomes

$$r_1 r_2 - r_3^2 \leqslant \pi - 1.$$

Put $$r_1 = \eta_1 + r_3, \quad r_2 = \eta_2 + r_3,$$

and this becomes $$(\eta_1+\eta_2) r_3 + \eta_1 \eta_2 \leqslant \pi - 1,$$

and, except when $\eta_1 = \eta_2 = 0$, we can deduce limits for r_1, r_2, r_3. In this case we have $n > 3r_3$, and hence, from (5), we have

$$n \leqslant 2\pi - 2 + r_1(r_1 - r_3) + r_2(r_2 - r_3),$$

and, as limits have been obtained for all the quantities on the right-hand side, we can deduce a limit for n in terms of π.

If, however, $\eta_1 = \eta_2 = 0$, we have $r_1 = r_2 = r_3 = R$, but no limit for R. We have $n \geqslant 3R$. If $n > 3R$, equation (5) gives

$$n < 2\pi - 2.$$

Now consider the case in which $n = 3R$. Equations (1) and (3) become

$$3R^2 + \sum_1 r_i^2 = 9R^2 - 2\pi + 2,$$

$$3R + \sum_4 r_i = 9R,$$

and hence

$$\sum_4 r_i(R - r_i) = 2\pi - 2.$$

We suppose that $\pi > 1$. Since $r_i \leqslant R$, we at once deduce from the equation that $R \leqslant 2\pi - 1$.

We have therefore established a limit for n for all systems in which $D \geqslant 2\pi - 2$. To solve the remaining case, we shall establish a limit for the number of adjoint systems which may exist. The proof will then follow as before.

In deducing a limit for i for systems in which $D < 2\pi - 2$, let D_i, π_i be the grade and genus of $|F_i|$. Remove the simple base points of $|F|$. Then, if $D + \alpha_1 \geqslant 2\pi - 2$, we proceed as in the foregoing. If $D + \alpha_1 < 2\pi - 2$, the system $|F_2|$ cuts on a curve F a series of order

$$4(\pi - 1) - D - \alpha_1 > 2\pi - 2,$$

which is therefore not special. Hence

$$\tfrac{1}{2}(n-3)(n-6) - \tfrac{1}{2}\sum_2 (r-1)(r-2)\,\alpha_r \leqslant 4(\pi-1) - D - \alpha_1 - \pi,$$

i.e.

$$\sum_2 \alpha_r \geqslant 10.$$

Now

$$\begin{aligned} D_1 - 2\pi_1 + 2 &= 3n - 9 - \sum_2 (r-1)\,\alpha_r \\ &= (D + \alpha_1 - 2\pi + 2) + \left(\sum_2 \alpha_r - 9\right) \\ &\geqslant (D + \alpha_1 - 2\pi + 2) + 1. \end{aligned}$$

If $D_1 + \alpha_2 \geqslant 2\pi_1 - 2$, we can establish a limit for the number of adjoint systems of $|F_1|$, and hence of $|F|$, in terms of $\pi_1 \leqslant 3(\pi - 1)$. If this relation does not hold, we proceed another stage and repeat the reasoning. Then clearly, for some value of i,

$$i \leqslant 2\pi - 2 - D - \alpha_1 \leqslant 2\pi - 2,$$

we obtain an adjoint system for which

$$D_i + a_{i+1} \geqslant 2\pi_i - 2;$$

and

$$\pi_i \leqslant 3^i(\pi - 1).$$

Hence we can always establish a limit for the number of adjoint systems that may exist.

Thus we have proved that *there are only a finite number of birationally independent linear systems of curves of genus π, with only ordinary base points.*

II. *Linear systems of curves on a cone.*

1. In considering linear systems of curves on a ruled surface, we take as our projective representation of the surface a cone of order n and genus p $(p > 0)$, in space of three dimensions. The terms grade, genus, dimension, etc., used have the same meaning as in Section I, and we make conventions similar to those of § 1 of that section.

We first prove that any curve on a cone can be obtained as the complete intersection of the cone with a surface whose only other intersection with the cone consists of a number of generators. Let α, β be two plane sections of the cone not through the vertex O. Let K be the curve considered, and let it meet the planes of α and β in sets A and B of m points. Through the set A draw any curve of order l in the plane of α, adjoint to α, meeting it in the multiple points, the set A, and a residual set Γ. We have to prove that the generators of the cone through Γ meet β in a set of points Γ', which, together with B, makes up the complete intersection of β with an adjoint curve of order l which does not contain β as a part, other than at the multiple points. This will be true if the generators through A and B form co-residual sets. But there is a $(1, k)$ correspondence between the generators and the points of the curve K, where k is the number of points, other than O, in which a generator meets K. Also A and B are co-residual on K, hence the corresponding sets of generators are co-residual. Thus an adjoint curve of order l passes through Γ' and B.

Now let q be any line, not meeting K or the generators through Γ Any plane through q contains ∞^s $(s \geqslant 0)$ curves of order l adjoint to the section of the cone, meeting this section in the points of K and the points of the generators through Γ which lie in the plane, one passing through s arbitrary fixed points of q. As the plane turns about q this curve describes a surface of order $l+r$, passing r times through q. This surface meets the cone in the multiple lines, the curve K, the generators

through Γ, and a residual curve K' of order rn, which must, however, meet any plane through q only in points on q. Thus K' must consist of a number of plane sections. But it is clear that this can only be so if K' consists of the section by the plane qO counted r times. Thus K is the section of the cone by a surface whose residual intersection with the cone consists of generators. Further, when l is taken sufficiently large, this surface will satisfy all the conditions of adjointness to the cone.

Through K construct any surface of order N, adjoint to the cone, which meets the cone only in K and along generators. Denote by G the residual intersection of the surface with the cone. Then, by the "Restsatz" for surfaces, the system of curves cut by all the surfaces of order N through G is the complete system to which K belongs. Suppose that these surfaces have an R-fold point at O. Then a particular surface of the system consists of a cone of order R through G, and any surface of order $k = N-R$. The former cuts the cone in G, and a set G_s of s generators, co-residual to the s tangents to K at O. The latter determines a curve of the system $|kC|$, where $|C|$ is the system of plane sections of the cone, and k is the number of points in which a generator meets K. Thus we may denote the complete system $|K|$, to which K belongs, by $|kC+G_s|$, and any linear system of curves on the cone can be deduced from a system of this type by the imposition of base points.

2. We now obtain a formula for the equivalence of multiple lines common to three surfaces. Let F_1, F_2, F_3 be three surfaces of order l_1, l_2, l_3 respectively; then, if their intersections are quite general, they have $l_1 l_2 l_3$ points in common. If the surfaces have a point O in common of multiplicities λ_1, λ_2, λ_3 for the surfaces F_1, F_2, F_3 respectively, $\lambda_1\lambda_2\lambda_3$ intersections are absorbed there. Suppose that through O there is a line r common to the three surfaces and multiple for them to the orders ρ_1, ρ_2, ρ_3 respectively. We suppose that the three surfaces have no point in common on r of multiplicity greater than the rest of the line, except O.

Consider the curve of intersection of F_1 and F_2. This consists of the line r, counted $\rho_1\rho_2$ times, and a curve Γ of order $l_1 l_2 - \rho_1\rho_2$, which has a $(\lambda_1\lambda_2 - \rho_1\rho_2)$-fold point at O. Consider any plane through r. This meets F_1 and F_2 in the line r, counted ρ_1 and ρ_2 times respectively, and in curves of order $l_1-\rho_1$ and $l_2-\rho_2$ passing $\lambda_1-\rho_1$ and $\lambda_2-\rho_2$ times through O. These therefore meet in

$$(l_1-\rho_1)(l_2-\rho_2)-(\lambda_1-\rho_1)(\lambda_2-\rho_2) = l_1 l_2 - \lambda_1\lambda_2 - \rho_1(l_2-\lambda_2) - \rho_2(l_1-\lambda_1)$$

points not on r. This gives the number of intersections of Γ with the

plane which are not on r. Hence the number of intersections of Γ and r is

$$(l_1 l_2 - \rho_1\rho_2) - \{l_1 l_2 - \lambda_1\lambda_2 - \rho_1(l_2-\lambda_2) - \rho_2(l_1-\lambda_1)\},$$

of which $\lambda_1\lambda_2 - \rho_1\rho_2$ are at O. Hence Γ meets r in

$$\rho_1(l_2-\lambda_2)+\rho_2(l_1-\lambda_1)$$

points not at O. Γ therefore meets F in

$$l_3(l_1 l_2-\rho_1\rho_2)-\lambda_3(\lambda_1\lambda_2-\rho_1\rho_2)-\rho_3\{\rho_1(l_2-\lambda_2)+\rho_2(l_1-\lambda_1)\}$$
$$= l_1 l_2 l_3 - \lambda_1\lambda_2\lambda_3 - \sum_{1,2,3}\rho_2\rho_3(l_1-\lambda_1)$$

points not on r.

3. We apply this to the calculation of the grade of $|kC+G_s|$. Let F_1 be the cone of order n, having lines of multiplicity ρ_i, and let F_2 and F_3 be two surfaces of order N with an R-fold point at the vertex O of the cone, having in a ρ_i-ple line of the cone a line of multiplicity σ_i. Then

$$k = N-R, \quad s+\sum_i \rho_i\sigma_i = nR,$$

hence

$$nk+s+\sum_i \rho_i\sigma_i = Nn.$$

The grade of the system is therefore

$$nN^2-nR^2-2(N-R)\Sigma\rho_i\sigma_i = nk^2+2ks.$$

Similarly we show that a curve of the system $|kC+G_s|$ meets a curve of the system $|k'C+G_{s'}|$ in

$$nkk'+ks'+k's$$

points.

4. To evaluate the genus of a curve of $|kC+G_s|$, we take our surfaces F_1 and F_2, as in the previous paragraph. We construct F_3 as follows. Let

$$ax+by+cz+dt = 0,$$
$$\alpha x+\beta y+\gamma z+\delta t = 0$$

be any line l. Then

$$\begin{vmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ U_x & U_y & U_z & U_t \\ V_x & V_y & V_z & V_t \end{vmatrix} = 0,$$

where $U_x = \partial F_1/\partial x$ and $V_x = \partial F_2/\partial x$, etc., is a surface F_3 of order $N+n-2$, which is the locus of points whose polar planes with regard to F_1 and F_2 intersect in a line meeting l. The points common to this surface and F_1 and F_2, which are not at multiple points of these surfaces are the points at which the tangents to the curve common to F_1 and F_2 meet l. Now this surface has O as an $(n+R-2)$-fold point, and passes $\rho_i+\sigma_i-2$ times through the common generators of F_1 and F_2. We have seen in § 2 that the curves of $|kC+G_s|$ meet one of these generators in $k\rho_i$ points, and the surfaces F_1 and F_2 have at each of these points a common tangent plane which is tangent to F_3. Hence F_1, F_2, F_3 meet in

$$nN(n+N-2)-nR(n+R-2)-(N-R)\Sigma\{\rho_i(\rho_i+\sigma_i-2)+\rho_i\sigma_i\}-k\Sigma\rho_i$$

points outside the common generators of F_1 and F_2. This is therefore the class of the curves $|kC+G_s|$. The order of the curves is $nN-\Sigma\rho_i\sigma_i$, and hence the genus is

$$\tfrac{1}{2}\{n^2k+nk(N+R-2)-k\Sigma\rho_i(\rho_i-1)-2k\Sigma\rho_i\sigma_i\}-nN+\Sigma\rho_i\sigma_i+1.$$

Now
$$(n-1)(n-2)-\Sigma\rho_i(\rho_i-1) = 2p,$$

where p is the genus of the cone; hence the genus of the curves is

$$\pi = \tfrac{1}{2}k(k-1)n+(k-1)(s-1)+kp.$$

5. Let r denote the effective dimension of the system $|kC+G_s|$ and r_m the dimension of $|mC+G_s|$ $(m \leqslant k)$. The system $|kC+G_s|$ cuts on a plane section of the cone a complete series of order $kn+s$; hence

$$r-r_{k-1} = kn+s-p+i_k+1,$$

where i_k is the index of speciality of this series. Similarly

$$r_{k-1}-r_{k-2} = (k-1)n+s-p+i_{k-1}+1, \ \ldots$$

$$r_0 = s-p+i_0,$$

where i_0 is the index of speciality of the group G_s of generators. Hence

$$r = \tfrac{1}{2}k(k+1)n+(k+1)(s+1)-(k+1)p-1+\sum_0^k i,$$

$$= D-\pi+1-p+\sum_0^k i,$$

D and π being the grade and genus of $|kC+G_s|$. We call

$$r' = D-\pi+1-p$$

the *virtual dimension* of the system, and $\sum_0^k i$ the *superabundance*. When the superabundance is zero, the system is said to be *regular*.

The system adjoint to $|kC+G_s|$ is $|k'C+G_{s'}|$, where

$$k' = k-2 \quad \text{and} \quad s' = s+n+2p-2 \quad (k \geqslant 2).$$

For such a system $\sum_0^k i'$ is zero, and the system is regular. This corresponds to a theorem due to Severi*.

6. We can now consider problems relating to the grade and genus of a linear system of curves $|K|$ which is obtained from the system $|kC+G_s|$ by imposing a_1 simple base points, a_2 double base points, ..., a_k k-fold base points. We may now suppose our cone transformed into one in hyperspace without multiple generators, so that the base points of $|K|$ are at simple points of the surface. Then the grade D and genus π of $|K|$ are given by

$$D = nk^2+2ks-\sum_1^k r^2 a_r, \tag{1}$$

$$\pi = \tfrac{1}{2}k(k-1)n+(s-1)(k-1)+kp-\tfrac{1}{2}\sum_1^k r(r-1)a_r. \tag{2}$$

We notice that if $k=1$, $a_r=0$ $(r>1)$, and, for all values of a_1 and s, $\pi = p$ and D may be as large as we please.

We cannot have $\pi < p$ except for the generators. For if such a system existed, on any curve of it there would be an involution of genus p, which is impossible if $\pi < p$. If $\pi = p$, we must have $k=1$, unless $p=0$, or $p=1$. The case in which $p=0$ belongs to Section I. On an elliptic cone we can show that any system of elliptic curves for which $k>1$ must be a pencil. For, from (2), we require

$$(k-1)(kn+2s) = \Sigma r(r-1)a_r$$

$$\leqslant (k-1)\Sigma r a_r.$$

Also $$D-2\pi+2 = D = kn+2s-\Sigma r a_r,$$

and hence $$D=0 \quad \text{and} \quad a_r = 0 \quad (r<k).$$

Indeed, such a system can be constructed on a cubic cone as follows. Let O be the vertex of the cone, and P any point on it. Let Q, R be two points not on the same generator, nor on OP, such that PQ, PR

* *Rend. Acc. dei Lincei* (5), 17 (1908), 690–694.

touch the cone in Q, R respectively. Consider the pencil of quadric cones with P as vertex touching POQ, POR, along PQ, PR respectively. These cut the cubic cone in a pencil of sextic curves, each of which meets a generator in two points, and has P, Q, and R as double points. Thus the curves of the pencil are of genus one.

If we eliminate s from equations (1) and (2), we obtain

$$D = 2(\pi-p)+\frac{2}{k-1}(\pi-p)-2k(p-1)-\frac{1}{k-1}\Sigma(k-r)\,ra_r;$$

hence, if $\pi > p$ and $k \geqslant 2$, we have

$$D \leqslant 4\pi+4-8p.$$

Thus on a ruled surface of genus p there is an infinite number of systems of curves of genus p; but, except for these, there are no systems of genus less than $2p-1$.

7. In enumerating the systems of curves of genus π on a cone, we first consider cones for which $p > 1$. We have the three equations

$$D = nk^2+2ks-\sum_1^k r^2 a_r, \tag{1}$$

$$\pi = \tfrac{1}{2}k(k-1)n+(s-1)(k-1)+kp-\tfrac{1}{2}\sum_1^k r(r-1)\,a_r, \tag{2}$$

$$D-2\pi+2 = k(n-2p+2)+2s-\sum_1^k ra_r. \tag{3}$$

From (1) and (3) we deduce that

$$k(D-2\pi+2) = D+\sum_1^k r^2 a_r-2k^2(p-1)-k\sum_1^k ra_r,$$

i.e.

$$2k^2(p-1)+k(D-2\pi+2)+\sum_1^k (k-r)\,ra_r = D. \tag{4}$$

If $D \geqslant 2\pi-2$, none of the terms on the left-hand side is negative, and we have therefore

$$k^2 \leqslant \frac{D}{2(p-1)} \leqslant \frac{2\pi+2-4p}{p-1} = \lambda^2,$$

and hence

$$2 \leqslant k \leqslant \lambda.$$

Now, in (4), give to k any value within these limits and to D any consistent value

$$2\pi-2 \leqslant D \leqslant 4\pi-4-8p,$$

and we deduce an equation for $a_1, a_2, \ldots, a_{k-1}$, in which the coefficients of these quantities are all positive integers. There is a finite number of such equations, and each of these can be solved in a finite number of ways for given values of $a_1, a_2, \ldots, a_{k-1}$ which are greater than or equal to zero; a_k, however, remains arbitrary. Taking any one of this finite number of sets of solutions, equation (2) gives

$$s = 1+\frac{1}{k-1}\left\{\pi-\tfrac{1}{2}k(k-1)n-kp+\tfrac{1}{2}\sum_{1}^{k-1} r(r-1)a_r\right\}+\tfrac{1}{2}ka_k.$$

We therefore determine which sets of solutions give an integral value of $s(\geqslant 0)$ for some values of a_k and the lowest not negative value of a_k for which this is so. Each such set of solutions determines a system of genus π and grade greater than or equal to $2\pi-2$. Further, each of these systems determines a family of systems of the same grade, genus, and virtual dimension. The systems of a family are obtained from a system already found by introducing new k-fold base points, and increasing s so that $(2s-ka_k)$ remains constant. Further, any system of genus π, and grade greater than or equal to $2\pi-2$, belongs to one of these families.

We see, moreover, that (even though $D<2\pi-2$), corresponding to any given value of k, there is only a finite number of simply infinite families of genus $\pi(\pi>p)$, obtained in the manner described. Hence, if we can establish limits for k we shall have established that on any cone of genus p there is a finite number of simply infinite families of systems of curves of given genus, greater than p, the parameter of the family being the number of generators of the cone which are fundamental for the systems.

To obtain the required limit for k, let D_1 and π_1 be the grade and genus of the system $|K_1|$, obtained by subtracting the generators which are fundamental for $|K|$ from the system adjoint to $|K|$. Then we have

$$\begin{aligned} D_1-2\pi_1+2 &= D-2\pi+2+8(p-1)+\sum_{1}^{k-1} a_r \\ &\geqslant D-2\pi+2+8(p-1). \end{aligned}$$

Denoting by $|K_2|$ the system obtained from $|K_1|$ in the same way, and so on, we have

$$D_2-2\pi_2+2 \geqslant D-2\pi+2+16(p-1), \quad \ldots,$$
$$D_i-2\pi_i+2 \geqslant D-2\pi+2+8i(p-1).$$

Hence we can find some i,

$$i \leqslant \tfrac{1}{4}\frac{\pi-1}{p-1} = \mu,$$

such that $$D_i \geqslant 2\pi_i - 2,$$

or for some value of i which does not exceed μ the systems $|K_i|$ cease to exist. In the latter case we must have $k \leqslant 2\mu - 1$, and a limit is obtained for k. In the former case, a limit is obtained for k in terms of π_i. But

$$\begin{aligned}\pi_1 &= \pi - (D - 2\pi + 2) - 8(p-1) - \sum_1^{k-1} a_r, \\ &\leqslant \pi - (D - 2\pi + 2) - 8(p-1),\end{aligned}$$

and we obtain by induction

$$\begin{aligned}\pi_i &\leqslant \pi - i(D - 2\pi + 2) - 4i(i+1)(p-1) \\ &\leqslant (2i+1)(\pi - 1) - 4i(i+1)(p-1) + 1,\end{aligned}$$

and so a limit is obtained for k in terms of π in all cases.

8. We return to the case of elliptic cones. We may confine ourselves to systems for which $\pi > 1$. Our equations are now

$$D = nk^2 + 2ks - \Sigma r^2 a_r, \tag{1}'$$

$$\pi = \tfrac{1}{2}k(k-1)n + (k-1)(s-1) + k - \tfrac{1}{2}\Sigma r(r-1)a_r, \tag{2}'$$

$$D - 2\pi + 2 = kn + 2s - \Sigma r a_r, \tag{3}'$$

$$k(D - 2\pi + 2) + \sum_1^k r(k-r)a_r = D. \tag{4}'$$

The method will be the same as in the previous paragraph, once we have established limits for k. We therefore confine ourselves to doing this.

If $D \geqslant 2\pi - 2$, we have, from (4)',

$$k \leqslant \frac{D}{D - 2\pi + 2} \leqslant 2\pi - 1.$$

If $D \leqslant 2\pi - 2$, we have, from (4)', $a_r > 0$, for some $r (r < k)$, except when

$$D = 2\pi - 2, \quad D = 0, \quad i.e. \quad \pi = 1,$$

which is a case already considered. Leaving this aside, we have, if $D \leqslant 2\pi - 2$,

$$\sum_1^{k-1} a_r > 0.$$

Defining D_1 and π_1 as in the previous paragraph, we have

$$D_1-2\pi_1+2 = D-2\pi+2+\sum_1^{k-1}\alpha_r$$

$$\geqslant (D-2\pi+2)+1.$$

If $D_1 > 2\pi_1-2$, we deduce, as before, a limit for k. If not, we proceed another stage. Finally, we see that, for some value of i less than or equal to $2\pi-1$, we have

$$D_i > 2\pi_i-2,$$

or for some value of i not exceeding this limit $|K_i|$ ceases to exist. In the latter case we have, as in the case in which $p \geq 1$, the inequality $k \leqslant 4\pi-3$. In the former case we establish a limit for k in terms of π_i, and since

$$\pi_i \leqslant (2i+1)(\pi-1)+1,$$

we have a limit for k in terms of π.

Thus we have now completely established the theorem:

On a ruled surface of genus $p > 0$, any system of curves of genus $\pi > p$ belongs to one of a finite number of simply infinite families of linear systems of curves of genus π.

A SERIES OF BESSEL FUNCTIONS CONNECTED WITH THE THEORY OF LATTICE POINTS

By J. R. WILTON.

[Received 21 April, 1927.—Read 28 April, 1927.—Received in revised form 6 February, 1928.]

IN its present form this paper owes very much to the helpful criticism of Mr. Oppenheim to whom the original manuscript was submitted. The most important of the changes which he suggested was the relegation of the "deep" Lemma 8, which appeared originally as Lemma 2, to as late a position as possible, so as to make it clear just where its use becomes necessary. Other modifications are (i) the addition of a brief historical paragraph, (ii) a rearrangement of the order in which the theorems are stated, and (iii) an expansion of the argument in the proof of Lemma 5*.

1. I begin with an explanation of the notation employed. Let

$$P \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} m_i m_j \quad (a_{ij} = a_{ji})$$

be a positive-definite quadratic form whose coefficients a_{ii} and $2a_{ij}$ $(i \neq j)$ are integers†, so that P takes only positive integer values when the variables m_i are integers, not all zero. Let $D = D_P$ be the discriminant of P, and let DQ be the form adjoint to P, so that the discriminant of Q is $D_Q = 1/D$, and $D_Q P$ is adjoint to Q.

By Σ is meant the repeated sum over all positive and negative integer values of m_i $(i = 1, 2, \ldots, n)$ with the sole exception of the set $m_1 = m_2 = \ldots = m_n = 0$; the sums with respect to the m_i are to be taken in a definite order, say m_1 followed by m_2, ..., followed by m_n;

* I am also indebted to Mr. Oppenheim for valuable criticism on several points of detail.

† The restriction to integers is unessential; the coefficients must be rational.

moreover, if in (1.5) $\mu \leqslant \frac{1}{2}n$—and therefore in (1.3)—the sum with regard to m_n is to be interpreted by the convention

$$\sum_{m_n=-\infty}^{\infty} = \lim_{N\to\infty} \sum_{m_n=-N}^{N},$$

and, if $\mu \leqslant \frac{1}{2}n-\frac{1}{2}$, the sum with regard to m_{n-1} is to be interpreted by the same convention. By Σ_P is meant the sum over all those values of m_i for which $0 < P \leqslant x$, and it is also understood that terms for which $P = x$ are to be halved. Further

$$\sum_{k=1}^{x}{}' f(k) \text{ denotes } \sum_{k=1}^{x-1} f(k)+\tfrac{1}{2}f(x) \text{ or } \sum_{k=1}^{[x]} f(k)$$

according as x is, or is not, an integer.

With these preliminary explanations let $r(k) = r_P(k)$ be the number of integer solutions of the equation $P = k$, and let

$$R(x) = R_P(x) = \sum_{k=1}^{x}{}' r(k) = \Sigma_P 1, \tag{1.1}$$

$$R(x) = \frac{\pi^{\frac{1}{2}n} x^{\frac{1}{2}n}}{\sqrt{D}\,\Gamma(\frac{1}{2}n+1)} - 1 + \frac{\psi(x)}{\sqrt{D}}; \tag{1.2}$$

then*

$$\psi(x) = \psi_P(x) = x^{\frac{1}{4}n} \Sigma Q^{-\frac{1}{4}n} J_{\frac{1}{2}n}\{2\pi\sqrt{(Qx)}\}, \tag{1.3}$$

$J_{\frac{1}{2}n}$ being the Bessel function of order $\frac{1}{2}n$. Further, let

$$R^{(\mu-\frac{1}{2}n)}(x) = R_P^{(\mu-\frac{1}{2}n)}(x) = \{\Gamma(\mu-\tfrac{1}{2}n+1)\}^{-1} \Sigma_P (x-P)^{\mu-\frac{1}{2}n} \tag{1.4}$$

$$= \frac{\pi^{\frac{1}{2}n} x^{\mu}}{\sqrt{D}\,\Gamma(\mu+1)} - \frac{x^{\mu-\frac{1}{2}n}}{\Gamma(\mu-\frac{1}{2}n+1)} + \frac{\psi^{(\mu-\frac{1}{2}n)}(x)}{\sqrt{D}};$$

then

$$\psi^{(\mu-\frac{1}{2}n)}(x) = \psi_P^{(\mu-\frac{1}{2}n)}(x) = \pi^{\frac{1}{2}n-\mu} x^{\frac{1}{2}\mu} \Sigma Q^{-\frac{1}{2}\mu} J_{\mu}\{2\pi\sqrt{(Qx)}\}, \tag{1.5}$$

provided that $\mu > \frac{1}{2}n-1$; but if $\frac{1}{2}n > \mu > \frac{1}{2}n-1$ the sum of the series has infinite discontinuities at the points where $Q = x$, *i.e.* Dx must not be an integer†.

---

* J. R. Wilton, "The lattice-points of an n-dimensional ellipsoid", *Journal London Math. Soc.*, 2 (1927), 227–233. I take this opportunity of expressing my gratitude to Prof. Hardy for his kindness in seeing this paper through the press for me. The two footnotes in square brackets are his.

† The series of (1.3) and (1.5) are convergent (with the meaning of Σ explained above), Wilton, *loc. cit.* The integrated functions (1.4) are employed by Walfisz, *Math. Zeitschrift*, 26 (1927), 106–124.

In (1 . 4) and (1 . 5), when $\mu-\frac{1}{2}n$ is an integer the index indicates that $R(x)$ or $\psi(x)$ is to be integrated $\mu-\frac{1}{2}n$ times from 0 to x; and when $\mu-\frac{1}{2}n$ is not an integer the index may be interpreted in the same way by making use of the theory of fractional integration, *e.g.* $(\mu > \frac{1}{2}n)$

$$(1.51) \qquad R^{(\mu-\frac{1}{2}n)}(x) = \frac{1}{\Gamma(\mu-\frac{1}{2}n)} \int_0^x R(u)(x-u)^{\mu-\frac{1}{2}n-1}\, du.$$

The finite sum of (1 . 4) will serve as the definition of $R^{(\mu-\frac{1}{2}n)}(x)$ and $\psi^{(\mu-\frac{1}{2}n)}(x)$ when $\mu < \frac{1}{2}n$.

In (1 . 5) P and Q may be interchanged; and if $\mu > n-\frac{1}{2}$, the series is absolutely convergent, and therefore, as $x \to \infty$,

$$(1.6) \qquad \psi^{(\mu-\frac{1}{2}n)}(x) = O(x^{\frac{1}{2}\mu-\frac{1}{4}}) \quad (\mu > n-\tfrac{1}{2});$$

and, further, the absolutely convergent multiple series may be rearranged as a single series converging to the same sum*.

LEMMA 1. *If* $\mu > n-\frac{1}{2}$ $(n \geqslant 2)$

$$(1.7) \qquad \pi^{\mu-\frac{1}{2}n}\psi_Q^{(\mu-\frac{1}{2}n)}(x) = \sum_{p=1}^{\infty} r_P(p)\left(\frac{x}{p}\right)^{\frac{1}{2}\mu} J_\mu\{2\pi\sqrt{(xp)}\} = O(x^{\frac{1}{2}\mu-\frac{1}{4}}).$$

2. The aim of the paper may now be stated in a few words; it is, briefly, the determination of the behaviour with regard to summability of the series (1 . 7) when $\mu \leqslant n-\frac{1}{2}$, by the examination of the partial sums. The main conclusions are contained in Theorems C and D, both of which are consequences of an explicit formula for the partial sum stated as Theorem B.

The problem is, of course, by no means new; nor is the method new, except in so far as it employs the repeated series (1 . 5). It is, in fact, the natural extension of a method given by Hardy† for the case $n = 2$, $\mu = 1$. For arbitrary n the summability of the series (1 . 7) has, so far as I am aware, been considered only by Walfisz‡ and Oppenheim§.

* In case $\mu-\frac{1}{2}n$ is an integer (1 . 6) and (1 . 7) are due to Landau, *Sitzungsberichte der Berliner Akad.*, 31 (1915), 458–476 (472); the general formulae are due to Müntz, *Sitzungsberichte der Berliner Math. Ges.*, 24 (1925), 81–93. I am dependent for these references upon Walfisz (*loc. cit.*, 110 and 119).

† G. H. Hardy, "The lattice points of a circle", *Proc. Royal Soc.* (A), 107 (1925), 623–635.

‡ A. Walfisz, "Über die summatorischen Funktionen einiger Dirichletscher Reihen", *Inaugural Dissertation* (Göttingen, 1922), 1–55, Satz 6.

§ A. Oppenheim, "Some identities in the theory of numbers", *Proc. London Math. Soc.* (2), 26 (1927), 295–350. References to earlier papers dealing with cases of convergence of (1 . 7) are given by Oppenheim (297).

Walfisz proves that, when $\mu = \frac{1}{2}n$, the series is summable $(C, [\frac{1}{2}n - \frac{1}{2}])$. Oppenheim considers only the lattice points of the hypersphere, and for this case, when $\mu = \frac{1}{2}n$, improves the result obtained by Walfisz by showing that (1 . 7) is summable

$$(C,\ \tfrac{1}{2}n - \tfrac{3}{2} + \delta) \quad (\delta > 0).$$

That is to say, he establishes my Theorem D in this case. He also proves the corresponding result when $\mu = \frac{1}{2}n - 1$*. Walfisz has also proved theorems analogous to D and D'' for certain sums connected with the general corpus of algebraic numbers†.

LEMMA 2. *If* $n \geqslant 2$, $\psi(x) = O(x^{\frac{1}{2}n - \frac{1}{2}})$.

Very much more than this crude result is known‡. For the case of the circle it goes back to Gauss; four proofs are given by Landau in his *Vorlesungen über Zahlentheorie*§.

LEMMA 3. *If* $\phi(u)$ *is L-integrable and bounded over* $(0, x)$ *and if, when* $\gamma > 0$ *and* $x > 0$,

$$\phi^{(\gamma)}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \phi(u)(x-u)^{\gamma-1}\, du;$$

and if, further,

$$|\phi(x)| < V, \quad |\phi^{(\gamma)}(x)| < W, \quad 0 < \gamma' < \gamma,$$

where V *and* W *are increasing (non-decreasing) functions of* x,

then
$$|\phi^{(\gamma)}(x)| < \gamma'' V^{1-(\gamma'/\gamma)} W^{(\gamma'/\gamma)} \quad (\gamma'' > 0),$$

where γ'' *depends only on* γ *and* γ'.

This is a known theorem due to Marcel Riesz‖.

---

* Oppenheim, *loc. cit.*, Theorem 2 and Theorem 4.

† Walfisz, *Math. Zeitschrift*, 22 (1925), 153–188 (154); *ibid.*, 26 (1927), 487–494. Dr. Walfisz informs me that his only published work on the explicit representation of $R(x)$ in series form is contained in his *Dissertation*; he was, however, at the time of publication of the papers referred to in this foot-note, well aware that the method there employed would succeed in the case of the ellipsoid problem, that is to say, he was aware of the truth of my Theorems D and D'' when $\mu = \frac{1}{2}n$, and he would naturally conjecture their truth for other values of μ. But Oppenheim (*loc. cit.*, 297) is mistaken in attributing to Walfisz the corresponding theorem when the lattice points are weighted with a complex factor (as in Theorem D').

‡ See Lemmas 7 and 8 below.

§ Band II, 185–187.

‖ "Sur un théorème de la moyenne et ses applications", *Acta Univ. Hungaricae Franc.-Jos.*, 1 (1923), 114–126.

LEMMA 4. *If* $n \geqslant 2$, $\frac{1}{2}n-\frac{1}{2} \geqslant \kappa > 0$,

$$\psi^{(\kappa)}(x) = O(x^{\frac{1}{2}n-\frac{1}{2}+\epsilon}) \quad (\epsilon > 0).$$

From Lemmas 1 and 2, if $\gamma > \frac{1}{2}n-\frac{1}{2}$,

$$\psi^{(\gamma)}(x) = O(x^{\frac{1}{2}\gamma+\frac{1}{4}n-\frac{1}{4}}), \quad \psi(x) = O(x^{\frac{1}{2}n-\frac{1}{2}});$$

hence, from Lemma 3, if $0 < \kappa \leqslant \frac{1}{2}n-\frac{1}{2}$,

$$\begin{aligned}\psi^{(\kappa)}(x) &= O\left\{x^{(\frac{1}{2}n-\frac{1}{2})\{1-(\kappa/\gamma)\}+(\kappa/\gamma)(\frac{1}{2}\gamma+\frac{1}{4}n-\frac{1}{4})}\right\} \\ &= O(x^{\frac{1}{2}n-\frac{1}{2}+\epsilon}),\end{aligned}$$

for every $\epsilon > 0$*.

---

* Mr. Oppenheim sends me the following proof that, when κ is an integer not exceeding $\frac{1}{2}n-\frac{1}{2}$,

$$\psi^{(\kappa)}(x) = O(x^{\frac{1}{2}n-\kappa}\log x);$$

this proof depends on nothing more elaborate than the crude asymptotic expression for the Bessel function and Taylor's theorem—together, of course, with Lemmas 1 and 2. He refers, for the method, to Landau's paper already cited, 472-476.

If $n+\frac{1}{2} \geqslant \mu > n-\frac{1}{2}$, $x \geqslant z > 0$, it follows from (1.7) that

$$\pi^{\mu-\frac{1}{2}n}\left\{\psi_Q^{(\mu-\frac{1}{2}n)}(x+z)-\psi_Q^{(\mu-\frac{1}{2}n)}(x)\right\} = \sum_{p=1}^{\infty}\frac{r_P(p)}{p^{\frac{1}{2}\mu}}\left\{(x+z)^{\frac{1}{2}\mu}J_\mu\left(2\pi\sqrt{(x+z)}\sqrt{p}\right)-x^{\frac{1}{2}\mu}J_\mu\left(2\pi\sqrt{(xp)}\right)\right\}.$$

The expression within the curly brackets on the right is, from the asymptotic value of the Bessel function,

$$O(p^{-\frac{1}{4}}x^{\frac{1}{2}\mu-\frac{1}{2}});$$

and by the first theorem of the mean it is

$$z\frac{d}{d\xi}\left\{\xi^{\frac{1}{2}\mu}J_\mu\left(2\pi\sqrt{(\xi p)}\right)\right\} = \pi z p^{\frac{1}{2}}\xi^{\frac{1}{2}\mu-\frac{1}{2}}J_{\mu-1}\left\{2\pi\sqrt{(\xi p)}\right\} = O(p^{\frac{1}{2}}zx^{\frac{1}{2}\mu-\frac{3}{2}}) \quad (x \leqslant \xi \leqslant x+z).$$

Hence

$$\begin{aligned}\psi_Q^{(\mu-\frac{1}{2}n)}(x+z)-\psi_Q^{(\mu-\frac{1}{2}n)}(x) &= O\left\{\sum_{p=1}^{\infty}\frac{r(p)}{p^{\frac{1}{2}\mu}}\operatorname{Min}\left(p^{-\frac{1}{4}}x^{\frac{1}{2}\mu-\frac{1}{2}},\ p^{\frac{1}{2}}zx^{\frac{1}{2}\mu-\frac{3}{2}}\right)\right\} \\ &= O\left\{\sum_{p\leqslant xz^{-2}}\frac{r(p)}{p^{\frac{1}{2}\mu}}p^{\frac{1}{2}}zx^{\frac{1}{2}\mu-\frac{3}{2}}\right\}+O\left\{\sum_{p\geqslant xz^{-2}}\frac{r(p)}{p^{\frac{1}{2}\mu}}p^{-\frac{1}{4}}x^{\frac{1}{2}\mu-\frac{1}{4}}\right\} \\ &= O(z^{\mu+\frac{1}{2}-n}x^{\frac{1}{2}n-\frac{1}{2}}),\end{aligned}$$

by partial summation, using (1.2) and Lemma 2. In case $\mu = n+\frac{1}{2}$, the final expression must be multiplied by $\operatorname{Max}\{\log(x/z^2), 1\}$.

Let n be even, take $\mu = n$, and write ψ for ψ_Q; then

$$\psi^{(\frac{1}{2}n)}(x+z)-\psi^{(\frac{1}{2}n)}(x) = O(z^{\frac{1}{2}}x^{\frac{1}{2}n-\frac{1}{2}}). \tag{1.71}$$

Again

$$\psi^{(\frac{1}{2}n)}(x+z)-\psi^{(\frac{1}{2}n)}(x) = z\psi^{(\frac{1}{2}n-1)}(x)+\ldots+\frac{z^{\frac{1}{2}n-1}}{\frac{1}{2}n-1!}\psi^{(1)}(x)+\frac{z^{\frac{1}{2}n-1}}{\frac{1}{2}n-1!}\int_x^{x+\theta z}\psi(t)\,dt \quad (0<\theta<1) \tag{1.72}$$

by Taylor's theorem, since $\psi^{(1)}(x) = \int_0^x \psi(t)\,dt$.

By taking $0 < A \leqslant z \leqslant A'$ we deduce from (1.71) and (1.72) that, when x is large,

$$\psi^{(\kappa)}(x) = O(x^{\frac{1}{2}n-\frac{1}{2}}) \quad (\kappa = 1, 2, \ldots, \tfrac{1}{2}n-1) \tag{1.73}$$

for even $n(\geqslant 4)$. Taking $\mu = n+\frac{1}{2}$ the same proof holds for odd $n(\geqslant 3)$, except that there is now a factor $\log x$,

$$\psi^{(\kappa)}(x) = O(x^{\frac{1}{2}n-\frac{1}{2}}\log x) \quad (\kappa = 1, 2, \ldots, \tfrac{1}{2}n-\tfrac{1}{2}). \tag{1.74}$$

In this case also very much more is known*.

3. The proof of Theorem B depends upon an interesting particular case of the Weber-Schafheitlin discontinuous integral†. The case which I require is $(\mu > \frac{1}{2}n-1)$

$$(3.1)\quad 2\pi \int_0^\infty \frac{J_{\mu+\lambda}(2\pi t\sqrt{x})\, J_{\lambda+\frac{1}{2}n-1}(2\pi t\sqrt{Q})}{t^{\mu-\frac{1}{2}n}}\, dt = \frac{\pi^{\mu-\frac{1}{2}n} Q^{\frac{1}{2}(\lambda+\frac{1}{2}n-1)}}{x^{\frac{1}{2}(\lambda+\mu)}\Gamma(\mu-\frac{1}{2}n+1)}\,(x-Q)^{\mu-\frac{1}{2}n},$$

if $0 < Q < x$, while the integral is zero if $Q > x$. When $\mu = \frac{1}{2}n$ there is a finite discontinuity at $Q = x$, and the value of the integral at $Q = x$ is $\frac{1}{2}x^{-\frac{1}{2}}$. Hence, from (1.4), by recalling the definition of Σ,

$$(3.2)\quad \pi^{\mu-\frac{1}{2}n} R_Q^{(\mu-\frac{1}{2}n)}(x) = \pi\Sigma \int_0^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+\lambda)} J_{\mu+\lambda}\{2\pi\sqrt{(\sigma x)}\} \times \left(\frac{\sigma}{Q}\right)^{\frac{1}{2}(\lambda+\frac{1}{2}n-1)} J_{\lambda+\frac{1}{2}n-1}\{2\pi\sqrt{(\sigma Q)}\}\, d\sigma.$$

LEMMA 5. *If $\lambda > \frac{1}{2}(n+1)$, $\mu > \frac{1}{2}n-1$, change of order of summation and integration is permissible in* (3.2), *and*

$$(3.3)\quad R_Q^{(\mu-\frac{1}{2}n)}(x) = \pi^{\lambda-\mu+\frac{1}{2}n} \int_0^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+\lambda)} J_{\mu+\lambda}\{2\pi\sqrt{(\sigma x)}\}\, \psi^{(\lambda-1)}(\sigma)\, d\sigma.$$

When $\mu > \frac{1}{2}n$ the series of integrals in (3.2) remains convergent when the Bessel functions are replaced by their moduli, so that the change of order presents no difficulty in this case. If $\frac{1}{2}n \geqslant \mu > \frac{1}{2}n-1$, the change of order is clearly permissible when the integral is taken over any finite interval $(0, A)$; and if the function $\chi(\sigma)$ is defined by the

---

* In fact, by combining Lemmas 3 and 8, it follows at once that

$$\psi^{(\kappa)}(x) = O(x^{\frac{1}{2}n-1+\kappa/(n-1)+\epsilon}) \quad (\epsilon > 0,\ n \geqslant 4,\ 0 < \kappa \leqslant \tfrac{1}{2}n-\tfrac{1}{2}).$$

Cf. Walfisz, *Math. Zeitschrift*, 26 (1927), 106–124 (119). Walfisz proves also (*loc. cit.*, 115) that $\psi^{(\frac{1}{2})}(x) = O(x^{\frac{1}{2}n-1})$.

† G. N. Watson, *Theory of Bessel functions* (1922), 401, (2). The proof of Theorem B is the natural extension to the general case of the method given by Hardy (*loc. cit.*) for the case of the circle. Hardy, in fact, remarks the possibility of extension to the general case in the last sentence of his introductory section.

equation

$$J_{\mu+\lambda}\{2\pi\sqrt{(\sigma x)}\}\,J_{\lambda+\frac{1}{2}n-1}\{2\pi\sqrt{(\sigma Q)}\}$$
$$=\chi(\sigma)+\tfrac{1}{2}(\pi^2\sigma^{\frac{1}{2}}x^{\frac{1}{4}}Q^{\frac{1}{4}})^{-1}[\cos\{2\pi\sqrt{\sigma}(\sqrt{Q}+\sqrt{x})-(\lambda+\tfrac{1}{2}\mu+\tfrac{1}{4}n)\pi\}$$
$$+\cos\{2\pi\sqrt{\sigma}(\sqrt{Q}-\sqrt{x})+\tfrac{1}{2}(\mu-\tfrac{1}{2}n+1)\pi\}],$$

it is easily seen that $\chi(\sigma)=O(\sigma^{-1})$ as $\sigma\to\infty$, so that the integral

$$\int^{\infty}\sigma^{-\frac{1}{2}(\mu-\frac{1}{2}n+1)}\chi(\sigma)\,d\sigma$$

is absolutely convergent; hence it is sufficient to prove that, for every $T>A$, change of order is permissible in the case of the series ($\sigma=t^2$)

$$(3.30)\quad \Sigma Q^{-\frac{1}{2}(\lambda+\frac{1}{2}n-\frac{1}{2})}\int_A^T\frac{dt}{t^{\mu-\frac{1}{2}n+1}}[\cos\{2\pi t(\sqrt{Q}+\sqrt{x})-(\lambda+\tfrac{1}{2}\mu+\tfrac{1}{4}n)\pi\}$$
$$+\cos\{2\pi t(\sqrt{Q}-\sqrt{x})+\tfrac{1}{2}(\mu-\tfrac{1}{2}n+1)\pi\}]$$
$$=\frac{1}{A^{\mu-\frac{1}{2}n+1}}\Sigma Q^{-\frac{1}{2}(\lambda+\frac{1}{2}n-\frac{1}{2})}\int_A^{T'}[\cos\{\ldots\}+\cos\{\ldots\}]\,dt$$
$$(A\leqslant T'\leqslant T),$$

and the modulus of this series is not greater than

$$(3.301)\quad \frac{1}{\pi A^{\mu-\frac{1}{2}n+1}}\Sigma Q^{-\frac{1}{2}(\lambda+\frac{1}{2}n-\frac{1}{2})}\left\{\frac{1}{\sqrt{Q}+\sqrt{x}}+\frac{1}{|\sqrt{Q}-\sqrt{x}|}\right\}=o(1),$$

as $A\to\infty$, provided that

$$|\sqrt{Q}-\sqrt{x}|\geqslant\delta>0,$$

i.e. provided that there is no set of values of the m_i for which $Q=x$. Further, if $\mu=\frac{1}{2}n$, the second cosine in (3.30) vanishes when $Q=x$, and the modulus of (3.30) is still controlled by (3.301) if the terms for which $Q=x$ are omitted from the second part of the sum. Hence change of order of summation and integration in the series (3.30) is permissible* by Dini's test if $\mu\geqslant\frac{1}{2}n$; and if

$$\tfrac{1}{2}n>\mu>\tfrac{1}{2}n-1,$$

it is permissible except for values of x for which there are integer values of the m_i satisfying $Q=x$. For such values of x both sides of (3.2) diverge. This proves the lemma.

---

* Bromwich, *Theory of infinite series* (1926), 502, Theorem C.

If, as has been assumed, $\lambda > \frac{1}{2}n+\frac{1}{2}$,

$$\psi^{(\lambda-1)}(\sigma) = O(\sigma^{\frac{1}{2}\lambda+\frac{1}{4}n-\frac{3}{2}});$$

hence, on integrating (3.3) by parts, the integrated portion vanishes, and the integral is unchanged, except that $\lambda-1$ is written instead of λ. Hence in (3.3) it may be assumed that

$$\frac{1}{2}n+\frac{1}{2} \geqslant \lambda > \frac{1}{2}n-\frac{1}{2}. \tag{3.31}$$

In all that follows λ will be assumed to be an integer*. I therefore take $\lambda = \frac{1}{2}n$ when n is even, and $\lambda = \frac{1}{2}n+\frac{1}{2}$ when n is odd. These are the smallest permissible integer values of λ, if all that is assumed about μ is that $\mu > \frac{1}{2}n-1$. But as μ is increased λ may be diminished; for, if the integral of (3.3) is integrated by parts, the integrated portion, as $\sigma \to \infty$, is

$$O\{\sigma^{-\frac{1}{2}(\lambda+\mu-\frac{1}{2})}\psi^{(\lambda-1)}(\sigma)\} = O\{\sigma^{\frac{1}{2}(n-\lambda-\mu-\frac{1}{2})+\epsilon}\} = o(1)$$

as $\sigma \to \infty$, if $\lambda+\mu+\frac{1}{2} > n$ and $\epsilon < \lambda+\mu+\frac{1}{2}-n$.

Hence (3.3) remains true if $\lambda = [n-\mu-\frac{1}{2}]$, provided that $\mu \leqslant n-\frac{3}{2}$. For greater values of μ it is permissible to put $\lambda = 1$.

LEMMA 6†. *If $n \geqslant 2$ and $\mu > n-\frac{5}{2}$ (and also, if $n = 2$, $\mu > 0$),*

$$\sqrt{D}\psi_Q^{(\mu-\frac{1}{4}n)}(x) - \frac{x^{\mu-\frac{1}{2}n}}{\Gamma(\mu-\frac{1}{2}n+1)} \tag{3.32}$$

$$= \pi^{\frac{1}{4}n-\mu+1}\int_0^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}\mu+\frac{1}{4}} J_{\mu+1}\{2\pi\sqrt{(\sigma x)}\}\{\psi(\sigma)-\sqrt{D}\}\,d\sigma,$$

while, if $n-\frac{5}{2} \geqslant \mu > \frac{1}{2}n-1$, the equation (3.3) is true if

$$\lambda = [n-\mu-\tfrac{1}{2}].$$

This follows, by recalling that $D_Q = 1/D_P = 1/D$, from (1.4), since

$$\pi^{\frac{1}{4}n-\mu+1}\int_0^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}\mu+\frac{1}{4}} J_{\mu+1}\{2\pi\sqrt{(\sigma x)}\}\,d\sigma = \frac{\pi^{\frac{1}{2}n}x^\mu}{\Gamma(\mu+1)}.$$

3A. This section interrupts the thread of the argument, but since Theorems A and A′ are direct consequences of Lemma 6, they may fittingly find their place here.

* There is no difficulty, but there seems to be no gain, in taking account of non-integral values of λ.

† For the case of the hypersphere a result equivalent to Lemma 6 is given by Oppenheim, *loc. cit.* (4.71); the notation which he uses is explained in his equations (3.53) and (3.51).

When $n = 2$, 3, or 4 it is permissible to put $\mu = \frac{1}{2}n$ in (3 . 32); and if

$$P = Q = m_1^2 + \ldots + m_n^2$$

(*i.e.* in the case of the hypersphere) so that $D = 1$, this leads to an integral equation for $\psi(x)$ in these three cases.

THEOREM A. *If*

$$\phi_n(x) \equiv x^{-\frac{1}{4}n-\frac{1}{2}}\{\psi(x)-1\},$$

then, in the case of the circle, sphere, and hypersphere ($n = 4$),

$$\phi_n(x) = \pi\int_0^\infty \phi_n(\sigma) J_{\frac{1}{2}n+1}\{2\pi\sqrt{(\sigma x)}\}\, d\sigma. \tag{3 . 4}$$

A continuous solution of (3 . 4) is easily seen to be*

$$\phi_n(x) = Ax^{-\frac{1}{2}} + Bx^{\frac{1}{4}n+\frac{1}{2}} e^{-\pi x},$$

a result which holds if only $\Re(n) > -4$.

THEOREM A′. *More generally, in the case of the hypersphere for which* $n = 4k+1$, $4k+2$, $4k+3$, *or* $4k+4$, *if*

$$\phi_n(x) \equiv x^{-(\frac{1}{4}n+k+\frac{1}{2})}\left\{\psi^{(k)}(x) - \frac{x^k}{\Gamma(k+1)}\right\};$$

then

$$\phi_n(x) = \pi\int_0^\infty \phi_n(\sigma) J_{\frac{1}{2}n+2k+1}\{2\pi\sqrt{(\sigma x)}\}\, d\sigma. \tag{3 . 41}$$

In other words, $\sqrt{x}\,.\,\phi_n(x^2/2\pi)$ is its own Hankel transform for this particular order of the Bessel function.

If we assume the truth of Lemma 8, and therefore of the order of $\psi^{(\kappa)}(x)$ stated in the foot-note to the last line of § 2, it is easy to show that Theorem A is true when $n = 5$ and $n = 6$; and, more generally, that Theorem A′ is true if $n \leqslant 4k+6$†.

4. The proof of Theorem B is obtained from Lemma 6 by dividing the path of integration in (3 . 3) into $(0, N)$ and (N, ∞), and integrating

* Watson, *loc. cit.*, 391 and 394. [*Added December* 1928.—In the e of the circle (3.4) has been stated recently by Hardy, *Messenger of Math.*, 57 (1927, published July, 1928), 113–120 (120)].

† Lemma 7 enables us to prove Theorem A when $n = 5$, and Theorem A′ when $n \leqslant 4k+5$.

the finite integral by parts until $\psi(\sigma)$ is obtained under the integral sign. The result of this process is

$$\pi^{\mu-\frac{1}{2}n} R_Q^{(\mu-\frac{1}{2}n)}(x) = \pi^\lambda \int_N^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+\lambda)} J_{\mu+\lambda}\{2\pi\sqrt{(\sigma x)}\}\,\psi^{(\lambda-1)}(\sigma)\,d\sigma$$

$$-\sum_{k=1}^{\lambda-1} \pi^{\lambda-k}\left(\frac{x}{N}\right)^{\frac{1}{2}(\mu+\lambda-k)} J_{\mu+\lambda-k}\{2\pi\sqrt{(Nx)}\}\,\psi^{(\lambda-k)}(N)$$

$$+\pi\sqrt{D}\int_0^N \left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+1)} J_{\mu+1}\{2\pi\sqrt{(\sigma x)}\}\,R(\sigma)\,d\sigma + \frac{\pi^\mu x^\mu \sqrt{D}}{\Gamma(\mu+1)}$$

$$-\sqrt{D}\left(\frac{x}{N}\right)^{\frac{1}{2}\mu} J_\mu\{2\pi\sqrt{(Nx)}\}$$

$$-\frac{\pi^{\frac{1}{2}n+1}}{\Gamma(\frac{1}{2}n+1)}\int_0^N \sigma^{\frac{1}{2}n}\left(\frac{x}{\sigma}\right)^{\frac{1}{2}\mu+\frac{1}{2}} J_{\mu+1}\{2\pi\sqrt{(\sigma x)}\}\,d\sigma,$$

by substituting for $\psi(\sigma)$ from (1 . 2).

The last integral may be evaluated by a reduction formula, the result being slightly different according as n is odd or even. And the integral containing $R(\sigma)$ may be evaluated by rewriting it as

$$\sum_{p=1}^{[N]} \int_{p-1}^{p} + \int_{[N]}^{N}$$

and employing (1 . 1).

THEOREM B. *If* $n \geqslant 2$, $\mu > \frac{1}{2}n-1$, $\lambda = 1$ *if* $\mu > n-\frac{5}{2}$, *and otherwise* $\lambda = [n-\mu-\frac{1}{2}]$,

$$(4\,.\,1)\quad \sqrt{D}\left\{\pi^{\mu-\frac{1}{2}n}\psi_Q^{(\mu-\frac{1}{2}n)}(x) - \sum_{p=1}^{N}{}' \frac{r_1(p)}{p^{\frac{1}{2}\mu}} x^{\frac{1}{2}\mu} J_\mu\left(2\pi\sqrt{(px)}\right)\right\}$$

$$= \pi^\lambda \int_N^\infty \left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+\lambda)} J_{\mu+\lambda}\{2\pi\sqrt{(\sigma x)}\}\,\psi^{(\lambda-1)}(\sigma)\,d\sigma$$

$$-\sum_{k=0}^{\lambda-1} \pi^k \left(\frac{x}{N}\right)^{\frac{1}{2}(\mu+k)} J_{\mu+k}\{2\pi\sqrt{(Nx)}\}\,\psi^{(k)}(N)$$

$$+\sum_{k=1}^{[\frac{1}{2}n]} \frac{\pi^{\frac{1}{2}n-k}}{\Gamma(\frac{1}{2}n-k+1)} N^{\frac{1}{2}n-k}\left(\frac{x}{N}\right)^{\frac{1}{2}(\mu-k)} J_{\mu-k}\{2\pi\sqrt{(Nx)}\} + \Theta_n(x),$$

where $\Theta_n(x) = 0$ *when* n *is even, and when* n *is odd*

$$\Theta_n(x) = (2\pi x)^{\mu-\frac{1}{2}n}\sqrt{\left(\frac{2}{\pi}\right)}\int_{2\pi\sqrt{(Nx)}}^\infty \frac{J_{\mu-\frac{1}{2}n+\frac{1}{2}}(t)}{t^{\mu-\frac{1}{2}n+\frac{1}{2}}}\,dt.$$

The following is then a corollary of Theorem B and Lemmas 2, 4, and 6, the last of which establishes the convergence of the integral containing $\psi^{(\lambda-1)}(\sigma)$.

THEOREM C. *If* $n \geqslant 2$, $n-\frac{1}{2} \geqslant \mu > \frac{1}{2}n-1$, *and*

$$S_N = S(\mu, N) = \sum_{p=1}^{N} \frac{r(p)}{p^{\frac{1}{2}\mu}} x^{\frac{1}{2}\mu} J_\mu\{2\pi\sqrt{(px)}\},$$

then

$$S_N = \pi^{\mu-\frac{1}{2}n}\psi_Q^{(\mu-\frac{1}{2}n)}(x)+o(1)+O(N^{\frac{1}{2}n-\frac{1}{2}\mu-\frac{3}{4}}),$$

where the constants involved in o and O depend upon x. In particular, the series $S = S(\mu, \infty)$ *is convergent if* $\mu > n-\frac{3}{2}$, *and* $S(n-\frac{3}{2}, N)$ *is bounded as* $N \to \infty$.

When $n = 2$ and $1 > \mu > \frac{1}{2}$, the series S is convergent, but there are infinite discontinuities at the points where $Q = x$, *i.e.* (with exceptions) when Dx is an integer.

In the particular case when $\mu = n-\frac{3}{2}$ rather more can be said about S_N by refining upon the crude formula of Lemma 2.

LEMMA 7. *If* $n \geqslant 2$ *and* $\epsilon > 0$,

$$\psi(x) = O(x^{\frac{1}{2}n-\{n/(n+1)\}+\epsilon}).$$

This approximation to the order of $\psi(x)$ is classical, and is due to Landau*. Considerably more is known† when $n \geqslant 4$.

Since, evidently, $r(x) = O\{\psi(x)\}$, it follows, from Theorem B and Lemma 7, that S_N oscillates when $\mu = n-\frac{3}{2}$; the precise expression for $S(n-\frac{3}{2}, N)$ is given by putting $\omega = N$ and $\mu = n-\frac{3}{2}$ in (7 . 0).

5. The present section contains an evident extension of Theorems B and C, depending upon known generalizations of the lemmas already used. The work is almost identical with that of previous sections, and I content myself with stating results.

Let $\mathbf{P}$ denote the result of writing $m_i+\mathbf{p}_i$ for m_i in $P(i = 1, 2, \ldots, n)$; and let it now be assumed that a_{ij} and $\mathbf{p}_i$ are rational numbers such that M is the least integer for which $M\mathbf{P}$ is a positive integer for all integer values of $m_1, \ldots, m_n$. It may evidently be supposed that $0 \leqslant \mathbf{p}_i < 1$.

---

* The earliest proof occurs in *Göttinger Nachrichten* (1912), 687–771. Subsequently, in a succession of papers—reference to which may be found in a paper by Walfisz, *Math. Zeitschrift*, 19 (1924), 300–307, foot-note (1)—Landau extended this result, and refined upon it by omission of ϵ; for my present purpose, however, this is immaterial.

† See Lemma 8.

Let $\mathbf{Q}$ be the result of writing $m_i+\mathbf{q}_i$ in Q, where $0 \leqslant \mathbf{q}_i < 1$, and $\mathbf{q}_i$ is real, though not necessarily rational. Further, let

$$\mathbf{r}(x) = \sum_{\mathbf{P}=r} e^{2\pi i \Sigma m\mathbf{q}},$$

where $\Sigma m\mathbf{q} = m_1\mathbf{q}_1+\ldots+m_n\mathbf{q}_n$; let

$$D^{-\frac{1}{2}}\psi_{\mathbf{P}}(x)-\eta_{\mathbf{p}} = \mathbf{R}(x) = \sum_{k=1}^{Mx}{}' \mathbf{r}\left(\frac{k}{M}\right) = \Sigma_{\mathbf{P}} e^{-2\pi i\Sigma m\mathbf{q}},$$

in which it is assumed that $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are not all zero, and that $\eta_{\mathbf{p}} = 1$ if $\mathbf{p}_1 = \mathbf{p}_2 = \ldots = \mathbf{p}_n = 0$, but is otherwise zero; and let

$$D^{\frac{1}{2}}\left\{\psi_{\mathbf{Q}}^{(\lambda)}(x)+\eta_{\mathbf{p}}\frac{\pi^{\frac{1}{2}n}x^{\lambda+\frac{1}{2}n}}{\Gamma(\lambda+\frac{1}{2}n+1)}\right\} = \mathbf{R}_{\mathbf{Q}}^{(\lambda)}(x) = \Sigma_{\mathbf{Q}}(x-\mathbf{Q})^{\lambda} e^{2\pi i\Sigma m\mathbf{p}}/\Gamma(\lambda+1).$$

Then Lemmas 2, 4, and 7 are true with these definitions of $\psi_{\mathbf{P}}$ and $\psi_{\mathbf{Q}}$ and since*

$$\pi^{\lambda-1}\psi_{\mathbf{P}}^{(\lambda-1)}(\sigma)\, e^{-2\pi i\Sigma\mathbf{pq}} = \Sigma\left(\frac{\sigma}{\mathbf{Q}}\right)^{\lambda+\frac{1}{2}n-\frac{1}{2}} J_{\lambda+\frac{1}{2}n-1}\left\{2\pi\sqrt{(\sigma\mathbf{Q})}\right\} e^{2\pi i\Sigma m\mathbf{p}},$$

where Σ has the same meaning (of repeated summation, etc.) as in § 1, except that the particular set of values

$$m_1 = m_2 = \ldots m_n = 0$$

is now included, it follows that Lemma 5 is true with the new definitions, while Lemma 6 is true in the form that in (3 . 3) we may put

$$\lambda = \mathrm{Max}(1, [n-\mu-\tfrac{1}{2}]).$$

The following theorem may now be proved precisely as before.

THEOREM B′. *If* (i) $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *are not all zero,* (ii) $\mu > \frac{1}{2}n-1$, *and* (iii) $\lambda = 1$ *if* $\mu > n-\frac{5}{2}$, *and otherwise* $\lambda = [n-\mu-\frac{1}{2}]$, $n \geqslant 2$, *then*

$$\sqrt{D}\left\{\pi^{\mu-\frac{1}{2}n} e^{2\pi i\Sigma\mathbf{pq}}\psi_{\mathbf{Q}}^{(\mu-\frac{1}{2}n)}(x) - \sum_{p=1}^{MN}{}' \mathbf{r}\left(\frac{p}{M}\right)\left(\frac{Mx}{p}\right)^{\frac{1}{2}\mu} J_{\mu}\left(2\pi\sqrt{\frac{px}{M}}\right)\right\}$$

$$= \pi^{\lambda}\int_N^{\infty}\left(\frac{x}{\sigma}\right)^{\frac{1}{2}(\mu+\lambda)} J_{\mu+\lambda}\left\{2\pi\sqrt{(\sigma x)}\right\}\psi_{\mathbf{P}}^{(\lambda-1)}(\sigma)\, d\sigma$$

$$-\sum_{k=0}^{\lambda-1}\pi^k\left(\frac{x}{N}\right)^{\frac{1}{2}(\mu+k)} J_{\mu+k}\left\{2\pi\sqrt{(Nx)}\right\}\psi_{\mathbf{P}}^{(k)}(N)$$

$$= O(N^{\frac{1}{2}n-\frac{1}{2}\mu-\frac{3}{4}})+o(1) \quad \text{(Lemmas 2 and 4).}$$

If Lemma 7 is employed instead of Lemma 2, the O term on the right may be replaced by

$$O(N^{\frac{1}{2}n-\frac{1}{2}\mu-\frac{3}{4}+\{1/(n+1)\}+\epsilon}).$$

---

* Wilton, *loc. cit.*

Hence the series for $\psi_{\mathbf{Q}}^{(\kappa)}(x)$ is convergent if

$$\kappa = \mu - \tfrac{1}{2}n > \tfrac{1}{2}n - \tfrac{5}{2} + \frac{2}{n+1}.$$

If $n > 3$, more is true.

THEOREM C'. *If* $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *are rational and not all zero, and if* $\mu > n - \frac{5}{2}$, $n \geqslant 4$,

$$\pi^{\mu-\frac{1}{2}n}\psi_{\mathbf{Q}}^{(\mu-\frac{1}{2}n)}e^{2\pi i\Sigma \mathbf{pq}} = \sum_{p=1}^{\infty} \mathbf{r}\left(\frac{p}{M}\right)\left(\frac{Mx}{p}\right)^{\frac{1}{2}\mu} J_\mu\left\{2\pi\sqrt{\left(\frac{px}{M}\right)}\right\}$$

is a convergent series; and, if $\mu = n - \frac{5}{2}$, $n \geqslant 5$, *the sum of* N *terms of the series on the right is bounded as* $N \to \infty$.

The proof requires the "deep" theorem quoted as Lemma 8, and also, for the second part, a slight refinement of Lemma 4, namely

$$\psi^{(1)}(x) = O(x^{\frac{1}{2}n-\frac{3}{2}}),$$

as in (1 . 73).

LEMMA 8. *If* $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *are rational, then when* $n = 4$,

$$\psi_{\mathbf{P}}(x) = O(x \log^2 x),$$

and when $n \geqslant 5$

$$\psi_{\mathbf{P}}(x) = O(x^{\frac{1}{2}n-1}).$$

For $n \geqslant 8$, this theorem, which lies very deep, is due to Walfisz*; the remaining cases are due to Landau†.

Lemma 8 has itself recently been extended in various directions. Landau‡ (attributing the theorem to Jarník) shows that, when $n \geqslant 5$, the order of $\psi_{\mathbf{P}}(x)$ is given precisely by Lemma 8, *i.e.*

$$\psi_{\mathbf{P}}(x) \neq o(x^{\frac{1}{2}n-1}).$$

More precise information, in the case of $\psi(x)$, is obtained by Müntz§, Petersson‖, and Walfisz¶, all of whom prove theorems which show that $\psi(x)$ is both**

$$\Omega_R(x^{\frac{1}{2}n-1}) \quad \text{and} \quad \Omega_L(x^{\frac{1}{2}n-1}).$$

* *Math. Zeitschrift*, 19 (1924), 300–307 (303).

† *Ibid.*, 21 (1924), 126–132. See also Kloosterman, *ibid.*, 24 (1925), 519–529.

‡ *Ibid.*, 24 (1925), 298–310 (306).

§ *Ibid.*, 25 (1926), 150–165.

‖ *Hamburg. Math. Abhandlungen*, 5 (1926), 116–150. Petersson obtains, when $n \geqslant 5$, formulae for $\psi(x)$, which involve $[\frac{1}{4}(n-1)]$ absolutely and uniformly convergent series, with an error term $O(x_{1_n}\log x)$.

¶ *Math. Zeitschrift*, 26 (1927), 106–124.

** Reference should also be made to two recent papers by Jarník, *Math. Zeitschrift*, 27 (1927), 154–160; *ibid.*, 28 (1928), 311–316.

6. Theorems B and B′ may be employed to establish the following results* concerning the summability of the series S and of the corresponding series $\mathbf{S}$ in which $\mathbf{r}$ takes the place of r.

THEOREM D. *If* $n=2$ *and* $\frac{1}{2} \geqslant \mu > \frac{1}{6}$, $n=3$ *and* $\frac{3}{2} \geqslant \mu > 1$, *or if* $n \geqslant 4$ *and* $n-\frac{3}{2} \geqslant \mu > \frac{1}{2}n-1$, *the series*

$$S = \sum_{p=1}^{\infty} \frac{r(p)}{p^{\frac{1}{2}\mu}} x^{\frac{1}{2}\mu} J_{\mu}\{2\pi\sqrt{(px)}\}$$

is summable $(R, p, n-\mu-\frac{3}{2}+\delta)$ *for every* $\delta\ (>0)$ *to the sum*

$$\pi^{\mu-\frac{1}{2}n}\, \psi_Q^{(\mu-\frac{1}{2}n)}(x).$$

Moreover, if $0 < x \leqslant \Delta$, *the summability is uniform with respect to* x *if* $\mu > \frac{1}{2}n$; *and if*

$$\tfrac{1}{2}n \geqslant \mu > \tfrac{1}{2}n-1$$

it is uniform in any closed interval of values of x *which does not include a point for which* Dx *is an integer.*

Summability (R, p, κ)—necessarily written for the more usual (R, n, κ)—is, of course, equivalent to summability (C, κ).

The question of the uniformity of summability presents no difficulty, for the only possible failure of uniformity lies in the integral on the right of (4 . 1), and, as in the proof of Lemma 5, it is readily seen that this integral is uniformly convergent if $\mu > \frac{1}{2}n$, and if $\frac{1}{2}n \geqslant \mu > \frac{1}{2}n-1$ provided that $Q \neq x$; and this is equivalent to the condition stated.

THEOREM D′. *If* $n \geqslant 4$ *and* $n-\frac{5}{2} \geqslant \mu > \frac{1}{2}n-1$, *if* $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *are rational numbers, and* $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *are real numbers not all zero, the series*

$$\mathbf{S} = \sum_{p=1}^{\infty}{}' \mathbf{r}\left(\frac{p}{M}\right)\left(\frac{Mx}{p}\right)^{\frac{1}{2}\mu} J_{\mu}\left(2\pi\sqrt{\frac{px}{M}}\right)$$

is summable $(R, p, [n-\mu-\frac{3}{2}])$ *to the sum*

$$\pi^{\mu-\frac{1}{2}n}\, \psi_{\mathbf{Q}}^{(\mu-\frac{1}{2}n)}(x)\, e^{2\pi i \Sigma \mathbf{pq}}.$$

---

* Theorem D, when $\mu = \frac{1}{2}n$, in the case of the hypersphere, is due to Oppenheim, *loc. cit.*, Theorem 2. The condition $\mu > \frac{1}{2}n-1$ is essential to the proof of all these theorems, but it would appear to be an artificial restriction in D and D′; and in fact Oppenheim's Theorem 4 agrees with the result of putting $\mu = \frac{1}{2}n-1$ in D—for, from (1 . 4), if Dx is not an integer

$$\psi_a^{(-1)}(x) = -\frac{\pi^{\frac{1}{2}n} x^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n)}.$$

Oppenheim has also proved the theorem when $\mu - \frac{1}{2}n = -p$, where p is a positive integer (*loc. cit.*, 332, last line).

Theorem D′ will appear as a corollary in the course of the proof of D; it is the easier of the two.

In the particular case $\mu = n-\frac{5}{2}$ $(n \geqslant 5)$ Theorem D′ may be improved. For, from C′, *if* $\mathbf{q}_1, \ldots, \mathbf{q}_n$ *are rational,* $\mathbf{S}(n-\frac{5}{2}, N)$ *oscillates finitely as* $N \to \infty$; in fact,

$$\mathbf{S}(n-\tfrac{5}{2}, N) = \pi^{\frac{1}{2}(n-5)} \psi_{\mathbf{Q}}^{\frac{1}{2}(n-5)}(x)\, e^{2\pi i \Sigma \mathbf{pq}}$$

$$-\pi^{-1} D^{-\frac{1}{2}} x^{\frac{1}{2}(n-3)} N^{-\frac{1}{2}n+1} \psi_{\mathbf{P}}(N) \cos\{2\pi\sqrt{(Nx)} - \tfrac{1}{2}n\pi\} + o(1);$$

and, from D′, $\mathbf{S}(n-\frac{5}{2}, \infty)$ is summable $(C, 1)$. Hence (when $n \geqslant 5$, $\mu = n-\frac{5}{2}$), by a known theorem*, $\mathbf{S}$ is summable (C, δ) *for every* $\delta > 0$.

It is remarkable that Theorem D requires for its complete proof the assumption of Lemma 8, whereas Lemma 2 is sufficient for the proof of Theorem D′. On the other hand, Theorem C requires only Lemma 2. while C′ requires Lemma 8. The difficulty is, in fact, precisely the same in the two cases; it consists in bridging the interval of values of μ,

$$n-\tfrac{3}{2} \geqslant \mu > n-\tfrac{5}{2}. \tag{6.0}$$

For all other values $(n-\frac{1}{2} \geqslant \mu > \frac{1}{2}n-1)$ Lemma 2 is sufficient. Lemma 7 serves to cover part of the interval (6.0), but still leaves†

$$n-\tfrac{5}{2}+\frac{2}{n+1} \geqslant \mu > n-\tfrac{5}{2}, \tag{6.01}$$

and to bridge this gap Lemma 8 is required. That Lemma 8 is actually necessary is easily seen; for, if we note that, in consequence of Lemmas 3 and 7,

$$\psi_{\mathbf{p}}^{(1)}(x) = o(x^{\frac{1}{2}n-\frac{1}{2}}),$$

Lemma 8 is a corollary of Theorems B′ and C′ when $\mu = n-\frac{5}{2}$.

The most convenient form of the definition of the Riesz sum‡ (R, p, κ) of the series S is

$$\lim_{\omega \to \infty} \kappa\omega^{-\kappa} \int_0^{\omega} (\omega-\tau)^{\kappa-1} S(\mu, \tau)\, d\tau,$$

in which it is assumed that $\kappa > 0$. This involves the determination,

* Andersen, *Over Cesàro's Summabilitetsmetode* (1921), 56–59 (Sætning 8).

† In case $n = 4$, $\mu = \frac{1}{2}n = 2$ falls within the interval (6.0), but not within (6.01). Oppenheim (*loc. cit.*) proves Theorem D for the hypersphere, when $\mu = \frac{1}{2}n$ $(n \geqslant 3)$, without making explicit use of Lemma 7. But it is easy to deduce Lemma 7 from Theorem B by means of Lemmas 2 and 4 [Wilton, *Proc. Royal Soc.* (A), 120 (1928), 358–369)]; so that the difference is more apparent than real.

‡ G. H. Hardy and Marcel Riesz, *The general theory of Dirichlet series* (1915), 21.

when they exist, of the two limits

$$(6.11)\quad \lim_{\omega\to\infty} \kappa\omega^{-\kappa} \sum_{k=0}^{\lambda-1} \pi^k \int_0^\omega (\omega-\tau)^{\kappa-1} \left(\frac{x}{\tau}\right)^{\frac{1}{2}(\mu+k)} J_{\mu+k}\{2\pi\sqrt{(\tau x)}\}\, \psi^{(k)}(\tau)\, d\tau,$$

$$(6.12)\quad \lim_{\omega\to\infty} \kappa\omega^{-\kappa} \sum_{k=1}^{\lambda} \frac{\pi^{\frac{1}{2}n-k} x^{\frac{1}{2}(\mu-k)}}{\Gamma(\frac{1}{2}n-k+1)} \int_0^\omega (\omega-\tau)^{\kappa-1} \tau^{\frac{1}{2}(n-\mu-k)} J_{\mu-k}\{2\pi\sqrt{(\tau x)}\}\, d\tau,$$

since the terms for which $k>\lambda$ in the second sum evidently vanish. It will be shown that (6.11) is zero if $\kappa=\lambda-1$, and that (6.12) vanishes if $\kappa>n-\mu-\frac{3}{2}$. The proof depends upon but does not use the full force of Lemma 9, which is stated with a view to the proof of Theorem D″.

LEMMA 9. *If $\phi(t)$* (i) *increases monotonely throughout the interval* $(0, 1)$, *and* (ii) *possesses a derivate on the left at $t=1$, and if $0<\kappa<1$, then, as $u\to\infty$,*

$$\int_0^1 (1-t)^{\kappa-1}\phi(t) \begin{matrix}\cos\\ \sin\end{matrix} ut\, dt = u^{-\kappa}\Gamma(\kappa)\,\phi(1) \begin{matrix}\cos\\ \sin\end{matrix} (u-\tfrac{1}{2}\kappa\pi)+O(u^{-\frac{1}{2}\kappa-\frac{1}{2}}).$$

It will be sufficient to consider the cosine. If $u>1$,

$$\begin{aligned}
&\int_0^1 (1-t)^{\kappa-1}\phi(t)\cos ut\, dt\\
&\quad=\int_0^1 t^{\kappa-1}\phi(1-t)\cos u(1-t)\, dt\\
&\quad=\phi(1)\int_0^{u^{-\frac{1}{2}}} t^{\kappa-1}\cos u(1-t)\, dt-\int_0^{u^{-\frac{1}{2}}} t^{\kappa-1}\{\phi(1)-\phi(1-t)\}\cos u(1-t)\, dt\\
&\qquad\qquad+\int_{u^{-\frac{1}{2}}}^1 t^{\kappa-1}\phi(1-t)\cos u(1-t)\, dt\\
&\quad=I_1\phi(1)-I_2+I_3, \quad \text{say.}
\end{aligned}$$

By the second theorem of the mean,

$$\begin{aligned}
I_3&=\phi(1-u^{-\frac{1}{2}})\,u^{\frac{1}{2}(1-\kappa)}\int_{u^{-\frac{1}{2}}}^{\delta}\cos u(1-t)\, dt+\phi(1)\int_{\delta}^1\cos u(1-t)\, dt \qquad (u^{-\frac{1}{2}}\leqslant\delta\leqslant 1)\\
&=O(u^{-\frac{1}{2}-\frac{1}{2}\kappa}).\\
I_2&=\{\phi(1)-\phi(1-u^{-\frac{1}{2}})\}\int_{\delta'}^{u^{-\frac{1}{2}}} t^{\kappa-1}\cos u(1-t)\, dt \quad (0\leqslant\delta'\leqslant u^{-\frac{1}{2}})\\
&=O(u^{-\frac{1}{2}-\frac{1}{2}\kappa}).
\end{aligned}$$

And, finally,

$$I_1 = u^{-\kappa}\left(\int_0^\infty - \int_{u^{\frac{1}{2}}}^\infty\right) t^{\kappa-1}\cos(u-t)\,dt$$

$$= u^{-\kappa}\Gamma(\kappa)\cos(u-\tfrac{1}{2}\kappa\pi) + O(u^{-\frac{1}{2}-\frac{1}{2}\kappa}).$$

Returning now to (6.11) and (6.12), write $\omega\tau$ instead of τ, and consider the expressions ($k = 0, 1, \ldots, \lambda-1$)

$$(6.21)\quad \omega^{-(\mu+k+m)}\int_0^1 (1-\tau)^{\kappa-1}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu+k)} J_{\mu+k}\{2\pi\sqrt{(\omega x\tau)}\}\,\frac{d^m}{d\tau^m}\,\psi^{(m+k)}(\omega\tau)\,d\tau$$

$$= \omega^{-(\mu+k+m)}\int_0^1 \psi^{(m+k)}(\omega\tau)$$

$$\times\left(-\frac{d}{d\tau}\right)^m\left\{(1-\tau)^{\kappa-1}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu+k)} J_{\mu+k}\left(2\pi\sqrt{(\omega x\tau)}\right)\right\}d\tau,$$

provided that $m < \kappa$, and

$$(6.22)\quad \omega^{\frac{1}{2}\mu-\mu}\int_0^1 (1-\tau)^{\kappa-1}\,\tau^{\frac{1}{2}\mu-k}\,\frac{d^m}{d\tau^m}\left\{\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu-k-m)} J_{\mu-k-m}\left(2\pi\sqrt{(\omega x\tau)}\right)\right\}d\tau$$

$$= \omega^{\frac{1}{2}\mu-\mu}\int_0^1\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu-k-m)} J_{\mu-k-m}\{2\pi\sqrt{(\omega x\tau)}\}$$

$$\times\frac{d^m}{d\tau^m}\{\tau^{\frac{1}{2}\mu-k}(1-\tau)^{\kappa-1}\}\,d\tau,$$

provided that both $m < \kappa$ and $k+m < \frac{1}{2}n+1$, in order that the integrated parts may vanish at both the upper and the lower limits. But if $k+m = \frac{1}{2}n+1$ (n therefore even) the non-vanishing term in the last step of the process of integration by parts is $O(\omega^{-1})$. And, on dividing the interval of integration into the parts $(0, \Delta\omega^{-1})$ and $(\Delta\omega^{-1}, 1)$, it is readily seen that, if

$$k+m = \tfrac{1}{2}n+\tfrac{1}{2},$$

the expression on the right of (6.22) is

$$O(\omega^{-\frac{1}{2}}) + O\{\omega^{\frac{1}{2}(n-\mu-k-m-\frac{1}{2})}\} = O(\omega^{-\frac{1}{2}}) + O\{\omega^{-\frac{1}{2}(\mu-\frac{1}{2}n+1)}\} = o(1).$$

It follows that, if, in (6.22), it be assumed that $\lambda > \kappa \geq \lambda-1$, it is permissible to put $m = \lambda-1$, provided that those terms, if any, of

$$\frac{d^m}{d\tau^m}\{\tau^{\frac{1}{2}n-k}(1-\tau)^{\kappa-1}\},$$

which contain negative powers of τ are simply ignored. With this understanding, let $m = \lambda-1$, and separate out the term involving $(1-\tau)^{\kappa-\lambda}$,

i.e. (except for a constant factor)

$$(6.31)\quad \omega^{\frac{1}{2}n-\mu}\int_0^1 (1-\tau)^{\kappa-\lambda}\tau^{\frac{1}{2}n-k}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu-\lambda-k+1)} J_{\mu-\lambda-k+1}\{2\pi\sqrt{(\omega x\tau)}\}\,d\tau.$$

The remaining terms of (6.22) may be integrated by parts once more, and if the term in $(1-\tau)^{\kappa-\lambda}$ is again separated out, the effect is merely to change k in (6.31) into $k+1$. The process may be repeated, and the final result is a set of terms of the same form, with $k=1, 2, \ldots, [\frac{1}{2}n]$. The part of (6.31) which arises from the integral over the interval $(0, \Delta\omega^{-1})$ is $O(\omega^{-\lambda})$; and over the interval $(\Delta\omega^{-1}, 1)$ the Bessel function is $O(\omega x\tau)^{-\frac{1}{2}}$. Hence, as $\omega\to\infty$, the order of magnitude of the whole expression is certainly not greater than

$$(6.32)\qquad \omega^{\frac{1}{2}(n-\mu-\frac{1}{2}-\lambda)+\frac{1}{2}(1-k)}\int_{\Delta/\omega}\tau^{\frac{1}{2}(n-\mu-\frac{3}{2})+\frac{1}{2}(\lambda-k)}\,d\tau+O(\omega^{-\lambda}).$$

The index of ω is negative when $k\geqslant 2$, and at the lower limit of the integral the whole expression is $O(\omega^{-\lambda})$. Thus the only case requiring attention is $k=1$; moreover, in this case, the Bessel function may be replaced by the dominant term of its asymptotic expansion, for the error involved in the substitution is $O(\omega x\tau)^{-\frac{3}{4}}$, and it is readily seen that this is negligible—the corresponding term is, in fact, equivalent to (6.32) with $k=2$. It follows that, when $k=1$, the order of (6.31) is equal to that of

$$(6.33)\quad \omega^{\frac{1}{2}(n-\mu-\frac{1}{2}-\lambda)}\int_0^1 (1-\tau)^{\kappa-\lambda}\tau^{\frac{1}{2}(\lambda+n-\mu-\frac{1}{2})-1} \times\cos\{2\pi\sqrt{(\omega x\tau)}-\tfrac{1}{2}(\mu-\lambda+\tfrac{1}{2})\pi\}\,d\tau$$
$$= O\{\omega^{\frac{1}{2}(n-\mu-\frac{3}{2}-\kappa)}\} = O(\omega^{-\frac{1}{2}\delta}),$$

by Lemma 9, if $\kappa=n-\mu-\frac{3}{2}+\delta$. This establishes Theorem D in so far as it depends upon (6.12). (Incidentally it is clear that, except for the term $k=1$, it would be possible to take $\kappa=n-\mu-\frac{3}{2}+\delta$ for every $\delta>0$.) The remainder of the proof follows much the same course.

When μ lies in the interval (6.0), $\lambda=1$. In order to postpone as long as possible the use of Lemma 8, I therefore assume, in the first place, that $\lambda\geqslant 2$.

The part of the expression on the right of (6.21) due to the integral over $(0, \Delta\omega^{-1})$ is $O(\omega^{-1})$ for every k, and therefore, as before, the asymptotic expansion of the Bessel function may be employed. In case

$$(6.4)\qquad m+k>\tfrac{1}{2}n-\tfrac{1}{2}$$

it follows from (1 . 6) that every term on the right of (6 . 21) is

$$O\{\omega^{-\frac{1}{2}(\mu-\frac{1}{2}n+1)}\} = o(1). \tag{6 . 41}$$

Let $\kappa = \lambda-1$ and $m = \lambda-2$ in (6 . 21) and integrate once more by parts. The integrated part does not now vanish, but gives rise to a term

$$\Gamma(\lambda-1)\,\omega^{-(\mu+\lambda+k-1)}\psi^{(\lambda+k-1)}(\omega)\,(\omega x)^{\frac{1}{2}(\mu+k)}J_{\mu+k}\{2\pi\sqrt{(\omega x)}\} = o(\omega^{1-\frac{1}{2}\lambda}) = o(1),$$

in every case, provided that, in the statement of Lemma 4, we take

$$\epsilon < \tfrac{1}{2}\{[n-\mu-\tfrac{1}{2}]-(n-\mu-\tfrac{3}{2})\}. \tag{6 . 42}$$

Integration by parts may be continued; for any value of m the non-vanishing term of the integrated part is a multiple of

$$\omega^{-\frac{1}{2}(\mu+\lambda+k+m)}\psi^{(m+k+1)}(\omega)\,J_{\mu+m+k-\lambda+2}\{2\pi\sqrt{(\omega x)}\}.$$

If $m+k+1 \geqslant \frac{1}{2}n-\frac{1}{2}$, the order of this expression is given by (6 . 41); and, if the condition is not satisfied, the order is ($m \geqslant 0$)

$$O\{\omega^{\frac{1}{2}(n-\mu-\frac{3}{2}-\lambda)-\frac{1}{2}k-\frac{1}{2}m+\epsilon}\} = o(\omega^{-\frac{1}{2}m}) = o(1),$$

if $\epsilon > 0$ is chosen to satisfy (6 . 42). The process of integration by parts may be continued until (6 . 4) is satisfied.

This completes the proof* of Theorem D′ and proves Theorem D except when μ lies in the interval (6 . 0).

When μ lies in (6 . 0), and therefore $\lambda = 1$, the terms of (6 . 11) reduce to the single one $k = 0$, *i.e.*

$$\kappa\omega^{-\kappa}\int_0^\omega(\omega-\tau)^{\kappa-1}\left(\frac{x}{\tau}\right)^{\frac{1}{2}\mu}J_\mu\{2\pi\sqrt{(\tau x)}\}\,\psi(\tau)\,d\tau \tag{6 . 5}$$

$$= O\left\{\omega^{-\frac{1}{2}\mu-\frac{1}{4}}\operatorname*{Max}_{\tau\leqslant\omega}|\psi(\tau)|\right\},$$

for every κ in the interval $0 \leqslant \kappa < 1$, where, of course, the interpretation of (6 . 5), when $\kappa = 0$, is

$$\left(\frac{x}{\omega}\right)^{\frac{1}{2}\mu}J_\mu\{2\pi\sqrt{(\omega x)}\}\,\psi(\omega) = O\{\omega^{-\frac{1}{2}\mu-\frac{1}{4}}|\psi(\omega)|\}. \tag{6 . 51}$$

It may therefore be assumed that $\kappa = 0$.

(i) If Lemma 2 is employed it is not proved that (6 . 51) is $o(1)$ for any μ in (6 . 0).

---

* This assumes the "theorem of consistency", that a series summable (R, p, κ) is summable (R, p, κ') to the same sum if $\kappa' > \kappa$. (Here $\kappa \geqslant 0$.)

(ii) If Lemma 7 is employed, (6 . 51) is seen to be

$$O\left\{\omega^{\frac{1}{2}\left\{n-\frac{3}{2}-\mu+2/(n+1)\right\}+\epsilon}\right\} = o(1),$$

when μ, in (6 . 0), is not within the interval (6 . 01), provided that we choose

$$\epsilon < \tfrac{1}{2}\left\{\mu-\left(n-\tfrac{5}{2}+\frac{2}{n+1}\right)\right\}.$$

(iii) Finally, employing Lemma 8, we see that (6 . 51) is ($n \geqq 4$)

$$O\left\{\omega^{-\frac{1}{2}(\mu-n+\frac{3}{2})}\log^2\omega\right\} = o(1)$$

in the interval (6 . 01).

The proof of Theorem D is thus completed.

7. By means of Lemma 9 it may now be shown that Theorem D is the best possible of its kind.

THEOREM D″. *If* $n-\frac{3}{2} \geqq \mu > \frac{1}{2}n-1$, *the series S is not summable* $(R, p, n-\mu-\frac{3}{2})$. *More precisely, under the conditions imposed on n and* μ *by Theorem D, the typical mean corresponding to the sum* $(R, p, n-\mu-\frac{3}{2})$ *oscillates; and its actual value is*

$$(7\,.\,0)\quad \pi^{\mu-\frac{1}{2}n}\,V_Q^{(\mu-\frac{1}{2}n)}(x) + \frac{\Gamma(n-\mu-\frac{1}{2})}{\sqrt{(\pi D)}\,\Gamma(\frac{1}{2}n)}\,(\pi x)^{\mu-\frac{1}{2}n}\cos\left\{2\pi\sqrt{(x\omega)}-\tfrac{1}{2}n\pi\right\}+o(1),$$

provided that, when $\frac{1}{2}n > \mu > \frac{1}{2}n-1$, Dx *is not an integer.*

When $\mu = n-\frac{3}{2}$ the result is evident from Theorem B and Lemma 7. When $\mu < n-\frac{3}{2}$,

$$\kappa = n-\mu-\tfrac{3}{2} > 0,$$

and every term of (6 . 11) vanishes, and also every term of (6 . 12), except that for which $k = 1$; and, when this term is integrated by parts $\lambda-1$ times, the only part of it which survives is the limit of

$$(7\,.\,1)\quad (-)^{\lambda-1}\,\frac{\pi^{\frac{1}{2}n-\lambda}\,\Gamma(\kappa+1)}{\Gamma(\frac{1}{2}n)\,\Gamma(\kappa-\lambda+1)}\,\omega^{\frac{1}{2}n-\mu}$$

$$\times\int_0^1 (1-\tau)^{\kappa-\lambda}\,\tau^{\frac{1}{2}n-1}\left(\frac{\omega x}{\tau}\right)^{\frac{1}{2}(\mu-\lambda)} J_{\mu-\lambda}\left\{2\pi\sqrt{(\omega x\tau)}\right\}d\tau$$

$$\sim (-)^{\lambda-1}\,\frac{\pi^{\frac{1}{2}n-\lambda-1}\,\Gamma(n-\mu-\frac{1}{2})}{\Gamma(\frac{1}{2}n)\,\Gamma(n-\mu-\frac{1}{2}-\lambda)}\,x^{\frac{1}{2}(\mu-\lambda-\frac{1}{2})}\,\omega^{\frac{1}{2}(n-\mu-\frac{1}{2}-\lambda)}$$

$$\times\int_0^1 (1-\tau)^{n-\mu-\frac{3}{2}-\lambda}\,\tau^{\frac{1}{2}(n-\mu-\frac{1}{2}+\lambda)-1}\cos\left\{2\pi\sqrt{(\omega x\tau)}-\tfrac{1}{2}(\mu-\lambda+\tfrac{1}{2})\pi\right\}d\tau.$$

An exceptional case arises when $n-\mu-\frac{1}{2}$ is an integer (λ), for then $\kappa = \lambda-1$. The expression corresponding to (7 . 1) in this case is the non-vanishing integrated part of (6 . 22)—multiplied by the appropriate constant factor—when $m = \lambda-2$, namely

$$(-)^{\lambda-1}\pi^{\frac{1}{2}n-\lambda}\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}n)}\omega^{\frac{1}{2}\mu-\mu}(\omega x)^{\frac{1}{2}(\mu-\lambda)}J_{\mu-\lambda}\{2\pi\sqrt{(\omega x)}\},$$

$$\sim(-)^{\lambda-1}\pi^{\frac{1}{2}n-\lambda-1}\frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}n)}x^{\frac{1}{2}(\mu-\lambda-\frac{1}{2})}\cos\{2\pi\sqrt{(\omega x)}-\tfrac{1}{2}(\mu-\lambda+\tfrac{1}{2})\pi\}.$$

On reference to (4 . 1) this is seen to be the result stated in Theorem D″ when $\lambda = n-\mu-\frac{1}{2}$; and the general case follows, from (7 . 1), by means of Lemma 9. For, putting $\tau = t^2$ in (7 . 1),

$$\phi(t) = 2(1+t)^{n-\mu-\frac{3}{2}-\lambda}t^{\lambda+n-\mu-\frac{3}{2}}$$

has a definite positive differential coefficient throughout the interval (0, 1).

BIHARMONIC ANALYSIS, ESPECIALLY IN A RECTANGLE, AND ITS APPLICATIONS TO THE THEORY OF ELASTICITY

By A. E. H. LOVE.

[Received 19 March, 1928.—Read 26 April, 1928.]

The object of this paper is to develop a method of constructing functions which shall be biharmonic in a simply-connected plane region, inside a specified boundary, and shall also satisfy certain special conditions at the boundary, and to exemplify the application of the analysis to questions that arise in the Theory of Elasticity. The paper is arranged in four chapters. A general description of the analytical method is given in Chapter 1. Chapter 2 is occupied with special developments for a rectangular boundary. The algebraic details of a certain process, essential to the construction of the functions, are placed in Chapter 3. Chapter 4 is devoted to the discussion of three particular applications to the Theory of Elasticity.

1. *Biharmonic functions.*

1.1. We are concerned with real functions of two real variables x and y, which are regarded as cartesian rectangular coordinates of a point in a plane, and with a simply-connected region of the plane, the region being inside a simple closed contour. The functions are to be "biharmonic," a designation which means that:

(i) Any such function is continuous, and has continuous derivatives of the first four orders with respect to x and y, at every point in the region.

(ii) Any such function χ satisfies the partial differential equation

$$\nabla^4\chi = 0$$

at every point of the region. In this statement $\nabla^4\chi$ means $\nabla^2(\nabla^2\chi)$, and ∇^2 denotes the operator $\partial^2/\partial x^2+\partial^2/\partial y^2$.

The boundary conditions imposed upon the biharmonic function χ will usually be taken to be that the boundary values of χ and of its outward normal derivative are prescribed. Parts of the theory would apply also to other types of boundary conditions. For example, the boundary values of χ and $\nabla^2\chi$ might be prescribed.

1.2. Let ν indicate the direction of the outward normal to the boundary. It can be proved that, if either χ and $\partial\chi/\partial\nu$, or χ and $\nabla^2\chi$, are given at any point of the boundary, the biharmonic function χ is uniquely determinate, in the sense that, if one function, biharmonic in the region, satisfies the boundary conditions, no other can. The proof follows easily from the generalized Green's equation

$$\iint \nabla^2\chi\nabla^2\chi'\,dx\,dy = \iint \chi\nabla^4\chi'\,dx\,dy + \int\left\{\frac{\partial\chi}{\partial\nu}\nabla^2\chi' - \chi\frac{\partial}{\partial\nu}(\nabla^2\chi')\right\}ds$$

$$= \iint \chi'\nabla^4\chi\,dx\,dy + \int\left\{\frac{\partial\chi'}{\partial\nu}\nabla^2\chi - \chi'\frac{\partial}{\partial\nu}(\nabla^2\chi)\right\}ds, \qquad (1)$$

in which χ and χ' are any two functions free from singularity [in the sense of 1.1 (i)], but not necessarily biharmonic, in the region, the double integrals are taken over the area of the region, and the single integrals are line integrals taken round the boundary, ds denoting an element of arc of the boundary, supposed to be described in the positive sense (so that $d\nu$ and ds are directions to which the positive directions of x and y can be brought by rotating the axes of x and y in their plane).

1.3. We shall now suppose the boundary conditions imposed upon the biharmonic function χ to be

$$\chi = \theta, \quad \frac{\partial\chi}{\partial\nu} = \psi, \qquad (1)$$

where θ and ψ are functions of some parameter specifying a point of the boundary. We shall always suppose these functions to be of quite simple kinds, *e.g.* such as admit of expansion in Fourier series.

The boundary conditions (1) admit of reduction to a simpler type. Let

$$\chi = \omega_1 + \omega_2, \qquad (2)$$

where both ω_1 and ω_2 are biharmonic in the region and the boundary conditions for them are

$$\omega_1 = 0, \quad \frac{\partial\omega_1}{\partial\nu} = \psi; \quad \omega_2 = \theta, \quad \frac{\partial\omega_2}{\partial\nu} = 0. \qquad (3)$$

Let $\Phi(\theta)$ denote† the function which is harmonic in the region and equal to θ on the boundary, and let us put

$$\omega_2 = \Phi(\theta) + \omega'_2. \tag{4}$$

Then ω'_2 is biharmonic in the region, and satisfies the boundary conditions

$$\omega'_2 = 0, \quad \frac{\partial \omega'_2}{\partial \nu} = -\frac{\partial}{\partial \nu}\Phi(\theta). \tag{5}$$

Thus the problem of determining ω_2 becomes a somewhat special case of the kind of problem that has to be solved in order to determine ω_1, the special feature being that the prescribed boundary value of the normal derivative of ω'_2 is the boundary value of the normal derivative of a harmonic function.

This special feature always presents itself if the line integral

$$\int \psi ds,$$

taken round the boundary of the region, vanishes. If

$$\int \psi ds \neq 0,$$

the general problem of determining ω_1 may always be reduced to the more special case in which the prescribed boundary value of the normal derivative is equal to that of a harmonic function, by putting

$$\omega_1 = C\{x^2+y^2-\Phi(x^2+y^2)\} + \omega'_1, \tag{6}$$

and determining the constant C by the equation

$$\int \psi ds = C \int \frac{\partial}{\partial \nu}(x^2+y^2)\, ds = 4AC, \tag{7}$$

where A is the area of the region. Then ω'_1 is biharmonic in the region and vanishes at the boundary, and the boundary value of its normal derivative is equal to that of a harmonic function.

Although the reduction can always be made it is sometimes more convenient not to make it.

1.4. To construct a biharmonic function χ we observe that $\nabla^2\chi$ will have to be harmonic in the region. Let

$$\nabla^2\chi = V,$$

† The notation Φ will be used frequently.

and let U be the harmonic function to which V is conjugate, so that $U+iV$ is a function of the complex variable z $(\equiv x+iy)$. This function is holomorphic at every point of the region. The point z being inside the region, a function $u+iv$ of z can be determined by the formula

$$u+iv = \tfrac{1}{4}\int_0^z (U+iV)\,dz. \tag{2}$$

Then, provided that the path of integration does not go outside the region, $u+iv$ also is a holomorphic function of z at every point of the region, and we have

$$U+iV = 4\frac{d}{dz}(u+iv) = 4\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right). \tag{3}$$

Hence
$$V = 4\frac{\partial v}{\partial x} = 2\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{4}$$

so that
$$\nabla^2\chi = \nabla^2(xv-yu), \tag{5}$$

and χ must be of the form

$$\chi = xv-yu+v_1, \tag{6}$$

where v_1 is harmonic in the region. It follows that any function of the form

$$xv-yu-\Phi(xv-yu), \tag{7}$$

in which $u+iv$ is a holomorphic function of z at every point of the region, is biharmonic in the region and vanishes at the boundary, and, conversely, that every function, biharmonic in the region and vanishing at the boundary, can be expressed in this form.

1.5. The way is now open for the construction of two families of functions, χ_0, χ_1, ..., and χ'_0, χ'_1, ..., all biharmonic in the region and vanishing at the boundary, and such that any function, having these properties, can be expressed as a sum, or it may be an infinite series, of the form

$$\Sigma(A_n\chi_n+A'_n\chi'_n) \tag{1}$$

with suitable coefficients. For this purpose the region in the plane of z, supposed to be simply connected, is represented conformally on the region within a circle, $|\zeta|=1$, in the plane of an auxiliary complex variable ζ. It may be assumed that this transformation can be effected. Then any function, such as $u+iv$, holomorphic at every point in the z region, becomes a function of ζ, holomorphic at every point within the

circle. In particular, z is such a function, and it is possible to express z by a power series in ζ, say

$$z = \sum_{n=0}^{\infty} a_{n+1}\zeta^{n+1}, \tag{2}$$

the origins in the two planes being corresponding points, and the series representing the function z at every point inside the circle. In general, the coefficients a need not be real. When the z boundary is a rectangle we shall find that (2) can be replaced by

$$z = -i\sum_{n=0}' a_{2n+1}\zeta^{2n+1}, \tag{3}$$

where all the coefficients a are real, and the equation holds at every point on the circle $|\zeta| = 1$, as well as at every point inside it. The general method will be described for the more special case where a formula of the type (3) holds, but it will be obvious that by suitable simple modifications it could be adapted to the general case represented by (2).

Every function that is harmonic in the z region is transformed into a function that is harmonic in the ζ region. Of course, a function that is biharmonic in the z region is not, in general, biharmonic in the ζ region.

We shall write

$$\zeta = \rho e^{i\phi}, \tag{4}$$

so that ρ is the absolute value and ϕ the argument of ζ. Then ρ and ϕ are the polar coordinates of the point in the ζ region that corresponds to the point (x, y) in the z region, and $\rho = 1$ at every point of the boundary.

1 . 51. To construct a family of functions, biharmonic in the z region and vanishing at the boundary, we take

$$\chi_n = X_n - \Phi(X_n), \quad X_n = R(-iz^*\zeta^{n+1}), \quad (n = 0, 1, 2, \ldots), \tag{1}$$

where R denotes "the real part of", and z^* denotes the complex number conjugate† to z. We should not obtain a function of the family by taking ζ^n instead of ζ^{n+1} and putting $n = 0$, but an expression which is identically zero. To construct a related family we put

$$\chi'_n = X'_n - \Phi(X'_n), \quad X'_n = R(z^*\zeta^{n+1}), \quad (n = 0, 1, 2, \ldots). \tag{2}$$

† This asterisk notation will be used frequently.

Since, in the notation of § 1 . 4,

$$xv-yu = R\{-iz^*(u+iv)\},$$

and $u+iv$ is a holomorphic function of ζ at every point inside the circle $\rho = 1$, it follows that every function which is biharmonic in the z region, and vanishes at the boundary, can be expressed in the form 1 . 5 (1).

1 . 52. To form $\Phi(X_n)$ and $\Phi(X'_n)$ we shall suppose that z can be expressed in terms of ζ by the series in the right-hand member of 1 . 5 (3). We have

$$X_0 = R(-iz^*\zeta) = R(iz\zeta^*). \tag{1}$$

Now on the boundary, where $|\zeta| = 1$, $\zeta^* = 1/\zeta$. Also the function z/ζ is holomorphic at every point inside the circular boundary, including the point $\zeta = 0$, and thus we have

$$\Phi(X_0) = R\left(\frac{iz}{\zeta}\right). \tag{2}$$

When n is even and greater than 0, we put $2n$ for n, and have

$$X_{2n} = R(-iz^*\zeta^{2n+1}) = R(iz\zeta^{*2n+1}); \tag{3}$$

and, on the boundary,

$$R(iz\zeta^{*2n+1}) = R(iz\zeta^{-(2n+1)})$$

$$= R\left(\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n+1}}\right.$$

$$\left.+a_1\zeta^{*2n}+a_3\zeta^{*2n-2}+\ldots+a_{2n-1}\zeta^{*2}\right).$$

Since all the coefficients a are real,

$$\Phi(X_{2n}) = R\left(\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n+1}}\right)$$

$$+R(a_1\zeta^{2n}+a_3\zeta^{2n-2}+\ldots+a_{2n-1}\zeta^2). \tag{4}$$

When n is odd, we put $2n-1$ for n, and have

$$X_{2n-1} = R(-iz^*\zeta^{2n}) = R(iz\zeta^{*2n}); \tag{5}$$

and, on the boundary,

$$R(iz\zeta^{*2n}) = R(iz\zeta^{-2n})$$

$$= R\left(\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n}}\right.$$

$$\left.+a_1\zeta^{*2n-1}+a_3\zeta^{*2n-3}+\ldots+a_{2n-1}\zeta^*\right).$$

Then $$\Phi(X_{2n-1}) = R\left(\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n}}\right)$$
$$+R(a_1\zeta^{2n-1}+a_3\zeta^{2n-3}+\ldots+a_{2n-1}\zeta). \quad (6)$$

1 . 521. The harmonic function $\Phi(X'_n)$ is formed in the same way. We have
$$X'_0 = R(z^*\zeta) = R(z\zeta^*), \quad (1)$$
and, on the boundary, $R(z\zeta^*) = R(z/\zeta)$,
whence
$$\Phi(X'_0) = R\left(\frac{z}{\zeta}\right). \quad (2)$$

When n is even and greater than 0, we put $2n$ for n, and have
$$X'_{2n} = R(z^*\zeta^{2n+1}) = R(z\zeta^{*2n+1}) = R\{(-i(iz\zeta^{*2n+1})\}, \quad (3)$$
whence
$$\Phi(X'_{2n}) = R\left(-i\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n+1}}\right)$$
$$+R\{i(a_1\zeta^{2n}+a_3\zeta^{2n-2}+\ldots+a_{2n-1}\zeta^2)\}. \quad (4)$$

When n is odd, we put $2n-1$ for n, and have
$$X'_{2n-1} = R(z^*\zeta^{2n}) = R(z\zeta^{*2n}) = R\{-i(iz\zeta^{*2n})\}, \quad (5)$$
whence
$$\Phi(X'_{2n-1}) = R\left(-i\frac{iz-a_1\zeta-a_3\zeta^3-\ldots-a_{2n-1}\zeta^{2n-1}}{\zeta^{2n}}\right)$$
$$+R\{i(a_1\zeta^{2n-1}+a_3\zeta^{2n-3}+\ldots+a_{2n-1}\zeta)\}. \quad (6)$$

1 . 53. In 1 . 52 (1) and (2) we put $\zeta = \rho e^{i\phi}$, $\zeta^* = \rho e^{-i\phi}$, and substitute for iz from 1 . 5 (3), getting
$$\chi_0 = \sum_{s=0}' a_{2s+1}(\rho^2-1)\rho^{2s}\cos 2s\phi, \quad (1)$$
where it may be observed that χ_0 is biharmonic in the ζ region as well as in the z region.

Treating 1 . 52 (3) and (4) in the same way, we get
$$\chi_{2n} = a_{2n+1}(\rho^{4n+2}-1)$$
$$+\sum_{s=1}^{n}\{a_{2n-2s+1}(\rho^{4n-2s+2}-\rho^{2s})+a_{2n+2s+1}(\rho^{4n+2}-1)\rho^{2s}\}\cos 2s\phi$$
$$+\sum_{s=n+1}^{\infty} a_{2n+2s+1}(\rho^{4n+2}-1)\rho^{2s}\cos 2s\phi. \quad (2)$$

In like manner we obtain the formulae

$$\chi_{2n-1} = \sum_{s=0}^{n-1} \{a_{2n-2s-1}(\rho^{4n-2s-1} - \rho^{2s+1}) + a_{2n+2s+1}(\rho^{4n} - 1)\rho^{2s+1}\} \cos(2s+1)\phi + \sum_{s=n}^{\infty} a_{2n+2s+1}(\rho^{4n} - 1)\rho^{2s+1} \cos(2s+1)\phi, \quad (3)$$

$$\chi'_0 = \sum_{s=1}^{\infty} a_{2s+1}(\rho^2 - 1)\rho^{2s} \sin 2s\phi, \quad (4)$$

$$\chi'_{2n} = \sum_{s=1}^{n} \{-a_{2n-2s+1}(\rho^{4n-2s+2} - \rho^{2s}) + a_{2n+2s+1}(\rho^{4n+2} - 1)\rho^{2s}\} \sin 2s\phi + \sum_{s=n+1}^{\infty} a_{2n+2s+1}(\rho^{4n+2} - 1)\rho^{2s} \sin 2s\phi, \quad (5)$$

$$\chi'_{2n-1} = \sum_{s=0}^{n-1} \{-a_{2n-2s-1}(\rho^{4n-2s-1} - \rho^{2s+1}) + a_{2n+2s+1}(\rho^{4n} - 1)\rho^{2s+1}\} \sin(2s+1)\phi + \sum_{s=n}^{\infty} a_{2n+2s+1}(\rho^{4n} - 1)\rho^{2s+1} \sin(2s+1)\phi. \quad (6)$$

Again it may be observed that χ'_0 is biharmonic in the ζ region.

We see that each of the functions χ_n and χ'_n can be expressed as an infinite series of the form

$$\sum_{s=0}^{\infty} f_{n,s}(\rho) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} s\phi,$$

in which n and s are always of the same parity. Thus each of the families $\chi_0, \chi_1, \ldots$, and $\chi'_0, \chi'_1, \ldots$ breaks up into two sub-families, those with even suffixes being expansible in series of cosines, or sines, of even multiples of ϕ, and those with uneven suffixes being expansible in series of cosines, or sines, of uneven multiples of ϕ. Further, the coefficient $f_{n,s}(\rho)$ of any cosine, or sine, is a polynomial in ρ, containing at most three terms, with coefficients which are either particular members of the set $a_1, a_3, \ldots$, with possibly a change of sign or, at worst, the sum or difference of two of these.

1.54. The function χ'_0 is redundant. There exists a linear relation

$$\sum_{n=0}^{\infty} a_{2n+1}\chi'_{2n} = 0. \quad (1)$$

This holds because z^*z $(\equiv x^2+y^2)$ is real, and therefore

$$R(z^* iz) = 0, \quad (2)$$

or

$$R\left\{z^* \sum_{n=0}^{\infty} (a_{2n+1}\zeta^{2n+1})\right\} = 0, \quad (3)$$

which is the relation (1).

There can be no other redundant function in the two families, for there is no function $f(z)$, other than the product of z and a real constant, having the property

$$R\{z^* if(z)\} = 0. \tag{4}$$

For this property can be expressed in the form

$$\arg z^* + \tfrac{1}{2}\pi + \arg f(z) = \tfrac{1}{2}\pi + n\pi, \tag{5}$$

where n may be zero, or any positive or negative integer. Whence

$$\arg f(z) = \arg z + n\pi; \tag{6}$$

from which it follows that $f(z)$ is Cz, where C is real.

1.6. We have the result that every function, biharmonic in the z region and vanishing at the boundary, can be expressed in the form of a series

$$A_0\chi_0 + \sum_{n=1}^{\infty}{}' (A_n\chi_n + A'_n\chi'_n) \tag{1}$$

with appropriate coefficients, the functions χ_n, χ'_n being those constructed and discussed in §§ 1.5–1.54.

By means of these two families we can construct a function χ, biharmonic in the region and vanishing at the boundary, to satisfy the boundary condition $\nabla^2\chi = \omega$, where ω is a function of ϕ which can be expressed by a Fourier series of the form

$$b_0 + \sum_{s=1}^{\infty} (b_s \cos s\phi + b'_s \sin s\phi). \tag{2}$$

We know that we must have at every point in the region

$$\nabla^2\chi = \Phi(\omega) = b_0 + \sum_{s=1}^{\infty}{}' \rho^s (b_s \cos s\phi + b'_s \sin s\phi). \tag{3}$$

Now $$\nabla^2\chi_n = \nabla^2 X_n = \nabla^2 R(-iz^*\zeta^{n+1}) = \nabla^2(xv_n - yu_n) = 4\frac{\partial v_n}{\partial x}, \tag{4}$$

where $$u_n + iv_n = \zeta^{n+1}.$$

Equation (4) is $$\nabla^2\chi_n = 4R\left\{-i\frac{d(u_n+iv_n)}{dz}\right\},$$

or $$\nabla^2\chi_n = 4(n+1)\,R\left(-i\frac{d\zeta}{dz}\zeta^n\right).$$

Also $d\zeta/dz$ is a holomorphic function of ζ, expressible in the form

$$\frac{d\zeta}{dz} = i\sum_{s=0}^{\infty}{}' c_{2s}\zeta^{2s}, \tag{5}$$

where the coefficients c_{2s} are real. It follows that

$$\nabla^2\chi_n = 4(n+1)\sum_{s=0}^{\infty}{}' c_{2s}\rho^{n+2s}\cos(n+2s)\phi. \tag{6}$$

In like manner it may be proved that

$$\nabla^2\chi'_n = -4(n+1)\sum_{s=0}^{\infty} c_{2s}\rho^{n+2s}\sin(n+2s)\phi. \tag{7}$$

Then, if the required function is assumed to have the form (1), the coefficients

$$A_0, A_2, \ldots, A_{2n}, \ldots; A_1, A_3, \ldots, A_{2n-1}, \ldots;$$
$$A'_2, A'_4, \ldots, A'_{2n}, \ldots, A'_1, A'_3, \ldots, A'_{2n-1}, \ldots,$$

are determined successively from four sets of equations, of which we write down the first three of the first set, viz. :

$$\left.\begin{aligned} 4A_0c_0 &= b_0, \\ 4(A_0c_2+3A_2c_0) &= b_2, \\ 4(A_0c_4+3A_2c_2+5A_4c_0) &= b_4. \end{aligned}\right\} \tag{8}$$

The determination of χ from conditions of this type is usually effected by quite different methods†, and it is not suggested that the method here indicated is really any improvement on the procedure that is customary when $\nabla^2\chi$ is expressed in terms of x and y.

1 . 61. The analysis of § 1 . 6 enables us to determine the form of a biharmonic function χ' so that, everywhere in the rectangle,

$$\nabla^2\chi' = \rho^{2n}\cos 2n\phi. \tag{1}$$

For, by what precedes, the function must be expressible in the form

$$V+\tfrac{1}{4}\sum_{s=0}^{\infty}(2n+2s+1)^{-1}\beta_{2s}X_{2n+2s}, \tag{2}$$

where V is harmonic, and the coefficients β satisfy the equations

$$\left.\begin{aligned} \beta_0c_0 &= 1, \\ \beta_2c_0+\beta_0c_2 &= 0, \\ \beta_4c_0+\beta_2c_2+\beta_0c_4 &= 0, \\ \ldots \quad & \quad \ldots \quad \quad \ldots \end{aligned}\right\} \tag{3}$$

Now from 1 . 5 (3) we have, inside the circle $\rho = 1$,

$$\frac{dz}{d\zeta} = -i\sum_{n=0}^{\infty}{}'(2n+1)a_{2n+1}\zeta^{2n},$$

† See, *e.g.*, my *Elasticity*, 4th ed., Art. 314, *c* (ii).

and from this and 1.6 (5) we get the equations determining the coefficients c in terms of the coefficients a by expressing the result that the product of the two series is equal to 1. These equations are

$$\left.\begin{aligned} a_1 c_0 &= 1, \\ 3a_3 c_0 + a_1 c_2 &= 0, \\ 5a_5 c_0 + 3a_3 c_2 + a_1 c_4 &= 0, \\ \cdots \quad & \cdots \quad \cdots \end{aligned}\right\} \tag{5}$$

It follows that $$\beta_{2s} = (2s+1)\, a_{2s+1}, \tag{6}$$

and thence that the desired function can be expressed in the form

$$\chi' = \tfrac{1}{4} \sum_{s=0}^{\infty} \frac{2s+1}{2n+2s+1} a_{2s+1} . X_{2n+2s}, \tag{7}$$

from which the harmonic function V is omitted, as not contributing anything to $\nabla^2 \chi'$.

The result may be written

$$\chi' = \tfrac{1}{4} R \left[\sum_{s=0}^{\infty} \left\{ \frac{2s+1}{2n+2s+1} a_{2s+1} (-iz^*)\, \zeta^{2n+2s+1} \right\} \right], \tag{8}$$

or compactly $$\chi' = \tfrac{1}{4} R \left(z^* \int_0^{\zeta} \zeta^{2n} \frac{dz}{d\zeta} d\zeta \right), \tag{9}$$

the integral being taken along any path which does not go outside the ζ region. Again, χ' can be arranged as a series proceeding by cosines of even multiples of 2ϕ, say

$$\chi' = \sum_{s=0}^{\infty} f_{2s}(\rho) \cos 2s\phi, \tag{10}$$

where, for example,

$$f_0(\rho) = \tfrac{1}{4} \sum_{s=0}^{\infty} \left(\frac{2s+1}{2n+2s+1} a_{2s+1}\, a_{2n+2s+1}\, \rho^{4n+4s+2} \right), \tag{11}$$

there being no difficulty in the necessary analytical steps.

This analysis will be useful presently for the purpose of expressing $\Phi(x^2+y^2)$ in terms of ρ and ϕ in the case of a rectangular boundary.

1.7. Although the functions χ_0, χ_1, ..., χ'_1, ... are immediately adaptable to the determination of a biharmonic function χ, vanishing at the boundary, from the boundary values of $\nabla^2 \chi$, they require modification for the purpose of determining the function from the boundary values of its normal derivative. Selecting for simplicity of exposition

the functions of the sub-family χ_{2n}, we may say that the process of modification is equivalent to replacing the monomial ζ^{2n+1} in the defining equation

$$\chi_{2n} = R(-iz^*\zeta^{2n+1}) - \Phi\{R(-iz^*\zeta^{2n+1})\} \tag{1}$$

by a polynomial $T_{2n+1}(\zeta)$ of the form

$$T_{2n+1}(\zeta) = \zeta^{2n+1} - r_{2n,1}\zeta - r_{2n,3}\zeta^3 - \ldots - r_{2n,2n-1}\zeta^{2n-1}, \tag{2}$$

where the coefficients are adjusted so that the modified function, denoted by $\chi_{2n}^{(2n)}$, satisfies the conditions

$$\int_0^{2\pi} \left(\frac{\partial}{\partial\rho}\chi_{2n}^{(2n)}\right)_{\rho=1} \cos 2s\phi\, d\phi = 0, \quad (s = 0, 1, \ldots, n-1). \tag{3}$$

We may describe the property of $\chi_{2n}^{(2n)}$ that is expressed by (3) by saying that $\chi_{2n}^{(2n)}$ is of "rank" $2n$, and name the process of modifying the function χ_{2n} the process of "promotion of rank". It will be discussed in detail in Chapter 3.

Precisely similar processes may be applied to the functions of the sub-family χ_{2n-1}, and to those of the two sub-families χ'_{2n} and χ'_{2n-1}. The process issues in the construction of two families of biharmonic functions, $\chi_n^{(n)}$ and $\chi'^{(n)}_n$, all vanishing at the boundary, all of the rank indicated by their suffixes, and having the property that any function χ, biharmonic in the z region and vanishing at the boundary, can be expressed by a series of the form

$$\chi = A_0\chi_0 + \sum_{n=1}^{\infty}{}' (A_n\chi_n^{(n)} + A'_n\chi'^{(n)}_n) \tag{4}$$

with appropriate coefficients. These coefficients can then be determined successively from the boundary condition, which, according to § 1 . 3, can be expressed in the form

$$\frac{\partial\chi}{\partial\rho} = \frac{\partial V}{\partial\rho}, \tag{5}$$

where V is a known harmonic function.

2. *Transformation from a rectangular to a circular boundary.*

2 . 1. We proceed to the detailed discussion of the case of a rectangular boundary given by the equations

$$x = \pm a, \quad y = \pm b, \tag{1}$$

working out the transformation of the rectangular region in the plane

of $z(\equiv x+iy)$ into a circular region in the plane of ζ $(\equiv \rho e^{i\phi})$, passing on to the transformation of certain harmonic functions, expressed by simple formulae in terms of x and y, into equivalent formulae in terms of ρ and ϕ, and obtaining, in particular, the ρ, ϕ form equivalent to $\Phi(x^2+y^2)$.

The region within the rectangular boundary in the z plane can be represented conformally on the half-plane $Y>0$ in the plane of an auxiliary complex variable $Z(\equiv X+iY)$ by the equation

$$Z = \sqrt{k_0}.\mathrm{sn}\left\{\frac{K_0}{a}(z+ib),\ k_0\right\}, \tag{1}$$

where the modulus k_0 of the elliptic function is real, and such that $0<k_0<1$, and is determined by the equation †

$$\frac{K'_0}{K_0} = \frac{2b}{a}, \tag{2}$$

in which $4K_0$ and $2iK'_0$ are the real and imaginary periods of the elliptic function.

The above half-plane can be represented conformally on the region inside the circle $|\zeta|=1$, in the plane of a second auxiliary complex variable ζ, by the equation

$$\zeta = \frac{i-Z}{i+Z}. \tag{3}$$

The equation obtained by eliminating Z between (1) and (3) effects the conformal transformation of the rectangular z region into the circular ζ region.

The correspondence of the vertices of the rectangle to points in the Z and ζ planes, and the values of the elliptic argument $K_0(z+ib)/a$ that answer to them, are shown in the following Table :

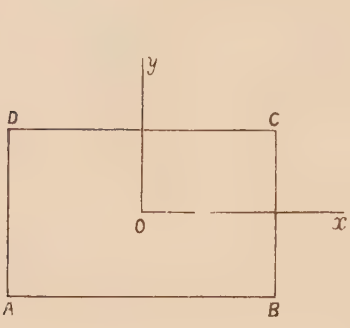
Vertex	z	Z	ζ	$K_0(z+ib)/a$
A	$-a-ib$	$-\sqrt{k_0}$	$e^{-i\alpha}$	$-K_0$
B	$a-ib$	$\sqrt{k_0}$	$e^{i\alpha}$	K_0
C	$a+ib$	$1/\sqrt{k_0}$	$e^{i(\pi-\alpha)}$	$K_0+iK'_0$
D	$-a+ib$	$-1/\sqrt{k_0}$	$e^{i(\pi+\alpha)}$	$-K_0+iK'_0$

† See, *e.g.*, A. R. Forsyth, *Theory of functions of a complex variable*, 2nd ed. (Cambridge, 1900), 639.

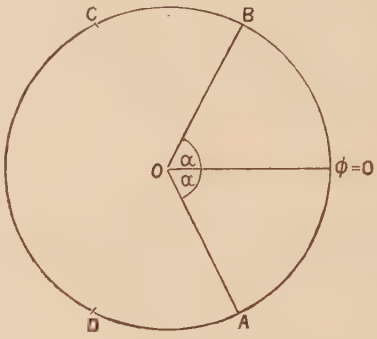
Here α is real, $0 < \alpha < \frac{1}{2}\pi$, and

$$\tan \tfrac{1}{2}\alpha = \sqrt{k_0}. \tag{4}$$

The same correspondence is shown in Fig. 1.



z region.



ζ region.

Z region.

Fig. 1.

2 . 11. The equations 2 . 1 (1) and 2 . 1 (3) yield the equations

$$\frac{K_0 dz}{a} = \left\{\left(1-\frac{Z^2}{k_0}\right)(1-k_0 Z^2)\right\}^{-\frac{1}{2}} \frac{dZ}{\sqrt{k_0}} \tag{1}$$

and

$$dZ = -2i(1+\zeta)^{-2}\, d\zeta, \tag{2}$$

with the conditions $z = 0$ and $\zeta = 0$ at $Z = i$. Elimination of Z and dZ yields the equation

$$dz = -\frac{2ia}{K_0(1+k_0)} \frac{d\zeta}{(1-2\zeta^2 \cos 2\alpha+\zeta^4)^{\frac{1}{2}}}. \tag{3}$$

Landen's transformation†, expressed by the equations

$$\left.\begin{aligned} &k_0 = (1-k')/(1+k'),\ K_0 = \tfrac{1}{2}(1+k')\,K,\ K'_0 = (1+k')\,K', \\ &v = u/(1+k'),\ \operatorname{sn}(u, k_0) = (1+k')\operatorname{sn}(v, k)\operatorname{cn}(v, k)/\operatorname{dn}(v, k), \end{aligned}\right\} \tag{4}$$

† Cf. Whittaker and Watson, *Modern analysis*, 2nd ed. (Cambridge, 1915), 500.

then gives
$$dz = -\frac{2ia}{K}\frac{d\zeta}{(1-2\zeta^2\cos 2\alpha+\zeta^4)^{\frac{1}{2}}}, \tag{5}$$

where K is the real quarter-period of sn (v, k) and

$$k = \sin\alpha. \tag{6}$$

The relation between a and b/a is now

$$\frac{K'}{K} = \frac{b}{a}, \tag{7}$$

$2iK'$ being the imaginary period of sn (v, k), so that k depends in a known way on the ratio of the length of the rectangle to its breadth.

We shall hereafter make no use of K_0 and k_0, but express our results in terms of K and k, or α.

Equation (5) gives

$$\left|\frac{dz}{d\zeta}\right| = \frac{2a}{K}\left|(1-2\zeta^2\cos 2\alpha+\zeta^4)^{-\frac{1}{2}}\right|, \tag{8}$$

and this formula expresses the scale at any point of the representation of the ζ region on the z region. On the bounding circle, $\rho = 1$, it gives

$$\left|\frac{dz}{d\zeta}\right| = \frac{a}{K}(\sin^2\alpha-\sin^2\phi)^{-\frac{1}{2}}, \quad \text{or} \quad \frac{a}{K}(\cos^2\alpha-\cos^2\phi)^{-\frac{1}{2}}, \tag{9}$$

according as the point $\zeta(\equiv e^{i\phi})$ is on one of the arcs AB, CD, or on one of the arcs BC, DA.

2 . 12. The equation 2 . 11(5) yields the formula

$$z = -\frac{2ia}{K}\sum_{n=0}^{\infty}\frac{1}{2n+1}P_n(\cos 2\alpha)\,\zeta^{2n+1}, \tag{1}$$

where P_n is the symbol for the Legendre coefficients. It appears that the coefficients a_{2n+1} of 1 . 5 (3) are expressed by the formula

$$a_{2n+1} = \frac{2a}{K}\frac{P_n(\cos 2\alpha)}{2n+1}. \tag{2}$$

Since $P_n(\cos 2\alpha)$ is $O(n^{-\frac{1}{2}})$, when n is great, a_{2n+1} is $O(n^{-\frac{3}{2}})$, and the series in the right-hand member of (1) converges absolutely and uniformly in the closed region bounded by the circle $|\zeta| = 1$, boundary included. This series represents the function z of ζ everywhere inside the circle and on its circumference.

It follows that in the same closed region

$$\left.\begin{aligned} x &= (2a/K)\sum_{n=0}^{\infty}(2n+1)^{-1}P_n(\cos 2a)\,\rho^{2n+1}\sin(2n+1)\phi, \\ y &= -(2a/K)\sum_{n=0}^{\infty}(2n+1)^{-1}P_n(\cos 2a)\,\rho^{2n+1}\cos(2n+1)\phi. \end{aligned}\right\} \quad (3)$$

It may be noted that the coefficients a_{2n+1} satisfy the difference equation

$$n(2n-1)\,a_{2n-1}+(n+1)(2n+3)\,a_{2n+3}-(2n+1)^2\,a_{2n+1}\cos 2a = 0, \quad (4)$$

and can be computed successively from this equation and the special values

$$a_1 = 2a/K, \quad a_3 = (2a/3K)\cos 2a, \quad (5)$$

if any case, not covered by existing Tables, should arise.

2 . 13. For the sake of completeness it may be noted that the coefficients c_{2s} of 1 . 6 and 1 . 61 can be evaluated explicitly for a rectangle. The series in 1 . 61 (4) has the sum

$$-i(2a/K)(1-2\zeta^2\cos 2a+\zeta^4)^{-\frac{1}{2}},$$

and therefore the series in 1 . 6 (5) has the sum

$$i(K/2a)(1-2\zeta^2\cos 2a+\zeta^4)^{\frac{1}{2}}.$$

Hence we have the identity

$$\sum_{s=0}^{\infty}c_{2s}\zeta^{2s} = (K/2a)(1-2\zeta^2\cos 2a+\zeta^4)\sum_{n=0}^{\infty}P_n(\cos 2a)\,\zeta^{2n},$$

which gives

$$\left.\begin{aligned} c_0 &= K/2a, \quad c_2 = -(K/2a)\cos 2a, \\ c_{2s} &= \frac{K}{2a}\frac{1}{2s-1}\{P_{s-2}(\cos 2a)-P_s(\cos 2a)\}, \quad (s\geqslant 2), \end{aligned}\right\} \quad (1)$$

by the use of the difference equation

$$(n+1)P_{n+1}(\cos 2a)+nP_{n-1}(\cos 2a) = (2n+1)\cos 2a\,P_n(\cos 2a).$$

2 . 2. It is necessary to transform some harmonic functions of x and y so as to express them in terms of ρ and ϕ.

Let U denote a function harmonic in the rectangle, and therefore also harmonic in the circle. The value of U on the boundary, where

$\rho = 1$, can be expressed by a Fourier series in the form

$$U = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi). \tag{1}$$

Then the value of U at any point inside or on the circle is

$$U = A_0 + \sum_{n=1}^{\infty} \rho^n (A_n \cos n\phi + B_n \sin n\phi). \tag{2}$$

When U is a given function of x and y, A_0 is determined at once as the value of U at $x = 0$, $y = 0$.

To determine A_n and B_n we use Green's equation

$$\int U \frac{\partial V}{\partial \nu} ds = \int V \frac{\partial U}{\partial \nu} ds, \tag{3}$$

ds denoting, as always, the element of arc of the (x, y) boundary, and ν indicating the direction of the outward normal to that boundary. We take V to be the harmonic function $\rho^n \cos n\phi$, or $\rho^n \sin n\phi$, and observe that, at the boundary, where $\rho = 1$,

$$\frac{\partial V}{\partial \nu} ds = \frac{\partial V}{\partial \rho} \rho d\phi = n \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} n\phi \, d\phi. \tag{4}$$

Then we have

$$n\pi A_n = \int_0^{2\pi} \left(\frac{\partial U}{\partial \nu}\right)_1 \left|\frac{dz}{d\zeta}\right|_1 \cos n\phi \, d\phi \tag{5}$$

and

$$n\pi B_n = \int_0^{2\pi} \left(\frac{\partial U}{\partial \nu}\right)_1 \left|\frac{dz}{d\zeta}\right|_1 \sin n\phi \, d\phi, \tag{6}$$

the suffix 1 denoting that the expression to which it is attached is to be evaluated at a point on the boundary. The right-hand member of (5) is the sum of four integrals :

$$\begin{aligned} &\frac{a}{K} \int_{-\alpha}^{\alpha} \frac{\cos n\phi}{\sqrt{(\sin^2 \alpha - \sin^2 \phi)}} \left(-\frac{\partial U}{\partial y}\right)_{y=-b} d\phi \\ &+ \frac{a}{K} \int_{\alpha}^{\pi-\alpha} \frac{\cos n\phi}{\sqrt{(\cos^2 \alpha - \cos^2 \phi)}} \left(\frac{\partial U}{\partial x}\right)_{x=a} d\phi \\ &+ \frac{a}{K} \int_{\pi-\alpha}^{\pi+\alpha} \frac{\cos n\phi}{\sqrt{(\sin^2 \alpha - \sin^2 \phi)}} \left(\frac{\partial U}{\partial y}\right)_{y=b} d\phi \\ &+ \frac{a}{K} \int_{\pi+\alpha}^{2\pi-\alpha} \frac{\cos n\phi}{\sqrt{(\cos^2 \alpha - \cos^2 \phi)}} \left(-\frac{\partial U}{\partial x}\right)_{x=-a} d\phi. \end{aligned} \tag{7}$$

In the same way the right-hand member of (6) is the sum of the four integrals obtained from (7) by replacing $\cos n\phi$ by $\sin n\phi$.

2 . 21. We apply this method to some examples.

First we take $$U = y.$$

Then $B_n = 0$ for all values of n. Also $A_0 = 0$, and $A_n = 0$ for all even values of n. When n is uneven, we replace it by $2n+1$, and have the equation

$$(2n+1)\pi A_{2n+1} = -\frac{4a}{K}\int_0^a \frac{\cos(2n+1)\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\,d\phi. \tag{1}$$

On comparing this result with those in 2 . 13 (2) and 2 . 13 (4) we obtain a formula

$$P_n(\cos 2a) = \frac{2}{\pi}\int_0^a \frac{\cos(2n+1)\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\,d\phi, \tag{2}$$

which is equivalent to Mehler's simplified form of Dirichlet's integral expression for a Legendre coefficient †.

We note that $$A_{2n+1} = -a_{2n+1},$$

where a_{2n+1} is defined by 2 . 12 (2).

2 . 22. As a second example we take

$$U = x^2 - y^2.$$

We get, when $n > 0$,

$$\frac{n\pi K A_n}{a} = 2a\int_\alpha^{\pi-\alpha} \frac{\cos n\phi + \cos n(\pi+\phi)}{\sqrt{(\cos^2\alpha - \cos^2\phi)}}\,d\phi$$

$$-2b\int_{-\alpha}^{\alpha} \frac{\cos n\phi + \cos n(\pi+\phi)}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\,d\phi \tag{1}$$

and

$$\frac{n\pi K B_n}{a} = 2a\int_\alpha^{\pi-\alpha} \frac{\sin n\phi + \sin n(\pi+\phi)}{\sqrt{(\cos^2\alpha - \cos^2\phi)}}\,d\phi$$

$$-2b\int_{-\alpha}^{\alpha} \frac{\sin n\phi + \sin n(\pi+\phi)}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\,d\phi. \tag{2}$$

B_n vanishes for all integral values of n, and A_n vanishes for all uneven integral values of n. Supposing n to be even, and putting $2n$ in place of it, we write

$$A_{2n} = a_{2n}, \tag{3}$$

and introduce the notation

$$a_{2n} = \int_0^\alpha \frac{\cos 2n\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\,d\phi, \quad b_{2n} = \int_\alpha^{\frac{1}{2}\pi} \frac{\cos 2n\phi}{\sqrt{(\cos^2\alpha - \cos^2\phi)}}\,d\phi. \tag{4}$$

† See, *e.g.*, Whittaker and Watson, *op. cit. ante.*, p. 309.

Then we have
$$a_{2n} = \frac{4a^2}{n\pi K^2}(Kb_{2n} - K'a_{2n}), \tag{5}$$
where use has been made of 2 . 11 (7).

To evaluate a_{2n} and b_{2n} we observe, ignoring the condition $n > 0$, and attending only to the definition in (4), that
$$a_0 = K, \quad b_0 = K'. \tag{6}$$
Again we can prove easily that
$$a_2 = 2E - K, \quad b_2 = K' - 2E'. \tag{7}$$
Then we note that
$$\frac{d}{d\phi}\{\sin 2n\phi \sqrt{(\sin^2 \alpha - \sin^2 \phi)}\} = \frac{n(\cos 2\phi - \cos 2\alpha)\cos 2n\phi - \frac{1}{2}\sin 2\phi \sin 2n\phi}{\sqrt{(\sin^2 \alpha - \sin^2 \phi)}}, \tag{8}$$
from which it appears that, when $n \geqslant 1$,
$$(2n-1)\, a_{2n-2} + (2n+1)\, a_{2n+2} = 4n a_{2n} \cos 2\alpha. \tag{9}$$
In like manner it may be proved that, when $n \geqslant 1$,
$$(2n-1)\, b_{2n-2} + (2n+1)\, b_{2n+2} = 4n b_{2n} \cos 2\alpha. \tag{10}$$
The result is that a_{2n} defined by (5) satisfies the difference equation
$$(n-1)(2n-1)\, a_{2n-2} + (n+1)(2n+1)\, a_{2n+2} = 4n^2 a_{2n} \cos 2\alpha, \tag{11}$$
with
$$a_0 = 0, \quad a_2 = -\frac{4a^2}{K^2}, \tag{12}$$
which follows from (7) and Legendre's relation
$$KE' + K'E - KK' = \tfrac{1}{2}\pi.$$
In particular,
$$a_4 = -\frac{8a^2}{3K^2}\cos 2\alpha. \tag{13}$$

It may be observed that the coefficients a_2 and a_4 can be found by squaring the series
$$-i \sum_{n=0}^{z} a_{2n+1} \zeta^{2n+1}$$
for z and selecting the real part. This process gives
$$x^2 - y^2 = -a_1^2 \rho^2 \cos 2\phi - 2a_1 a_3 \rho^4 \cos 4\phi - \ldots,$$
so that
$$a_2 = -a_1^2 = -4a^2/K^2, \quad a_4 = -2a_1 a_3 = -(8a^2/3K^2)\cos 2\alpha,$$
as before.

We have the result that

$$x^2-y^2 = \sum_{n=1}^{\infty} a_{2n}\rho^{2n}\cos 2n\phi, \tag{14}$$

where a_2 and a_4 are given by (12) and (13), and a_6, a_8, ... can be calculated successively from (11).

2 . 221. A discussion of the difference equations, 2 . 22 (9) and (11), with the special values found for small values of n and a general value of a, would lead to a rather long, and perhaps not very profitable, piece of analysis, and we shall content ourselves with obtaining a formula for the value of a_{2n} in terms of n when the rectangle is a square, so that $\cos 2a = 0$.

We put $$K'a_{2n}-Kb_{2n} = \pi\lambda_n \quad (n = 0, 1, 2, \ldots), \tag{1}$$

so that λ_n satisfies the difference equation

$$(2n-1)\lambda_{n-1}+(2n+1)\lambda_{n+1} = 0, \tag{2}$$

with the special values

$$\lambda_0 = 0, \quad \lambda_1 = 1. \tag{3}$$

It appears that, when n is even,

$$\lambda_n = 0, \tag{4}$$

and, when n is uneven,

$$\lambda_n = (-1)^{\frac{1}{2}(n-1)}\frac{3\cdot 7\ldots(2n-3)}{5\cdot 9\ldots(2n-1)}. \tag{5}$$

Hence in this case

$$\left.\begin{aligned} a_{4m} &= 0, \quad a_2 = -\frac{4a^2}{K^2}, \\ a_{4m+2} &= (-1)^{m+1}\frac{3\cdot 7\ldots(4m-1)}{5\cdot 9\ldots(4m+1)}\frac{1}{2m+1}\frac{4a^2}{K^2} \quad (m = 1, 2, \ldots). \end{aligned}\right\} \tag{6}$$

2 . 23. As a third example we take

$$U = x^4+y^4-6x^2y^2.$$

It is seen easily that all the coefficients B_n of § 2 . 2 vanish, and all the coefficients A_n with odd suffixes also vanish. When n is even we put $2n$ in place of it, and write

$$\beta_{2n} = A_{2n}. \tag{1}$$

Then we have, when $n > 0$,

$$\frac{2n\pi K\beta_{2n}}{a} = 16\int_0^{\alpha}(b^3-3bx^2)\,\frac{\cos 2n\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi$$

$$+16\int_{\alpha}^{\frac{1}{2}\pi}(a^3-3ay^2)\,\frac{\cos 2n\phi}{\sqrt{(\cos^2\alpha-\cos^2\phi)}}\,d\phi. \quad (2)$$

Putting

$$c_{2n} = \int_0^{\alpha} x^2\,\frac{\cos 2n\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi, \qquad d_{2n} = \int_{\alpha}^{\frac{1}{2}\pi} y^2\,\frac{\cos 2n\phi}{\sqrt{(\cos^2\alpha-\cos^2\phi)}}\,d\phi, \quad (3)$$

we have

$$\beta_{2n} = \frac{8a}{n\pi K}\{(b^3a_{2n}+a^3b_{2n})-3(bc_{2n}+ad_{2n})\}, \quad (4)$$

where a_{2n} and b_{2n} are defined by 2 . 22 (4).

To evaluate c_{2n} we observe that

$$c_{2n-2}-c_{2n+2} = 2\int_0^{\alpha} x^2\,\frac{\sin 2n\phi\,\sin 2\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi, \quad (5)$$

and that

$$\frac{d}{d\phi}\{x^2\sin 2n\phi\,\sqrt{(\sin^2\alpha-\sin^2\phi)}\}$$

$$= 2nx^2\cos 2n\phi\,\sqrt{(\sin^2\alpha-\sin^2\phi)}$$

$$-\tfrac{1}{2}x^2\,\frac{\sin 2n\phi\,\sin 2\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}+2x\,\frac{dx}{d\phi}\,\sin 2n\phi\,\sqrt{(\sin^2\alpha-\sin^2\phi)}, \quad (6)$$

and further that (5) and (6) give

$$c_{2n-2}-c_{2n+2} = 4n\int_0^{\alpha} x^2\,\frac{\cos 2n\phi(\cos 2\phi-\cos 2\alpha)}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi$$

$$+8\int_0^{\alpha} x\,\frac{dx}{d\phi}\,\sqrt{(\sin^2\alpha-\sin^2\phi)}\,\sin 2n\phi\,d\phi. \quad (7)$$

Now, along the arc $(0<\phi<\alpha)$ of the circle $\rho = 1$, $dx/d\phi$ is the same as $|(dz/d\zeta)|$, and therefore

$$\int_0^{\alpha} x\,\frac{dx}{d\phi}\,\sqrt{(\sin^2\alpha-\sin^2\phi)}\,\sin 2n\phi\,d\phi = \frac{a}{K}\int_0^{\alpha} x\sin 2n\phi\,d\phi$$

$$= -\frac{a^2}{K}\,\frac{\cos 2n\alpha}{2n}+\frac{a^2}{K^2}\int_0^{\alpha}\frac{\cos 2n\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi, \quad (8)$$

where use has been made of 2 . 11 (9). It follows then from (7) that

c_{2n} satisfies the difference equation

$$(2n-1)\,c_{2n-2}+(2n+1)\,c_{2n+2}-4nc_{2n}\cos 2\alpha = (4a^2/nK^2)(K\cos 2n\alpha - a_{2n}). \quad (9)$$

In the same way it can be proved that

$$(2n-1)\,d_{2n-2}+(2n+1)\,d_{2n+2}-4nd_{2n}\cos 2\alpha = -(4a^2/nK^2)(K'\cos 2n\alpha - b_{2n}). \quad (10)$$

From (4), (9), (10), combined with 2 . 22 (9) and (10), it follows that, when $n>1$,

$$(n-1)(2n-1)\,\beta_{2n-2}+(n+1)(2n+1)\,\beta_{2n+2}-4n^2\beta_{2n}\cos 2\alpha = \frac{96a^4}{n\pi K^4}(K'a_{2n}-Kb_{2n}). \quad (11)$$

The coefficients β_2, β_4 are found most easily by raising to the fourth power the series $-i\sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}$ for z, and selecting the real part. This process gives

$$x^4+y^4-6x^2y^2 = a_1^4\rho^4\cos 4\phi+\ldots,$$

so that

$$\beta_2 = 0, \quad \beta_4 = a_1^4 = \frac{16a^4}{K^4}. \quad (12)$$

We have the result that

$$x^4+y^4-6x^2y^2 = \sum_{n=1}^{\infty}\beta_{2n}\rho^{2n}\cos 2n\phi, \quad (13)$$

where β_2, β_4 are given by (12), and β_6, β_8, ... can be found successively from (11).

2 . 231. As in § 2 . 221 we proceed to express β_{2n} in terms of n when the rectangle is a square.

We put

$$\beta_{2n} = \frac{96a^4}{nK^4}x_n, \quad (1)$$

and have the equation

$$(2n-1)\,x_{n-1}+(2n+1)\,x_{n+1} = n^{-1}\lambda_n \quad (2)$$

with the special values

$$\left.\begin{aligned} &x_1 = 0, \quad x_2 = \tfrac{1}{3}, \quad \lambda_0 = 0, \quad \lambda_1 = 1, \\ &\lambda_2 = \lambda_4 = \ldots = 0, \quad \lambda_3 = -\tfrac{3}{5}, \quad \lambda_5 = \frac{3\,.\,7}{5\,.\,9}, \quad \lambda_7 = -\frac{3\,.\,7\,.\,11}{5\,.\,9\,.\,13}, \quad \ldots \end{aligned}\right\}. \quad (3)$$

We see that, when n is odd,

$$x_n = 0. \tag{4}$$

When n is even, and equal to $2m$, we put

$$x_{2m} = (-1)^{m-1} \frac{1 \cdot 5 \ldots (4m-3)}{3 \cdot 7 \ldots (4m-1)} u_m, \quad u_1 = 1. \tag{5}$$

Then
$$u_{m+1} - u_m = \frac{1}{2m+1} \left\{\frac{3 \cdot 7 \ldots (4m-1)}{5 \cdot 9 \ldots (4m+1)}\right\}^2, \tag{6}$$

so that
$$u_m = 1 + \sum_{r=2}^{m} \frac{1}{2r-1} \left\{\frac{3 \cdot 7 \ldots (4r-5)}{5 \cdot 9 \ldots (4r-3)}\right\}^2 \quad (m = 2, 3, \ldots). \tag{7}$$

Hence we have, when n is odd,

$$\beta_{2n} = 0, \tag{8}$$

and, when n is even,

$$\beta_{2n} = (-1)^{\frac{1}{2}n-1} \frac{96a^4}{nK^4} \frac{1 \cdot 5 \ldots (2n-3)}{3 \cdot 7 \ldots (2n-1)} \left[1 + \sum_{r=2}^{\frac{1}{2}n} \frac{1}{2r-1} \left\{\frac{3 \cdot 7 \ldots (4r-5)}{5 \cdot 9 \ldots (4r-3)}\right\}^2\right]. \tag{9}$$

2 . 3. We shall require the expansion of $\Phi(x^2+y^2)$ in the form

$$\Phi(x^2+y^2) = \sum_{n=0}^{\infty} \gamma_{2n} \rho^{2n} \cos 2n\phi, \tag{1}$$

in which it is clear from the symmetry that the series must involve cosines only, and even multiples only of ϕ.

To obtain this expansion we use the generalized Green's equation 1 . 2 (1) in the form

$$\int \left\{\chi \frac{\partial}{\partial \nu} (\nabla^2 \chi') - \nabla^2 \chi' \frac{\partial \chi}{\partial \nu} - \chi' \frac{\partial}{\partial \nu} (\nabla^2 \chi) + \nabla^2 \chi \frac{\partial \chi'}{\partial \nu}\right\} ds = 0, \tag{2}$$

taking χ to be the biharmonic function x^2+y^2, and χ' to be the function so denoted as in § 1 . 61, and the integral to be a line integral round the periphery of the rectangle. The first term becomes, on transformation to the circle, $2n\pi\gamma_{2n}$. The second term is transformed by the method of § 2 . 2, and found to be

$$8(a/K)(ba_{2n} + ab_{2n}).$$

The third term is identically zero, and the fourth term is

$$\int_0^{2\pi} 4 \left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} d\phi.$$

Thus we have

$$2n\pi\gamma_{2n} = \frac{8a}{K}(ba_{2n} + ab_{2n}) - 4 \int_0^{2\pi} \left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} d\phi, \tag{3}$$

and we require the value of the integral in the right-hand member of (3).

2 . 31. For this we must discuss the function χ' rather more minutely than in 1 . 61. The equation 1 . 61 (8) can be written

$$\chi' = R\Big[\sum_{s=0}^{\infty}\Big\{\frac{2s+1}{2n+2s+1}\,a_{2s+1}\,\rho^{2n+2s+1}\,e^{i(2n+2s+1)\phi}\Big\} \times \sum_{m=0}^{\infty}\{a_{2m+1}\,\rho^{2m+1}\,e^{-i(2m+1)\phi}\}\Big], \quad (1)$$

where the expression under the sign R is the sum of the double series

$$\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\Big\{\frac{2s+1}{2n+2s+1}\,a_{2s+1}\,a_{2m+1}\,\rho^{2m+2n+2s+2}\,e^{i(2n+2s-2m)\phi}\Big\}, \quad (2)$$

which is absolutely convergent for all values of $\rho<1$, the coefficients a being given by 2 . 12 (2). The expression for χ' can therefore, as stated in 1 . 61, be arranged as a simple series

$$\chi' = \sum_{s=0}^{\infty} f_{2s}(\rho)\cos 2s\phi, \quad (3)$$

where each of the functions $f_{2s}(\rho)$ is a power series in ρ, converging absolutely for all values of $\rho<1$. The series for χ' converges uniformly inside any circle $\rho=1-\epsilon$, where $0<\epsilon<1$. It can be differentiated term by term with respect to ρ, and the result can be integrated term by term with respect to ϕ, with limits of integration 0 and 2π. It follows that, if $0<\epsilon<1$,

$$\int_0^{2\pi}\Big(\frac{\partial\chi'}{\partial\rho}\Big)_{\rho=1-\epsilon}d\phi = 2\pi\{f_0'(\rho)\}_{\rho=1-\epsilon}. \quad (4)$$

Now

$$\int_0^{2\pi}\Big(\frac{\partial\chi'}{\partial\rho}\Big)_{\rho=1}d\phi = \int\frac{\partial\chi'}{\partial\nu}\,ds, \quad (5)$$

where the integral in the right-hand member is taken round the periphery of the rectangle in the (x, y) plane. This integral is the same as the double integral

$$\iint\nabla^2\chi'\,dx\,dy \quad \text{or} \quad \iint\rho^{2n}\cos 2n\phi\,dx\,dy \quad (6)$$

taken over the area within the rectangular boundary. When ϵ varies from 1 to 0 the curves $\rho=1-\epsilon$ in the (x, y) plane are a family of curves approximating to concentric circles around the origin when $\epsilon\to 1$, and to the rectangular boundary when $\epsilon\to 0$. Let ds' denote the element of

arc of such a curve, and let ν' indicate the direction of its outward normal. Since $\log \rho$ and ϕ are conjugate functions,

$$\int \frac{\partial \chi'}{\partial \nu'} ds' = \int_0^{2\pi} \left(\rho \frac{\partial \chi'}{\partial \rho}\right)_{\rho=1-\epsilon} d\phi. \tag{7}$$

We may take ϵ so small that the distance along the inward normal to the rectangular boundary from any point on that boundary to the neighbouring point on the curve $\rho = 1-\epsilon$ is less than a prescribed length η, however small. Then the difference

$$\int \frac{\partial \chi'}{\partial \nu} ds - \int \frac{\partial \chi'}{\partial \nu'} ds',$$

which is equal to the double integral

$$\iint \rho^{2n} \cos 2n\phi \, dx \, dy,$$

taken over the area between the rectangular boundary and the curve $\rho = 1-\epsilon$, is numerically less than $4(a+b)\eta$, and we have the result

$$\int_0^{2\pi} \left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} d\phi = \int \frac{\partial \chi'}{\partial \nu} ds = \lim_{\epsilon \to 0} \int_0^{2\pi} \left(\rho \frac{\partial \chi'}{\partial \rho}\right)_{\rho=1-\epsilon} d\phi = 2\pi \lim_{\epsilon \to 0} \{f'_0(\rho)\}_{\rho=1-\epsilon}. \tag{8}$$

2 . 311. In the equation 2 . 31 (8) the value of

$$\lim_{\epsilon \to 0} \{f_0'(\rho)\}_{\rho=1-\epsilon}$$

can be found by differentiating the series in the right-hand member of 1 . 61 (11) term by term. The result is that we have the equation

$$\int_0^{2\pi} \left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} d\phi = \frac{2\pi a}{K} \sum_{s=0}^{\infty} a_{2n+2s+1} P_s(\cos 2a), \tag{1}$$

in which

$$\frac{K}{2a} a_{2n+2s+1} = \frac{P_{n+s}(\cos 2a)}{2n+2s+1} = \frac{2}{\pi} \int_0^a \frac{\cos (2n+2s+1)\phi}{2n+2s+1} \frac{d\phi}{\sqrt{(\sin^2 a - \sin^2 \phi)}}, \tag{2}$$

by 2 . 12 (2) and 2 . 21 (2). Hence (1) gives

$$\int_0^{2\pi} \left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} d\phi = \frac{8a^2}{K^2} \int_0^a \sum_{s=0}^{\infty} \left\{P_s(\cos 2a) \frac{\cos (2n+2s+1)\phi}{2n+2s+1}\right\} \frac{d\phi}{\sqrt{(\sin^2 a - \sin^2 \phi)}}, \tag{3}$$

the order of the operations summation and integration being clearly interchangeable.

Now

$$\sum_{s=0}^{\infty} P_s(\cos 2\alpha)\,\frac{\cos(2n+2s+1)\phi}{2n+2s+1} = R\left(\sum_{s=0}^{\infty}\left\{P_s(\cos 2\alpha)\,\frac{\xi^{2n+2s+1}}{2n+2s+1}\right\}_{\rho=1}\right)$$

$$= R\left(\int_0^{e^{i\phi}} \frac{\xi^{2n}}{(1-2\xi^2\cos 2\alpha+\xi^4)^{\frac{1}{2}}}\,d\xi\right), \quad (4)$$

the integral being taken along any path from the centre to the point $\xi = e^{i\phi}$ on the circumference, provided that the path does not go outside the circle $\rho = 1$, or pass through a singular point such as $\xi = e^{i\alpha}$. In order to use this result in (3) it is necessary that the ϕ in the upper limit of integration should be in the interval $(0,\ \alpha)$.

To evaluate the integral in (4) we take it along the path, along which ζ is real, from 0 to 1, and then along the arc of the circle $\rho = 1$ from $\phi = 0$ to the final value of ϕ. The integral along the real axis is the real integral

$$\int_0^1 \frac{\rho^{2n}}{\sqrt{(1-2\rho^2\cos 2\alpha+\rho^4)}}\,d\rho \quad (= g_{2n} \text{ say}), \quad (5)$$

and the real part of the integral along the circular arc is

$$R\left(\int_0^{\phi} \frac{ie^{(2n+1)i\phi}}{(1-2e^{2i\phi}\cos 2\alpha+e^{4i\phi})^{\frac{1}{2}}}\,d\phi\right),$$

which is

$$-\tfrac{1}{2}\int_0^{\phi} \frac{\sin 2n\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi \quad (= -\tfrac{1}{2}h_{2n} \text{ say}). \quad (6)$$

We have therefore proved that

$$\int_0^{2\pi}\left(\frac{\partial\chi'}{\partial\rho}\right)_{\rho=1} d\phi = \frac{8a^2}{K^2}\int_0^{\alpha}(g_{2n}-\tfrac{1}{2}h_{2n})\,\frac{d\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}, \quad (7)$$

in which g_{2n} and h_{2n} are defined by (5) and (6).

2 . 312. We shall evaluate g_{2n} and h_{2n} for $n = 0$ and $n = 1$, and form difference equations from which their values for higher values of n can be found.

To evaluate g_0 we put $\rho = \tan\frac{1}{2}\psi$, and get

$$g_0 = \tfrac{1}{2}K'. \quad (1)$$

To evaluate g_2 we use the same transformation, getting

$$g_2 = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{1-\cos\psi}{1+\cos\psi}\,\frac{d\psi}{\sqrt{(1-\cos^2\alpha\sin^2\psi)}}$$

$$= \tfrac{1}{2}\int_0^{K'} \frac{1-\mathrm{cn}\,(u,\ k')}{1+\mathrm{cn}\,(u,\ k')}\,du. \quad (2)$$

From the identity

$$\frac{d}{du}\frac{\operatorname{sn}\frac{1}{2}u\,\operatorname{dn}\frac{1}{2}u}{\operatorname{cn}\frac{1}{2}u}=\tfrac{1}{2}\left(2\operatorname{dn}^2\tfrac{1}{2}u-1+\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\right), \tag{3}$$

where the modulus is k, we find that

$$\begin{aligned}\int_0^K\frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}\,du &= K-4\{Z(\tfrac{1}{2}K)+\tfrac{1}{2}E\}+2\,\frac{\operatorname{sn}\frac{1}{2}K\,\operatorname{dn}\frac{1}{2}K}{\operatorname{cn}\frac{1}{2}K}\\ &= 2k'+K-2E. \end{aligned}\tag{4}$$

Hence

$$g_2 = k+\tfrac{1}{2}K'-E'. \tag{5}$$

To form a difference equation for g_{2n}, we observe that

$$\begin{aligned}&\frac{d}{d\rho}\{\rho^{2n-1}\sqrt{(1-2\rho^2\cos 2\alpha+\rho^4)}\}\\ &\qquad=\frac{(2n-1)\rho^{2n-2}-4n\rho^{2n}\cos 2\alpha+(2n+1)\rho^{2n+2}}{\sqrt{(1-2\rho^2\cos 2\alpha+\rho^4)}},\end{aligned}\tag{6}$$

whence

$$(2n-1)g_{2n-2}+(2n+1)g_{2n+2}-4ng_{2n}\cos 2\alpha = 2\sin\alpha. \tag{7}$$

In particular, we have

$$g_4=\tfrac{2}{3}\sin\alpha(1+2\cos 2\alpha)-\tfrac{1}{6}K'(5-8k'^2)+\tfrac{4}{3}E'(1-2k'^2). \tag{8}$$

Proceeding to the calculation of h_{2n}, we find that

$$h_0=0,\quad h_2=2\{\sin\alpha-\sqrt{(\sin^2\alpha-\sin^2\phi)}\}. \tag{9}$$

Also we observe that

$$h_{2n+2}-h_{2n-2}=2\int_0^\phi\frac{\cos 2n\phi\sin 2\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}\,d\phi, \tag{10}$$

and that

$$\begin{aligned}&\frac{d}{d\phi}\{\cos 2n\phi\sqrt{(\sin^2\alpha-\sin^2\phi)}\}\\ &\qquad=-2n\sin 2n\phi\sqrt{(\sin^2\alpha-\sin^2\phi)}-\frac{\cos 2n\phi\sin 2\phi}{2\sqrt{(\sin^2\alpha-\sin^2\phi)}},\end{aligned}\tag{11}$$

from which and (10) we get the difference equation

$$\begin{aligned}&(2n-1)h_{2n-2}+(2n+1)h_{2n+2}-4nh_{2n}\cos 2\alpha\\ &\qquad=4\sin\alpha-4\cos 2n\phi\sqrt{(\sin^2\alpha-\sin^2\phi)}\}.\end{aligned}\tag{12}$$

In particular, we find that

$$\begin{aligned}h_4=&\tfrac{4}{3}\{\sin\alpha-\cos 2\phi\sqrt{(\sin^2\alpha-\sin^2\phi)}\}\\ &+\tfrac{8}{3}\cos 2\alpha\{\sin\alpha-\sqrt{(\sin^2\alpha-\sin^2\phi)}\}.\end{aligned}\tag{13}$$

2 . 32. Resuming the discussion of the coefficients γ_{2n}, we find from 2 . 3 (3) and 2 . 311 (7) that, when $n > 0$,

$$n\pi\gamma_{2n} = \frac{4a}{K}(ba_{2n}+ab_{2n}) - \frac{16a^2}{K^2}\int_0^a (g_{2n}-\tfrac{1}{2}h_{2n})\frac{d\phi}{\sqrt{(\sin^2\alpha-\sin^2\phi)}}. \quad (1)$$

From 2 . 22 (7) and 2 . 312 (5) and (9), along with 2 . 11 (7), we get

$$\pi\gamma_2 = \frac{4a^2}{K^2}[K'(2E-K)-K(2E'-K')-4\{(\tfrac{1}{2}K'-E')K+a\}],$$

which reduces by Legendre's relation to

$$\gamma_2 = \frac{4a^2}{K^2}\left(1-\frac{4a}{\pi}\right). \quad (2)$$

Again from 2 . 22 (6), (7), (9), and (10) we find

$$3a_4 = 8E(1-2k^2)-K(5-8k^2), \quad 3b_4 = 8E'(1-2k'^2)-K'(5-8k'^2), \quad (4)$$

and from 2 . 312 (8) and (13) we find

$$g_4-\tfrac{1}{2}h_4 = -\tfrac{1}{6}K'(5-8k'^2)+\tfrac{4}{3}E'(1-2k'^2) + \tfrac{2}{3}(\cos 2\phi+2\cos 2\alpha)\sqrt{(\sin^2\alpha-\sin^2\phi)}, \quad (5)$$

so that (1) gives

$$\pi\gamma_4 = \frac{2a^2}{3K^2}[K'\{8E(1-2k^2)-K(5-8k^2)\}+K\{8E'(1-2k'^2)-K'(5-8k'^2)\} + 2K\{K'(5-8k'^2)-8E'(1-2k'^2)\}-4(\sin 2\alpha+4\alpha\cos 2\alpha)],$$

which reduces by Legendre's relation to

$$\gamma_4 = \frac{8a^2}{3K^2}\left\{\left(1-\frac{4a}{\pi}\right)\cos 2\alpha-\frac{\sin 2\alpha}{\pi}\right\}. \quad (6)$$

Further, from (1), combined with 2 . 22 (9) and (10), and 2 . 312 (7) and (12) we get, when $n > 1$,

$$(n-1)(2n-1)\gamma_{2n-2}+(n+1)(2n+1)\gamma_{2n+2}-4n^2\gamma_{2n}\cos 2\alpha = -\frac{16a^2}{n\pi K^2}\sin 2n\alpha, \quad (7)$$

and from this difference equation and the values of γ_2 and γ_4 in (2) and (6) the values of γ_{2n} for $n > 2$ can be calculated successively.

The function $\Phi(x^2+y^2)$ is thus expressed in the form

$$\Phi(x^2+y^2) = \gamma_0+\sum_{s=1}^{\infty}\gamma_{2s}\rho^{2s}\cos 2s\phi, \quad (8)$$

and it may be noted that, at the boundary,

$$x^2+y^2 = \gamma_0 + \sum_{s=1}^{\infty} \gamma_{2s} \cos 2s\phi. \tag{9}$$

The constant γ_0 has not yet been determined.

2.321. As in § 2.221 and § 2.231 we proceed to express γ_{2n} in terms of n when the rectangle is a square.

In this case it is seen at once that $\gamma_{2n} = 0$ if n is odd.

When n is even we put $n = 2m$, and

$$\gamma_{4m} = \frac{(-1)^m}{m} \frac{8a^2}{\pi K^2} y_m. \tag{1}$$

Then we have

$$-(4m+1)\, y_m + (4m+3)\, y_{m+1} = 1/(2m+1), \tag{2}$$

and, if we put

$$y_m = \frac{1.5 \ldots (4m-3)}{3.7 \ldots (4m-1)} u_m, \tag{3}$$

we get

$$u_{m+1} - u_m = \frac{1}{2m+1} \frac{3.7 \ldots (4m-1)}{5.9 \ldots (4m+1)}, \quad u_1 = 1. \tag{4}$$

Hence, when $m > 1$,

$$u_m = 1 + \sum_{r=2}^{m} \frac{3.7 \ldots (4r-5)}{5.9 \ldots (4r-3)} \frac{1}{2r-1}. \tag{5}$$

Thus we find, when $m > 1$,

$$\gamma_{4m} = \frac{(-1)^m}{m} \frac{8a^2}{\pi K^2} \frac{1.5 \ldots (4m-3)}{3.7 \ldots (4m-1)} \left\{ 1 + \sum_{r=2}^{m} \frac{3.7 \ldots (4r-5)}{5.9 \ldots (4r-3)} \frac{1}{2r-1} \right\}. \tag{6}$$

2.33. The method of § 2.3 gives no rule for finding γ_0, the value of $\Phi(x^2+y^2)$ at the origin. If we form the product series for z and z^* from

$$z = -i \sum_{n=0}^{\infty} a_{2n+1} \rho^{2n+1} e^{(2n+1)i\phi}, \quad z^* = i \sum_{n=0}^{\infty} a_{2n+1} \rho^{2n+1} e^{-(2n+1)i\phi}, \tag{1}$$

we get a result of the form

$$x^2+y^2 = \sum_{s=0}^{\infty} F_{2s}(\rho) \cos 2s\phi, \tag{2}$$

where

$$F_0(\rho) = \sum_{n=0}^{\infty} (a_{2n+1})^2 \rho^{4n+2}. \tag{3}$$

The corresponding expression for $\Phi(x^2+y^2)$ is

$$\Phi(x^2+y^2) = \sum_{s=0}^{\infty} F_{2s}(1)\,\rho^{2s}\cos 2s\phi, \tag{4}$$

so that

$$\gamma_0 = \sum_{n=0}^{\infty} (a_{2n+1})^2. \tag{5}$$

This series converges fairly rapidly for any value of a.

Another way of finding γ_0 is to use the result† that

$$\Phi(x^2+y^2) = 2a^2-(x^2-y^2) - \frac{64a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\frac{\cosh\{(2n+1)\,\pi y/2a\}}{\cosh\{(2n+1)\,\pi b/2a\}}\cos\frac{(2n+1)\,\pi x}{2a}, \tag{6}$$

which gives

$$\gamma_0 = 2a^2-64\,\frac{a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3\cosh\{(2n+1)\,\pi b/2a\}}. \tag{7}$$

This series converges very rapidly when $b > a$. When $b < a$, a and b can be interchanged.

It may be noted that the expression, analogous to (2), for x^2+y^2 can always be found, for any form of boundary, when the equation 1 . 5 (2), expressing the conformal transformation of the z region into the ζ region, is known, and thus a formula of a sort for $\Phi(x^2+y^2)$ can be obtained in all such cases. The object of 2 . 3 – 2 . 32 was to obtain expressions for the coefficients, in the case of a rectangular boundary, which would be fairly easy to compute.

3. *Promotion of rank.*

3 . 1. We have now to discuss in detail the process sketched in 1 . 7.

Beginning with the case of the χ functions with even suffixes, from any function χ_{2n}, with $n \geqslant 1$, we form a new function $\chi^{(2)}_{2n}$ by the rule

$$\chi^{(2)}_{2n} = \chi_{2n} - q_{2n,0}\,\chi_0 \quad (n \geqslant 1), \tag{1}$$

where $q_{2n,0}$ is a constant determined by the condition that $\chi^{(2)}_{2n}$ is of rank 2. From 1 . 53 (2) we can write down

$$\left(\frac{\partial\chi_{2n}}{\partial\rho}\right)_{\rho=1} = \sum_{s=0}^{\infty} b_{2n,2s}\cos 2s\phi, \tag{2}$$

† See, *e.g.*, my *Elasticity*, Art. 221 (c). The formula was effectively obtained by Stokes, *Trans. Camb. Phil. Soc.*, 8 (1843), 105–137 (133).

where, for $n = 0, 1, 2, \ldots,$

$$\left.\begin{aligned} b_{2n,0} &= (4n+2)\, a_{2n+1} \\ b_{2n,2s} &= (4n-4s+2)\, a_{2n-2s+1} + (4n+2)\, a_{2n+2s+1} \quad (n \geqslant s \geqslant 1) \\ b_{2n,2s} &= (4n+2)\, a_{2n+2s+1} \quad (s \geqslant n+1) \end{aligned}\right\}. \tag{3}$$

The condition that $\chi^{(2)}_{2n}$ is of rank 2 is then

$$b_{2n,0} - q_{2n,0}\, b_{0,0} = 0 \quad (n \geqslant 1), \tag{4}$$

or
$$q_{2n,0} = (2n+1)\, a_{2n+1}/a_1 = P_n(\cos 2\alpha). \tag{5}$$

Then we write
$$\left\{\frac{\partial \chi^{(2)}_{2n}}{\partial \rho}\right\}_{\rho=1} = \sum_{s=0}^{\infty} b^{(2)}_{2n,2s} \cos 2s\phi, \tag{6}$$

where
$$b^{(2)}_{2n,2s} = b_{2n,2s} - q_{2n,0}\, b_{0,2s} \quad (s \geqslant 0), \tag{7}$$

and
$$b^{(2)}_{2n,0} = 0. \tag{8}$$

Then (8) is the same equation as (4). All the functions $\chi^{(2)}_{2n}$, with $n \geqslant 1$, are of rank 2.

From any function $\chi^{(2)}_{2n}$, with $n \geqslant 2$, we form a new function $\chi^{(4)}_{2n}$ by the rule

$$\chi^{(4)}_{2n} = \chi^{(2)}_{2n} - q_{2n,2}\, \chi^{(2)}_{2} \quad (n \geqslant 2), \tag{9}$$

and determine the constant $q_{2n,2}$ by the condition that $\chi^{(4)}_{2n}$ is of rank 4. From (6) we see that this condition is

$$b^{(2)}_{2n,2} - q_{2n,2}\, b^{(2)}_{2,2} = 0 \quad (n \geqslant 2), \tag{10}$$

We can write
$$\left\{\frac{\partial}{\partial \rho}\chi^{(4)}_{2n}\right\}_{\rho=1} = \sum_{s=1}^{\infty} b^{(4)}_{2n,2s} \cos 2s\phi, \tag{11}$$

where
$$b^{(4)}_{2n,2s} = b^{(2)}_{2n,2s} - q_{2n,2}\, b^{(2)}_{2,2s} \quad (s \geqslant 1), \tag{12}$$

and
$$b^{(4)}_{2n,2} = 0. \tag{13}$$

Then (13) is the same equation as (10). All the functions $\chi^{(4)}_{2n}$, with $n \geqslant 2$, are of rank 4.

3 . 11. This process can be continued, and it issues in the construction of a new sub-family of biharmonic functions $\chi^{(2n)}_{2n}$, all of them vanishing at the boundary, and each of them of the rank indicated by its suffix or superscript. We can write in general

$$\chi^{(2k)}_{2n} = \chi^{(2k-2)}_{2n} - q_{2n,2k-2}\, \chi^{(2k-2)}_{2k-2} \quad (k \leqslant n), \tag{1}$$

and
$$\left\{\frac{\partial}{\partial \rho}\chi^{(2k)}_{2n}\right\}_{\rho=1} = \sum_{s=k-1}^{\infty} b^{(2k)}_{2n,2s} \cos 2s\phi \quad (k \leqslant n), \tag{2}$$

where $$b^{(2k)}_{2n,\,2s} = b^{(2k-2)}_{2n,\,2s} - q_{2n,\,2k-2}\, b^{(2k-2)}_{2k-2,\,2s} \quad (s \geqslant k-1), \tag{3}$$

and $$b^{(2k)}_{2n,\,2k-2} = 0, \tag{4}$$

equation (4) serving to determine $q_{2n,\,2k-2}$, for all values of n, and all values of k which do not exceed n.

To complete the notation we make it a convention that $\chi^{(0)}_{2n}$ and $b^{(0)}_{2n,\,2s}$ mean χ_{2n} and $b_{2n,\,2s}$.

3.12. The algebraic equations giving $q_{2n,\,2k}$ and $b^{(2k)}_{2n,\,2s}$ can be solved in a very general way. From 3.11 (6) we can write down the equations

$$\left.\begin{array}{lllll} b^{(2k)}_{2n,\,2s} + q_{2n,\,2k-2}\, b^{(2k-2)}_{2k-2,\,2s} - b^{(2k-2)}_{2n,\,2s} = 0 \\ b^{(2k-2)}_{2n,\,2s} + q_{2n,\,2k-4}\, b^{(2k-4)}_{2k-4,\,2s} - b^{(2k-4)}_{2n,\,2s} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ b^{(2)}_{2n,\,2s} + q_{2n,\,0}\, b_{0,\,2s} - b_{2n,\,2s} = 0 \end{array}\right\}. \tag{1}$$

On adding the left-hand members we find the first of equations (2) below, and the rest of these equations (except the last one) are obtained from the first by changing $2n$ into $2k$ and then putting successively $2k-2$, $2k-4$, ... for $2k$. The equations are

$$\left.\begin{array}{rrrrrrrr} b^{(2k)}_{2n,\,2s} + q_{2n,\,2k-2}\, b^{(2k-2)}_{2k-2,\,2s} + q_{2n,\,2k-4}\, b^{(2k-4)}_{2k-4,\,2s} + \cdots + q_{2n,\,0}\, b_{0,\,2s} - b_{2n,\,2s} &= 0 \\ b^{(2k-2)}_{2k-2,\,2s} + q_{2k-2,\,2k-4}\, b^{(2k-4)}_{2k-4,\,2s} + \cdots + q_{2k-2,\,0}\, b_{0,\,2s} - b_{2k-2,\,2s} &= 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \cdots \\ b^{(2)}_{2,\,2s} + q_{2,\,0}\, b_{0,\,2s} - b_{2,\,2s} &= 0 \\ b_{0,\,2s} - b_{0,\,2s} &= 0 \end{array}\right\}, \tag{2}$$

where the last equation is an identity. From these we can write down the equation

$$b^{(2k)}_{2n,\,2s} = (-1)^k\, \Delta(2n,\, 2s,\, 2k), \tag{3}$$

where

$$\Delta(2n,\, 2s,\, 2k) = \begin{vmatrix} q_{2n,\,2k-2}, & q_{2n,\,2k-4}, & q_{2n,\,2k-6}, & \cdots, & q_{2n,\,2}, & q_{2n,\,0}, & b_{2n,\,2s} \\ 1, & q_{2k-2,\,2k-4}, & q_{2k-2,\,2k-6}, & \cdots, & q_{2k-2,\,2}, & q_{2k-2,\,0}, & b_{2k-2,\,2s} \\ 0, & 1, & q_{2k-4,\,2k-6}, & \cdots, & q_{2k-4,\,2}, & q_{2k-4,\,0}, & b_{2k-4,\,2s} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0, & 0, & 0, & \ldots, & 1, & q_{2,\,0}, & b_{2,\,2s} \\ 0, & 0, & 0, & \ldots, & 0, & 1, & b_{0,\,2s} \end{vmatrix}. \tag{4}$$

In this $q_{2n,0}$ is given by 3.1 (5). Let us suppose that the coefficients $q_{2n,2m}$ have been found for all values of m from 0 to $k-1$, where $k<n-1$. Then

$$q_{2n,2k}=\frac{b^{(2k)}_{2n,2k}}{b^{(2k)}_{2k,2k}}=\frac{\Delta(2n,2k,2k)}{\Delta(2k,2k,2k)}. \tag{5}$$

Thus the values of all the coefficients $q_{2n,2k}$ can be found. The determinants in (5) can be evaluated promptly by observing that, if

$$\delta_{2k}=\begin{vmatrix} q_{2k,2k-2}, & q_{2k,2k-4}, & \dots, & q_{2k,0} \\ 1, & q_{2k-2,2k-4}, & \dots, & q_{2k-2,0} \\ 0, & 1, & \dots, & q_{2k-4,0} \\ \vdots & \vdots & & \\ 0, & 0, & \dots, & q_{2,0} \end{vmatrix}, \tag{6}$$

then

$$\delta_{2k}=q_{2k,2k-2}\delta_{2k-2}-q_{2k,2k-4}\delta_{2k-4}+\dots(-1)^{k-2}q_{2k,2}\delta_2+(-1)^{k-1}q_{2k,0}, \tag{7}$$

and

$$\Delta(2k,2k,2k)=b_{0,2k}\delta_{2k}-b_{2,2k}\frac{\partial\delta_{2k}}{\partial q_{2,0}}+b_{4,2k}\frac{\partial^2\delta_{2k}}{\partial q_{2,0}\partial q_{4,2}}+\dots$$
$$+(-1)^{k-1}b_{2k-2,2k}\frac{\partial^{k-1}\delta_{2k}}{\partial q_{2,0}\partial q_{4,2}\dots\partial q_{2k-2,2k-4}}+(-1)^k b_{2k,2k}, \tag{8}$$

and further $\Delta(2n,2k,2k)$ can be obtained from $\Delta(2k,2k,2k)$ by replacing k by n wherever $2k$ occurs as a first suffix, whether attached to a q or a b.

3.13. The process so far described results in replacing χ_{2n} by a function $\chi^{(2n)}_{2n}$, which is necessarily a linear aggregate of the functions $\chi_{2n}, \chi_{2n-2}, \dots, \chi_0$. To form this aggregate let us write by analogy to 3.11 (1),

$$X^{(2k)}_{2n}=X^{(2k-2)}_{2n}-q_{2n,2k-2}X^{(2k-2)}_{2k-2} \quad (k\leqslant n), \tag{1}$$

taking $X^{(0)}_{2n}$ to be the same as X_{2n}, the function so denoted in 1.51. Then we have

$$\left.\begin{array}{lllll} X^{(2n)}_{2n} & +q_{2n,2n-2}X^{(2n-2)}_{2n-2} & -X^{(2n-2)}_{2n} & = 0 \\ X^{2n-2)}_{2n} & +q_{2n,2n-4}X^{(2n-4)}_{2n-4} & -X^{(2n-4)}_{2n} & = 0 \\ \dots & \dots & \dots & \dots & \dots \\ X^{(2)}_{2n} & +q_{2n,0} \quad X_0 & -X_{2n} & = 0 \end{array}\right\}. \tag{2}$$

From this, by an argument similar to that in 3 . 12, we obtain the result

$$X_{2n}^{(2n)} = X_{2n} - \frac{\partial^{n-1}\delta_{2n}}{\partial q_{2,0}\,\partial q_{4,2}\ldots\partial q_{2n-2,2n-4}} X_{2n-2} + \ldots + (-1)^{n-1}\frac{\partial \delta_{2n}}{\partial q_{2,0}} X_2 + (-1)^n \delta_{2n} X_0. \quad (3)$$

This equation shows that

$$\chi_{2n}^{(2n)} = X_{2n}^{(2n)} - \Phi\{X_{2n}^{(2n)}\}, \quad (4)$$

where

$$X_{2n}^{(2n)} = R\{-iz^* T_{2n+1}(\zeta)\}, \quad (5)$$

and

$$T_{2n+1}(\zeta) = \zeta^{2n+1} - \frac{\partial^{n-1}\delta_{2n}}{\partial q_{2,0}\,\partial q_{4,2}\ldots\partial q_{2n-2,2n-4}} \zeta^{2n-1} + \ldots + (-1)^n \delta_{2n}\zeta. \quad (6)$$

In the notation of 1 . 7 this is written

$$T_{2n+1}(\zeta) = \zeta^{2n+1} - r_{2n,2n-1}\zeta^{2n-1} - r_{2n,2n-3}\zeta^{2n-3} - \ldots - r_{2n,1}\zeta. \quad (7)$$

With a view to satisfying a boundary condition of the type

$$\frac{\partial \chi}{\partial \rho} = \frac{\partial V}{\partial \rho} \quad \text{at} \quad \rho = 1,$$

where V is harmonic, we note that

$$\left\{\frac{\partial}{\partial \rho}\chi_{2n}^{(2n)}\right\}_{\rho=1} = \sum_{s=n}^{\infty} b_{2n,2s}^{(2n)} \cos 2s\phi, \quad (8)$$

where $b_{2n,2s}^{(2n)}$ is found by writing n for k in 3 . 12 (3).

3 . 2. The process carried out step by step in 3 . 1 – 3 . 13 ensures the existence of the numbers $q_{2n,2k}$ with $k \leqslant n-1$, and of the numbers $r_{2n,2m+1}$, with $m \leqslant n-1$, and therefore also of the polynomials $T_{2n+1}(\zeta)$. It is, moreover, easy to compute the numbers $b_{2n,2s}^{(2s)}$ required for satisfying the boundary condition. For we have

$$b_{2n,2s}^{(2n)} = (-1)^n \left\{\delta_{2n} b_{0,2s} - \frac{\partial \delta_{2n}}{\partial q_{2,0}} b_{2,2s} + \frac{\partial^2 \delta_{2n}}{\partial q_{2,0}\,\partial q_{4,2}} b_{4,2s} - \ldots + (-1)^{n-1}\frac{\partial^{n-1}\delta_{2n}}{\partial q_{2,0}\,\partial q_{4,2}\ldots\partial q_{2n-2,2n-4}} b_{2n-2,2s} + (-1)^n b_{2n,2s}\right\}, \quad (1)$$

and the determinant δ_{2n} is calculated easily from 3 . 12 (7). But it is, of course, possible to obtain general formulae for the coefficients $r_{2n,2m+1}$ from the condition that $\chi_{2n}^{(2n)}$ is of rank $2n$, and then to write down expressions for the $b_{2n,2s}^{(2n)}$ in terms of these coefficients. We have,

in fact,

$$\chi_{2n}^{(2n)} = \chi_{2n} - \sum_{m=0}^{n-1} r_{2n,\,2m+1}\,\chi_{2m}, \tag{2}$$

and thence

$$\left\{\frac{\partial}{\partial\rho}\,\chi_{2n}^{(2n)}\right\}_{\rho=1} = \sum_{s=0}^{\infty} b_{2n,\,2s}\cos 2s\phi - \sum_{m=0}^{n-1}\sum_{s=0}^{\infty} r_{2n,\,2m+1}\,b_{2m,\,2s}\cos 2s\phi, \tag{3}$$

and the conditions that $\chi_{2n}^{(2n)}$ may be of rank $2n$ are the n equations

$$b_{2n,\,2k} - \sum_{m=0}^{n-1} r_{2n,\,2m+1}\,b_{2m,\,2k} = 0 \quad (k = 0,\ 1,\ \ldots,\ n-1). \tag{4}$$

From these equations any required r, whose first suffix is $2n$, can be written down as a quotient of two determinants, and then the coefficients $b_{2n,\,2s}^{(2n)}$ of 3 . 13 (8) can be written down in the form

$$b_{2n,\,2s}^{(2n)} = b_{2n,\,2s} - \sum_{m=0}^{n-1} r_{2n,\,2m+1}\,b_{2m,\,2s}. \tag{5}$$

3 . 3. So far we have been considering the χ functions with even suffixes. We now proceed to the χ functions with odd suffixes. We have

$$\left\{\frac{\partial}{\partial\rho}\,\chi_{2n-1}\right\}_{\rho=1} = \sum_{s=0}^{\infty} b_{2n-1,\,2s+1}\cos(2s+1)\,\phi, \tag{1}$$

where, by 1 . 53 (3),

$$\left.\begin{aligned} b_{2n-1,\,2s+1} &= (4n-4s-2)\,a_{2n-2s-1} + 4n a_{2n+2s+1} \quad (n-1 \geqslant s \geqslant 0) \\ b_{2n-1,\,2s+1} &= 4n a_{2n+2s+1} \qquad\qquad\qquad\qquad\quad (s \geqslant n) \end{aligned}\right\}. \tag{2}$$

In general all the functions χ_{2n-1} are of rank 1. We proceed by steps by putting

$$\chi_{2n-1}^{(2k-1)} = \chi_{2n-1}^{(2k-3)} - q_{2n-1,\,2k-3}\,\chi_{2k-3}^{(2k-3)} \quad (k = 2,\ 3,\ \ldots,\ n), \tag{3}$$

$$\left\{\frac{\partial}{\partial\rho}\,\chi_{2n-1}^{(2k-1)}\right\}_{\rho=1} = \sum_{s=k-2}^{\infty} b_{2n-1,\,2s+1}^{(2k-1)}\cos(2s+1)\,\phi, \tag{4}$$

$$b_{2n-1,\,2s+1}^{(2k-1)} = b_{2n-1,\,2s+1}^{(2k-3)} - q_{2n-1,\,2k-3}\,b_{2k-3,\,2s+1}^{(2k-3)}, \tag{5}$$

$$b_{2n-1,\,2k-3}^{(2k-1)} = 0 \quad (k = 2,\ 3,\ \ldots.\ n). \tag{6}$$

To complete the notation we note that it is implied that

$$\chi_{2n-1}^{(1)} = \chi_{2n-1}, \qquad b_{2n-1,\,2s+1}^{(1)} = b_{2n-1,\,2s+1}, \tag{7}$$

so that the superscript of a χ function always indicates its rank, and the superscript of a b coefficient always indicates the rank of the χ function from which it is derived.

Then (6) gives the equation

$$b^{(1)}_{2n-1,\,1} - q_{2n-1,\,1}\, b^{(1)}_{1,\,1} = 0 \quad (n \geqslant 2), \tag{8}$$

so that
$$q_{2n-1,\,1} = \{(2n-1)\, a_{2n-1} + 2n a_{2n+1}\} / (a_1 + 2a_3). \tag{9}$$

Also (5) gives the set of equations

$$\left.\begin{aligned}
b^{(2k-1)}_{2n-1,\,2s+1} + q_{2n-1,\,2k-3}\, b^{(2k-3)}_{2k-3,\,2s+1} - b^{(2k-3)}_{2n-1,\,2s+1} &= 0,\\
b^{(2k-3)}_{2n-1,\,2s+1} + q_{2n-1,\,2k-5}\, b^{(2k-5)}_{2k-5,\,2s+1} - b^{(2k-5)}_{2n-1,\,2s+1} &= 0,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots &\\
b^{(3)}_{2n-1,\,2s+1} + q_{2n-1,\,1}\quad b_{1,\,2s+1} \quad - b_{2n-1,\,2s+1} &= 0,
\end{aligned}\right\} \tag{10}$$

and from these we get the set of equations

$$\left.\begin{aligned}
b^{(2k-1)}_{2n-1,\,2s+1} + q_{2n-1,\,2k-3}\, b^{(2k-3)}_{2k-3,\,2s+1} & \\
+ q_{2n-1,\,2k-5}\, b^{(2k-5)}_{2k-5,\,2s+1} + \cdots + q_{2n-1,\,1}\, b_{1,\,2s+1} - b_{2n-1,\,2s+1} &= 0,\\
b^{(2k-3)}_{2k-3,\,2s+1} + q_{2k-3,\,2k-5}\, b^{(2k-5)}_{2k-5,\,2s+1} + \cdots + q_{2k-3,\,1}\, b_{1,\,2s+1} - b_{2k-3,\,2s+1} &= 0,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \\
b^{(3)}_{3,\,2s+1} + q_{3,\,1}\, b_{1,\,2s+1} - b_{3,\,2s+1} &= 0,\\
b_{1,\,2s+1} - b_{1,\,2s+1} &= 0,
\end{aligned}\right\} \tag{11}$$

of which the first is obtained by adding the left-hand members of (10); the second, third, ... are obtained by substituting k for n in the first, and then replacing k successively by $k-1$, $k-2$, ...; and the last is an identity. These equations give the formula

$$b^{(2k-1)}_{2n-1,\,2s+1} = (-1)^{k-1}\, \Delta(2n-1,\ 2s+1,\ 2k-1), \tag{12}$$

where

$$\Delta(2n-1,\ 2s+1,\ 2k-1)$$

$$= \begin{vmatrix}
q_{2n-1,\,2k-3}, & q_{2n-1,\,2k-5}, & q_{2n-1,\,2k-7}, & \cdots, & q_{2n-1,\,1}, & b_{2n-1,\,2s+1}\\
1, & q_{2k-3,\,2k-5}, & q_{2k-3,\,2k-7}, & \cdots, & q_{2k-3,\,1}, & b_{2k-3,\,2s+1}\\
0, & 1, & q_{2k-5,\,2k-7}, & \cdots, & q_{2k-5,\,1}, & b_{2k-5,\,2s+1}\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
0, & 0, & 0, & \cdots, & q_{3,\,1}, & b_{3,\,2s+1}\\
0, & 0, & 0, & \cdots, & 1, & b_{1,\,2s+1}
\end{vmatrix}. \tag{13}$$

Then the q's are given by the equation

$$q_{2n-1,\,2k-3} = b^{(2k-3)}_{2n-1,\,2k-3}/b^{(2k-3)}_{2k-3,\,2k-3} \quad (k = 2, 3, \ldots, n). \quad (14)$$

Also, if δ_{2k-1} denotes the determinant

$$\delta_{2k-1} = \begin{vmatrix} q_{2k-1,\,2k-3}, & q_{2k-1,\,2k-5}, & \ldots, & q_{2k-1,\,1} \\ 1, & q_{2k-3,\,2k-5}, & \ldots, & q_{2k-3,\,1} \\ 0, & 1, & \ldots, & q_{2k-5,\,1} \\ \vdots & \vdots & & \vdots \\ 0, & 0, & \ldots, & q_{3,\,1} \end{vmatrix}, \quad (15)$$

so that

$$\delta_{2k-1} = q_{2k-1,\,2k-3}\,\delta_{2k-3} - q_{2k-1,\,2k-5}\,\delta_{2k-5} + \ldots + (-1)^{k-2} q_{2k-1,\,1}, \quad (16)$$

we have

$$\Delta(2k-1,\ 2s+1,\ 2k-1)$$

$$= b_{1,\,2s+1}\,\delta_{2k-1} - b_{3,\,2s+1}\frac{\partial\delta_{2k-1}}{\partial q_{3,\,1}} + \ldots$$

$$+ (-1)^{k-2}\frac{\partial^{k-2}\delta_{2k-1}}{\partial q_{3,\,1}\,\partial q_{5,\,3}\ldots\partial q_{2k-3,\,2k-5}}\,b_{2k-3,\,2s+1} + (-1)^{k-1} b_{2k-1,\,2s+1}, \quad (17)$$

and $\Delta(2n-1,\ 2s+1,\ 2k-1)$ is obtained by substituting n for k wherever $2k-1$ occurs as a first suffix, whether attached to a q or a b.

3.4. The process described in 3.3 results in replacing χ_{2n-1} by a new function $\chi^{(2n-1)}_{2n-1}$, of rank $2n-1$. Just as in 3.13 we can write

$$\chi^{(2n-1)}_{2n-1} = X^{(2n-1)}_{2n-1} - \Phi\{X^{(2n-1)}_{2n-1}\}, \quad (1)$$

$$X^{(2n-1)}_{2n-1} = R\{-iz^* T_{2n}(\zeta)\}, \quad (2)$$

$$T_{2n}(\zeta) = \zeta^{2n} - \frac{\partial^{n-2}\delta_{2n-1}}{\partial q_{3,\,1}\,\partial q_{5,\,3}\ldots\partial q_{2n-3,\,2n-5}}\,\zeta^{2n-2} + \ldots + (-1)^{n-1}\delta_{2n-1}\,\zeta^2. \quad (3)$$

Again we can identify this with

$$T_{2n}(\zeta) = \zeta^{2n} - r_{2n-1,\,2n-2}\,\zeta^{2n-2} - r_{2n-1,\,2n-4}\,\zeta^{2n-4} - \ldots - r_{2n-1,\,2}\,\zeta^2, \quad (4)$$

and, if we prefer to do so, determine the coefficients r as quotients of determinants, without previously determining the coefficients q, by solving the equations

$$b_{2n+1,\,2k+1} - \sum_{m=1}^{n-1} r_{2n-1,\,2m}\,b_{2m-1,\,2k+1} = 0 \quad (k = 0, 1, \ldots, n-2). \quad (5)$$

Further, we shall have

$$\left\{\frac{\partial}{\partial\rho}\chi_{2n-1}^{(2n-1)}\right\}_{\rho=1} = \sum_{s=n-1}^{\infty} b_{2n-1,\,2s+1}^{(2n-1)} \cos(2s+1)\phi, \tag{6}$$

where the coefficients b are determined by putting n for k in 3 . 3 (12), or may also be determined by the equation

$$b_{2n-1,\,2s+1}^{(2n-1)} = b_{2n-1,\,2s+1} - \sum_{m=1}^{n-1} r_{2n-1,\,2m}\, b_{2m-1,\,2s+1}. \tag{7}$$

3 . 5. For the χ''s with even suffixes, χ'_0 being redundant, and χ'_2 of rank 2, that is to say, its ρ derivative at $\rho=1$ possessing an expansion in sines of even multiples of ϕ, beginning with $\sin 2\phi$, we put

$$\left\{\frac{\partial}{\partial\rho}\chi'_{2n}\right\}_{\rho=1} = \sum_{s=1}^{\infty} b'_{2n,\,2s}\sin 2s\phi, \tag{1}$$

where, by 1 . 53 (5),

$$\left.\begin{aligned} b'_{2n,\,2s} &= -(4n-4s+2)\,a_{2n-2s+1} + (4n+2)\,a_{2n+2s+1} \quad (s=1,\,2,\,\ldots,\,n), \\ b'_{2n,\,2s} &= (4n+2)\,a_{2n+2s+1} \quad (s \geqslant n+1). \end{aligned}\right\} \tag{2}$$

Then we proceed by steps, putting

$$\chi'^{(2k)}_{2n} = \chi'^{(2k-2)}_{2n} - q'_{2n,\,2k-2}\,\chi'^{(2k-2)}_{2k-2} \quad (k=2,\,3,\,\ldots,\,n), \tag{3}$$

$$\left\{\frac{\partial}{\partial\rho}\chi'^{(2k)}_{2n}\right\}_{\rho=1} = \sum_{s=k-1}^{\infty} b'^{(2k)}_{2n,\,2s}\sin 2s\phi, \tag{4}$$

$$b'^{(2k)}_{2n,\,2s} = b'^{(2k-2)}_{2n,\,2s} - q'_{2n,\,2k-2}\,b'^{(2k-2)}_{2k-2,\,2s}, \tag{5}$$

$$b'^{(2k)}_{2n,\,2k-2} = 0. \tag{6}$$

To complete the notation we must have

$$\chi'^{(2)}_{2n} = \chi'_{2n}, \quad b'^{(2)}_{2n,\,2s} = b'_{2n,\,2s}. \tag{7}$$

In particular, we have $q'_{2n,\,2} = b'_{2n,\,2}/b'_{2,\,2} \quad (n \geqslant 2)$,

or

$$q'_{2n,\,2} = \frac{(2n-1)\,a_{2n-1} - (2n+1)\,a_{2n+3}}{a_1 - 3a_5}. \tag{8}$$

As in previous cases, we can prove that

$$b'^{(2k)}_{2n,\,2s} = (-1)^{k-1}\,\Delta'(2n,\,2s,\,2k), \tag{9}$$

and that

$$q'_{2n,\,2k-2} = \frac{\Delta'(2n,\,2k-2,\,2k-2)}{\Delta'(2k-2,\,2k-2,\,2k-2)}, \tag{10}$$

where

$$\Delta'(2k,\ 2s,\ 2k) = \delta'_{2k}\, b'_{2,\ 2s} - \frac{\partial \delta'_{2k}}{\partial q'_{4,\ 2}}\, b'_{4,\ 2s} + \frac{\partial^2 \delta'_{2k}}{\partial q'_{4,\ 2}\, \partial q'_{6,\ 4}}\, b'_{6,\ 2s} - \ldots$$

$$+ (-1)^{k-2} \frac{\partial^{k-2} \delta'_{2k}}{\partial q'_{4,\ 2}\, \partial q'_{6,\ 4} \ldots \partial q'_{2k-2,\ 2k-4}}\, b'_{2k-2,\ 2s} + (-1)^{k-1}\, b'_{2k,\ 2s}, \quad (11)$$

δ'_{2k} being the determinant

$$\delta'_{2k} = \begin{vmatrix} q'_{2k,\ 2k-2}, & q'_{2k,\ 2k-4}, & q'_{2k,\ 2k-6}, & \ldots, & q'_{2k,\ 2} \\ 1, & q'_{2k-2,\ 2k-4}, & q'_{2k-2,\ 2k-6}, & \ldots, & q'_{2k-2,\ 2} \\ 0, & 1, & q'_{2k-4,\ 2k-6}, & \ldots, & q'_{2k-4,\ 2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0, & 0, & 0, & \ldots, & q'_{4,\ 2} \end{vmatrix} \quad (12)$$

and $\Delta'(2n,\ 2s,\ 2k)$ being obtained from (11) by substituting n for k wherever $2k$ occurs as a first suffix, whether attached to a q' or to a b'.

3.6. The process described in 3.5 results in the construction of a function $\chi'^{(2n)}_{2n}$, of rank $2n$. Just as in 3.13 and 3.4 we can write

$$\chi'^{(2n)}_{2n} = X'^{(2n)}_{2n} - \Phi\{X'^{(2n)}_{2n}\}, \quad (1)$$

$$X'^{(2n)}_{2n} = R\{z^* T'_{2n+1}(\xi)\}, \quad (2)$$

$$T'_{2n+1}(\xi) = \xi^{2n+1} - \frac{\partial^{n-2} \delta'_{2n}}{\partial q'_{4,\ 2}\, \partial q'_{6,\ 4} \ldots \partial q'_{2n-2,\ 2n-4}}\, \xi^{2n-1} + \ldots$$

$$+ (-1)^{n-2} \frac{\partial \delta'_{2n}}{\partial q'_{4,\ 2}}\, \xi^5 + (-1)^{n-1}\, \xi'^3. \quad (3)$$

Again, we can write this

$$T'_{2n+1}(\xi) = \xi^{2n+1} - \sum_{m=1}^{n-2} r'_{2n,\ 2m+1}\, \xi^{2m+1}, \quad (4)$$

and we can determine the coefficients r', either by identifying (3) and (4), or directly from the equations

$$b'_{2n,\ 2s} - \sum_{m=1}^{n-2} r'_{2n,\ 2m+1}\, b'_{2m,\ 2s} = 0 \quad (s = 1, 2, \ldots, n-1). \quad (5)$$

Further, with a view to satisfying boundary conditions, we note that

$$\left(\frac{\partial}{\partial \rho} \chi'^{(2n)}_{2n}\right)_{\rho=1} = \sum_{s=1}^{\infty} b'^{(2n)}_{2n,\ 2s} \sin 2s\phi, \quad (6)$$

where $b'^{(2n)}_{2n,\,2s}$ can be obtained from 2 . 5 (9) by writing n for k, or can be written in the form

$$b'^{(2n)}_{2n,\,2s} = b'_{2n,\,2s} - \sum_{m=1}^{n-2} r'_{2n,\,2m+1}\, b'_{2m,\,2s} \quad (s \geqslant n). \tag{7}$$

3 . 7. For the χ''s with odd suffixes we put

$$\left\{\frac{\partial}{\partial\rho}\chi'_{2n-1}\right\}_{\rho=1} = \sum_{s=0}^{\infty} b'_{2n-1,\,2s+1} \sin(2s+1)\phi, \tag{1}$$

where, by 1 . 53 (6),

$$\left.\begin{aligned} b'_{2n-1,\,2s+1} &= -(4n-4s-2)\, a_{2n-2s-1} + 4n a_{2n+2s+1} \quad (s = 0, 1, \ldots, n-1), \\ b'_{2n-1,\,2s+1} &= 4n a_{2n+2s+1} \quad (s \geqslant n). \end{aligned}\right\} \tag{2}$$

Then we proceed by steps, and there is no difference between the work to be done for this case and that which was done in 3 . 3 and 3 . 4, except that 3 . 3 (9) is replaced by

$$q'_{2n-1,\,1} = \frac{-(2n-1)\, a_{2n-1} + 2n a_{2n+1}}{-a_1 + 2a_3}, \tag{3}$$

$\cos(2s+1)\phi$ is replaced by $\sin(2s+1)\phi$, and all the letters χ, q, b, Δ, δ, r T are accented.

3 . 8. Having regard to the forms of the polynomials $T_{2n+1}(\xi)$, ..., and to the identity 1 . 54 (3), and after what was done in 1 . 51, we may conclude, without formal proof, that any function, biharmonic in the rectangle, and vanishing at the boundary, can be expressed by a series of the form

$$A_0\chi_0 + \sum_{n=1}^{\infty} \{A_n \chi_n^{(n)} + A'_n \chi'^{(n)}_n\} \tag{1}$$

with suitable coefficients. To satisfy a boundary condition of the form

$$\frac{\partial\chi}{\partial\nu} = \frac{\partial V}{\partial\nu}, \tag{2}$$

where V is harmonic in the rectangle, we may suppose V to be expressed in the form

$$V = C_0 + \sum_{s=1}^{\infty} \rho^s (C_s \cos s\phi + C'_s \sin s\phi), \tag{3}$$

and determine the constants A_0, ... by the equations

$$A_0 = 0, \tag{4}$$

$$\left.\begin{aligned} A_1 b_{1,1} &= C_1, \\ A_1 b_{1,3} + A_3 b^{(3)}_{3,3} &= 3C_3, \\ A_1 b_{1,5} + A_3 b^{(3)}_{3,5} + A_5 b^{(5)}_{5,5} &= 5C_5, \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} A_2 b^{(2)}_{2,2} &= 2C_2, \\ A_2 b^{(2)}_{2,4} + A_4 b^{(4)}_{4,4} &= 4C_4, \\ A_2 b^{(2)}_{2,6} + A_4 b^{(4)}_{4,6} + A_6 b^{(6)}_{6,6} &= 6C_6, \end{aligned}\right\} \tag{6}$$

and the equations derived from (5) and (6) by accenting all the letters, where it is to be remembered that, when the letters are accented, and the superscript is 2, the superscript may be omitted in accordance with 3 . 5 (7).

4. *Applications to the theory of elasticity.*

4 . 1. We take first the problem of the clamped rectangular plate bent by uniform pressure.

The plate is taken to be of uniform small thickness $2h$, and of uniform isotropic material, whose elastic quality is specified by its Young's modulus E and Poisson's ratio σ. The clamped edges are given by the equations

$$x = \pm a, \quad y = \pm b,$$

and the constant pressure is denoted by p. The deflexion w is the displacement of a point on the initially plane middle surface in the direction of the pressure. It is known† that w satisfies the partial differential equation

$$D\nabla^4 w = p \tag{1}$$

at every point within the rectangle, and the conditions

$$w = 0, \quad \frac{\partial w}{\partial \nu} = 0 \tag{2}$$

at the boundary. Here D denotes the flexural rigidity, given in terms of h, E, σ by the formula

$$D = \tfrac{2}{3}\frac{Eh^3}{1-\sigma^2}. \tag{3}$$

† See, *e.g.*, my *Elasticity*, Chap. xxii.

The history of the problem goes back to a famous memoir by Poisson of date 1928. Its later stages are traced in my *Elasticity*, 4th edn., Art. 314 *c* (iii). Reference may also be made to *Journal London Math. Soc.*, 3 (1928), 144.

4 . 11. To solve the problem we put

$$w = \frac{p}{8D}\{(a^2-x^2)(b^2-y^2)+\chi\}. \tag{1}$$

Then χ is biharmonic inside the rectangle, and vanishes at the boundary, and, at the boundary, it satisfies the condition

$$\frac{\partial \chi}{\partial \nu} = 2a(b^2-y^2)+2b(a^2-x^2). \tag{2}$$

In accordance with 1 . 3 (6) we put

$$\chi = \tfrac{1}{3}(a^2+b^2)\{x^2+y^2-\Phi(x^2+y^2)\}+\chi'. \tag{3}$$

Then χ' is biharmonic in the rectangle, and vanishes at the boundary, and also satisfies the condition

$$\left(\frac{\partial \chi'}{\partial \rho}\right)_{\rho=1} = \left[\frac{\partial}{\partial \rho}\{-\tfrac{2}{3}(a^2-b^2)(x^2-y^2)+\tfrac{1}{3}(a^2+b^2)\,\Phi(x^2+y^2) + \tfrac{1}{6}(x^4+y^4-6x^2y^2)\}\right]_{\rho=1}. \tag{4}$$

The function in { } is harmonic in the rectangle, and each of the three harmonic functions composing it has been expressed in terms of ρ and ϕ in Chapter 2. In accordance with results there obtained it can be expressed in the form

$$\sum_{s=0}^{\infty} C_{2s}\rho^{2s}\cos 2s\phi. \tag{5}$$

The required function χ' can be expressed in the form

$$\chi' = \sum_{n=1}^{\infty} A_{2n}\chi_{2n}^{(2n)}, \tag{6}$$

and the coefficients can be found from the equations 3 . 8 (6).

4 . 12. To test the practical usefulness of the solution it may be applied to calculate the central deflexion, that is the value w_0 of w at $x = 0$, $y = 0$, in the case of a square plate, in which $a = b$, $a = \frac{1}{4}\pi$, $K = 1.854074....$ In this case approximate numerical solutions, ob-

tained by Ritz† and Hencky‡, agree in giving

$$w_0 = (pa^4/8D)(0.162). \tag{1}$$

In this case we have

$$w_0 = \frac{p}{8D}\left[a^4 - \tfrac{2}{3}a^2\gamma_0 + \sum_{n=1}^{\infty} A_{2n}\{\chi_{2n}^{(2n)}\}_0\right], \tag{2}$$

where the suffix 0 indicates that the function is to be calculated for $\rho = 0$. The coefficients C_{2s} of 4 . 11 (5) are given by the formula

$$C_{2s} = \tfrac{2}{3}a^2\gamma_{2s} + \tfrac{1}{6}\beta_{2s}, \tag{3}$$

where β_{2s} and γ_{2s} can be obtained from results given in 2 . 231 and 2 . 321. It appears that C_{2s} vanishes unless s is even, and thence that in 4 . 11 (6) n must be even.

Also in this case we have, when n is uneven,

$$a_{2n+1} = 0 \quad (n = 1, 3, 5, \ldots), \tag{4}$$

and, when n is even,

$$a_1 = \frac{2a}{K}, \quad a_{2n+1} = (-1)^{\frac{1}{2}n}\frac{2a}{K}\frac{1.3\ldots(n-1)}{2.4\ldots n}\frac{1}{2n+1}, \quad (n = 2, 4, \ldots). \tag{5}$$

The value of γ_0, given by 2 . 33 (7), is approximately $(1.17874)a^2$.

The first three terms in the series for $\chi_4^{(4)}$ and $\chi_8^{(8)}$ proceeding by cosines of multiples of 4ϕ are

$$\text{for } \chi_4^{(4)}, \quad \{(\rho^{10}-1)-5(\rho^2-1)\}a_5 + \{(a_1 - 5a_5^2/a_1)(\rho^2-1)\rho^4 + a_9(\rho^{10}-1)\rho^4\}\cos 4\phi + \{a_{13}(\rho^{10}-1)\rho^8 - 5(a_5 a_9/a_1)(\rho^2-1)\rho^8\}\cos 8\phi, \tag{6}$$

$$\text{for } \chi_8^{(8)}, \quad \{\rho^{18}-1-9(\rho^2-1)\}a_9 - m\{(\rho^{10}-1)-5(\rho^2-1)\}a_5 + [\{(\rho^{10}-1)\rho^4 a_5 + (\rho^{18}-1)\rho^4 a_{13} - 9(a_5 a_9/a_1)(\rho^2-1)\rho^4\} - m\{(a_1 - 5a_5^2/a_1)(\rho^2-1)\rho^4 + a_9(\rho^{10}-1)\rho^4\}]\cos 4\phi + [\{(a_1 - 9a_9^2/a_1)(\rho^2-1)\rho^8 + a_{17}(\rho^{18}-1)\rho^8\} - m\{a_{13}(\rho^{10}-1)\rho^8 - 5(a_5 a_9/a_1)(\rho^2-1)\rho^8\}]\cos 8\phi, \tag{7}$$

where

$$m = \frac{5a_5 + 9a_{13} - 9a_5 a_9/a_1}{a_1 + 5a_9 - 5a_5^2/a_1}. \tag{8}$$

† W. Ritz, *Journal für Math.*, 135 (1909) 1-61; reprinted in *Ges. Werke Walther Ritz* (Paris, 1911), 192.

‡ H. Hencky, *Der Spannungszustand in rechteckigen Platten* (Diss.) (München, 1913).

Also we have

$$\left.\begin{aligned} C_4 &= \frac{8a^4}{3K^2}\left(\frac{1}{K^2}-\frac{2}{3\pi}\right), \\ C_8 &= -\frac{16a^4}{15K^2}\left(\frac{1}{K^2}-\frac{5}{7\pi}\right). \end{aligned}\right\} \tag{9}$$

The boundary condition for χ' gives approximately

$$\left.\begin{aligned} A_4\frac{2a}{K} &= (0{\cdot}105404)\,a^4, \\ A_8\frac{2a}{K} &= -(0{\cdot}061102)\,a^4, \end{aligned}\right\} \tag{10}$$

and therefore we have, as an approximation to w_0,

$$w_0 = (pa^4/8D)(1-0{\cdot}78583-0{\cdot}04216-0{\cdot}00604) = (pa^4/8D)(0{\cdot}16597). \tag{11}$$

The closeness of the approximation of this value to that in (1), three terms of the series being calculated and the rest ignored, suggests that the solution may be practically useful as well as analytically complete.

4 . 2. As a second application of our biharmonic analysis we take the problem of a clamped rectangular plate bent by concentrated pressure applied at one point.

The history of this problem also goes back to very early researches in the theory of elasticity. It has, for example, been known for a long time that the deflexion w must be expressible in the form

$$w = \frac{P}{8\pi D}(r_1^2 \log r_1 + \chi), \tag{1}$$

where P denotes the concentrated pressure applied at the point (x_1, y_1), r_1 denotes the distance between this point and the point (x, y), and χ is a function, biharmonic in the region inside the rectangular boundary, and satisfying at the boundary the two conditions arising from

$$w = 0, \quad \frac{\partial w}{\partial \nu} = 0. \tag{2}$$

All this would be the same for a more general boundary. The analytical problem of determining χ was discussed in a very general way by Hadamard† by means of analysis of a quite different kind from that used in this paper, and the special case of a rectangular boundary was dis-

† J. Hadamard, "Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées", *Mém. par divers savants*, 33 (Paris, 1908), No. 4.

cussed recently by A. C. Dixon† by means of analysis of the same kind as that used by Hadamard. A numerical solution for a square plate loaded at the centre has been given by H. Hencky (*op. cit. ante*).

4 . 21. Let us begin with the special case where the concentrated pressure is applied at the origin, so that, in accordance with 4 . 2 (1),

$$w = (P/8\pi D)(r^2 \log r + \chi), \tag{1}$$

where r denotes distance from the origin. Introducing the complex variable z and ζ, we observe that, by 1 . 5 (3), the function $\log(z/\zeta)$ is holomorphic in the region inside the rectangle, the function $\log(r/\rho)$ is harmonic in the region, and the function $(x^2+y^2)\log(r/\rho)$ is biharmonic in the region. We can put

$$w = (P/8\pi D)\{(x^2+y^2)\log\rho + \chi'\}. \tag{2}$$

Then χ' is biharmonic in the region, and vanishes at the boundary, and at the boundary it satisfies the condition

$$\frac{\partial\chi'}{\partial\rho} + \frac{\partial}{\partial\rho}\{(x^2+y^2)\log\rho\} = 0. \tag{3}$$

Since $\rho = 1$ at the boundary, this equation is

$$\left(\frac{\partial\chi'}{\partial\rho}\right)_{\rho=1} = -(x^2+y^2)_{\rho=1}. \tag{4}$$

The right-hand member of (4) has been expressed in 2 . 32 in the form $\sum_{s=0}^{\infty} -\gamma_{2s}\cos 2s\phi$, and we may put

$$\chi' = A_0\chi_0 + \sum_{n=0}^{\infty} A_{2n}\chi_{2n}^{(2n)}, \tag{5}$$

Then the coefficients A are determined by the equations

$$\left.\begin{aligned} A_0 b_{0,0} &= -\gamma_0, \\ A_0 b_{0,2} + A_2 b_{2,2}^{(2)} &= -\gamma_2, \\ A_0 b_{0,4} + A_2 b_{2,4}^{(2)} + A_4 b_{4,4}^{(4)} &= -\gamma_4, \end{aligned}\right\} \tag{6}$$

so that they can be found successively, or a general determinant formula can be written down for any one of them.

4 . 22. We proceed to the general case where the concentrated pressure is applied at any point (x_1, y_1) inside the rectangle. Let z_1 be the

† *Proc. London Math. Soc.* (2), 25 (1926), 417.

affix of this point in the z plane, ζ_1 the affix of the corresponding point in the ζ plane. We write

$$|z-z_1| = r_1, \quad |\zeta-\zeta_1| = \rho_1. \tag{1}$$

According to 4 . 2 (1) we are to have

$$w = (P/8\pi D)(r_1^2 \log r_1 + \chi), \tag{2}$$

where χ satisfies the conditions stated in § 4 . 2. Here again the function $(z-z_1)/(\zeta-\zeta_1)$ is holomorphic in the z region, the function $\log(r_1/\rho_1)$ is harmonic in that region, and in the ζ region, and the function $r_1^2 \log(r_1/\rho_1)$ is biharmonic in the z region; and we can replace (2) by

$$w = (P/8\pi D)(r_1^2 \log \rho_1 + \chi'), \tag{3}$$

where χ' is biharmonic in the z region, and the conditions 4 . 2 (2) are to be satisfied at the boundary.

We write $$\xi_1 = ce^{i\beta}. \tag{4}$$

Let ζ_2 be the affix of the point in the ζ plane which is inverse (with respect to the circle $\rho = 1$) to the point whose affix is ζ_1, so that

$$\xi_2 = |1/\xi_1| e^{i \arg \zeta_1} = (1/c) e^{i\beta}, \tag{5}$$

and let $$|\zeta-\zeta_2| = \rho_2. \tag{6}$$

Then ρ_1/ρ_2 has the same value at all points on the circle $\rho = 1$, and that value is $|\zeta_1|$, or c. Also $\log \rho_2$ is harmonic in the ζ region, and in the z region, and $r_1^2 \log \rho_2$ is biharmonic in the z region. We can therefore put

$$w = (P/8\pi D)\{r_1^2 \log(\rho_1/\rho_2 c) + \chi''\}. \tag{7}$$

Then χ'' is biharmonic in the z region, vanishes at the boundary, and must satisfy the condition

$$\left(\frac{\partial \chi''}{\partial \rho}\right)_{\rho=1} = -\left\{\frac{\partial}{\partial \rho}\left(r_1^2 \log \frac{\rho_1}{\rho_2 c}\right)\right\}_{\rho=1}$$

$$= -\left\{r_1^2 \frac{\partial}{\partial \rho}\left(\log \frac{\rho_1}{\rho_2 c}\right)\right\}_{\rho=1}. \tag{8}$$

This equation is

$$\left(\frac{\partial \chi''}{\partial \rho}\right)_{\rho=1} = -\left\{r_1^2\left(\frac{1}{\rho_1}\frac{\partial \rho_1}{\partial \rho} - \frac{1}{\rho_2}\frac{\partial \rho_2}{\partial \rho}\right)\right\}_{\rho=1}. \tag{9}$$

It is found immediately that, at $\rho = 1$,

$$\frac{\partial \rho_1}{\partial \rho} = \frac{1+\rho_1^2-c^2}{2\rho_1}, \quad \frac{\partial \rho_2}{\partial \rho} = \frac{1+\rho_2^2-1/c^2}{2\rho_2}, \tag{10}$$

and these, with $\rho_2 = \rho_1/c$, at $\rho = 1$, give

$$\left(\frac{1}{\rho_1}\frac{\partial \rho_1}{\partial \rho} - \frac{1}{\rho_2}\frac{\partial \rho_2}{\partial \rho}\right)_{\rho=1} = \frac{1-c^2}{\rho_1^2}. \tag{11}$$

Hence the boundary condition (9) becomes

$$\left(\frac{\partial \chi''}{\partial \rho}\right)_{\rho=1} = -(1-c^2)\left(\frac{r_1^2}{\rho_1^2}\right)_{\rho=1}. \tag{12}$$

4.23. In order to satisfy this condition it is necessary to expand the function $(r_1^2/\rho_1^2)_{\rho=1}$ in a Fourier series of cosines and sines of multiples of ϕ. We write down the equations

$$r_1^2 = x_1^2+y_1^2-2x_1x-2y_1y+(x^2+y^2), \tag{1}$$

and

$$(1/\rho_1^2)_{\rho=1} = 1+2\sum_{m=1}^{\infty} c^m \cos m(\beta-\phi). \tag{2}$$

Then

$$\{(x_1^2+y_1^2)/\rho_1^2\}_{\rho=1} = x_1^2+y_1^2+2(x_1^2+y_1^2)\sum_{s=1}^{\infty}(c^s \cos s\beta \cos s\phi + c^s \sin s\beta \sin s\phi). \tag{3}$$

Let the series for $(x/\rho_1^2)_{\rho=1}$ be

$$(x/\rho_1^2)_{\rho=1} = \xi_0 + \sum_{s=1}^{\infty}(\xi_s \cos s\phi + \xi'_s \sin s\phi). \tag{4}$$

Then

$$2\pi\xi_0 = \int_0^{2\pi}\left\{\sum_{n=0}^{\infty} a_{2n+1}\sin(2n+1)\phi\right\}\left\{1+2\sum_{m=1}^{\infty} c^m \cos m(\beta-\phi)\right\}d\phi$$

$$= \pi\sum_{n=0}^{\infty} a_{2n+1}c^{2n+1}\sin(2n+1)\beta = \pi x_1,$$

so that

$$\xi_0 = \tfrac{1}{2}x_1. \tag{5}$$

Again we have

$$\left.\begin{aligned}\pi\xi_s &= \int_0^{2\pi}\left\{\sum_{n=0}^{\infty} a_{2n+1}\sin(2n+1)\phi\right\}\left\{1+2\sum_{m=1}^{\infty} c^m \cos m(\beta-\phi)\right\}\cos s\phi\, d\phi\\ \pi\xi'_s &= \int_0^{2\pi}\left\{\sum_{n=0}^{\infty} a_{2n+1}\sin(2n+1)\phi\right\}\left\{1+2\sum_{m=1}^{\infty} c^n \cos m(\beta-\phi)\right\}\sin s\phi\, d\phi\end{aligned}\right\}. \tag{6}$$

Now we can write down the equations

$$\left\{1+2\sum_{m=1}^{\infty} c^m \cos m(\beta-\phi)\right\} \cos s\phi$$

$$= c^s \cos s\beta + \sum_{l=1}^{\infty} \{c^{|l-s|} \cos (l-s)\beta + c^{l+s} \cos (l+s)\beta\} \cos l\phi$$

$$+ \sum_{l=1}^{\infty} \{c^{|l-s|} \sin (l-s)\beta + c^{l+s} \cos (l+s)\beta\} \cos l\phi, \quad (7)$$

and

$$\left\{1+2\sum_{m=1}^{\infty} c^m \cos m(\beta-\phi)\right\} \sin s\phi$$

$$= c^s \sin s\beta + \sum_{l=1}^{\infty} \{-c^{|l-s|} \sin (l-s)\beta + c^{l+s} \sin (l+s)\beta\} \cos l\phi$$

$$+ \sum_{l=1}^{\infty} \{c^{|l-s|} \cos (l-s)\beta - c^{l+s} \sin (l+s)\beta\} \sin l\phi. \quad (8)$$

From these we find that

$$\xi_s = \sum_{n=0}^{\infty} a_{2n+1} \{c^{|2n-s+1|} \sin (2n-s+1)\beta + c^{2n+s+1} \sin (2n+s+1)\beta\} \quad (9)$$

and

$$\xi'_s = \sum_{n=0}^{\infty} a_{2n+1} \{c^{|2n-s+1|} \cos (2n-s+1)\beta - c^{2n+s+1} \cos (2n+s+1)\beta\}. \quad (10)$$

4.231. In the same way we can write down the Fourier series for $(y/\rho_1^2)_{\rho=1}$ and $\{(x^2+y^2)/\rho_1^2\}_{\rho=1}$. We put

$$(y/\rho_1^2)_{\rho=1} = \eta_0 + \sum_{s=1}^{\infty} (\eta_s \cos s\phi + \eta'_s \sin s\phi), \quad (1)$$

and

$$\{(x^2+y^2)/\rho_1^2\}_{\rho=1} = p_0 + \sum_{s=1}^{\infty} (p_s \cos s\phi + p'_s \sin s\phi), \quad (2)$$

and observe that

$$(y)_{\rho=1} = -\sum_{n=0}^{\infty} a_{2n+1} \cos (2n+1)\phi, \quad (3)$$

and

$$(x^2+y^2)_{\rho=1} = \sum_{n=0}^{\infty} \gamma_{2n} \cos 2n\phi. \quad (4)$$

Then we find that

$$\eta_0 = -\tfrac{1}{2} \sum_{n=0}^{\infty} a_{2n+1} c^{2n+1} \cos (2n+1)\beta = \tfrac{1}{2} y_1, \quad (5)$$

and

$$\eta_s = -\sum_{n=0}^{\infty} a_{2n+1}\{c^{|2n-s+1|}\cos(2n-s+1)\beta + c^{2n+s+1}\cos(2n+s+1)\beta\}, \quad (6)$$

and also

$$\eta'_s = -\sum_{n=0}^{\infty} a_{2n+1}\{-c^{|2n-s+1|}\sin(2n-s+1)\beta + c^{2n+s+1}\sin(2n+s+1)\beta\}. \quad (7)$$

Again we find that

$$p_0 = \gamma_0 + \sum_{n=1}^{\infty} \gamma_{2n} c^{2n}\cos 2n\beta = \{\Phi(x^2+y^2)\}_{x=x_1,\ y=y_1}, \quad (8)$$

and

$$p_s = 2\gamma_0 c^s \cos s\beta + \sum_{n=1}^{\infty} \gamma_{2n}\{c^{|2n-s|}\cos(2n-s)\beta + c^{2n+s}\cos(2n+s)\beta\}, \quad (9)$$

and also

$$p'_s = 2\gamma_0 c^s \sin s\beta + \sum_{n=1}^{\infty} \gamma_{2n}\{-c^{|2n-s|}\sin(2n-s)\beta + c^{2n+s}\sin(2n+s)\beta\}. \quad (10)$$

4 . 24. The four terms in the expression $(r_1^2/\rho_1^2)_{\rho=1}$ have now been expressed as Fourier series of cosines and sines of multiples of ϕ. The function χ'' of 4 . 22 can now be determined, in accordance with the equation 4 . 22 (12), by assuming that

$$\chi'' = A_0\chi_0 + \sum_{n=1}^{\infty}\{A_n\chi_n^{(n)} + A'_n\chi_n'^{(n)}\}, \quad (1)$$

and writing down the equations

$$\left.\begin{aligned} A_0 b_{0,0} &= -(1-c^2)p_0, \\ A_0 b_{0,2} + A_2 b_{2,4}^{(2)} &= -(1-c^2)\{2(x_1^2+y_1^2)c^2 - 2x_1\xi_2 - 2y_1\eta_2 + p_2\} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \end{aligned}\right\} \quad (2)$$

and so on.

It may be noted that there is no difficulty of principle preventing the extension of the solution to any form of boundary, enclosing a simply-connected region, if the equation of type 1 . 5 (2), expressing the conformal transformation of the z region to the ζ region, is known.

4 . 3. We shall now consider the question of determining the mean stresses in a rather long, rather deep, and rather narrow rectangular beam resting on two supports, placed symmetrically and at the same level, and loaded by a weight, placed midway between them. The

weight of the beam will be neglected, and the beam will be assumed to be in a state of "generalized plane stress", the plane of the mean stress being that of the length and depth.

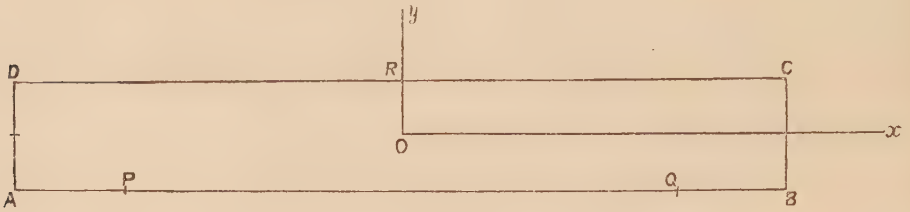


Fig. 2.

In Fig. 2 the rectangle $ABCD$ represents a section of the beam by a vertical plane, so that AB $(=2a)$ is its length and BC $(=2b)$ is its depth. The third dimension, the breadth, at right angles to the plane, will be denoted by $2h$. The load W, applied vertically downwards in a plane at right angles to the plane of the figure, and meeting CD at its middle point R, will be supposed to be distributed uniformly along that segment of a line through R at right angles to the plane of the figure, which is contained between the faces of the beam that are parallel to the plane of the figure. Upward vertical pressures, each equal to $\frac{1}{2}W$, will then be applied to the lower face of the beam along two lines, also at right angles to the plane of the figure, and meeting it in the points P, Q. These pressures also will be distributed uniformly. The mean concentrated pressures at R, P, Q will then be $W/2h$, $W/4h$, $W/4h$. The axes of x and y are chosen as the horizontal and vertical median lines of the rectangle. We shall take the points P, Q to be $(-l, -b)$ and $l, -b)$, so that the length PQ is $2l$.

The history of the problem, which has been solved very nearly completely by L. N. G. Filon*, is traced in my *Elasticity*, 3rd or 4th ed., Articles 245 and 245A.

4 . 31. In all problems of generalized plane stress, in a body strained by surface tractions only, the mean stresses $\bar{X}_x$, $\bar{Y}_y$, $\bar{X}_y$ are expressible in the forms

$$\bar{X}_x = \frac{\partial^2 \chi}{\partial y^2}, \quad \bar{Y}_y = \frac{\partial^2 \chi}{\partial x^2}, \quad \bar{X}_y = -\frac{\partial^2 \chi}{\partial x \partial y}, \tag{1}$$

where the function χ, called the "stress-function", must be a biharmonic function of x and y.

† L. N. G. Filon, *Phil. Trans. Royal Soc.* (A), 201 (1903).

In this problem χ must have singular points at the points P, Q, R in the figure. The nature of the singularity at R is that $\partial\chi/\partial x$ is discontinuous at R, with a *saltus* $W/2h$ as x changes from positive to negative values. It will be sufficient to take it to have the constant value $-W/4h$ along RC, and the constant value $W/4h$ along DR. Similar descriptions apply to the singularities at P and Q. It is sufficient to take $\partial\chi/\partial x$ to have the constant value $-W/4h$ along QB, the constant value $W/4h$ along AP, and the value zero along PQ.

We have still to satisfy the conditions

$$\frac{\partial^2\chi}{\partial x\,\partial y} = 0 \tag{2}$$

at all the edges, and

$$\frac{\partial^2\chi}{\partial y^2} = 0 \tag{3}$$

at the edges BC and DA.

It will be sufficient to take the boundary values of χ to be given by $\chi = -Wx/4h$ along $QBCR$, $\chi = Wx/4h$ along $RDAP$, $\chi = -Wl/4h$ along PQ, and further to take the boundary values of $\partial\chi/\partial\nu$ to be zero along AB and CD, and $-W/4h$ along BC and DA.

Discontinuity of $\partial\chi/\partial\nu$ at the vertices of the rectangle does not imply any forces concentrated at these vertices. Discontinuity of $\partial\chi/\partial x$ or $\partial\chi/\partial y$ would imply that such forces were present. For example, if χ were $\frac{1}{2}x^2$, a beam such as that represented by $ABCD$ in Fig. 2 would be under uniform tension parallel to the axis of y, $\partial\chi/\partial\nu$ would be discontinuous at the vertices, and there would be no concentrated forces at the vertices. If, on the other hand, we were to consider the plane region in the positive quadrant bounded by the axes of x and y as representing a plate in a state of generalized plane stress, expressed by a stress-function χ, equal to $y+x\tan^{-1}(x/y)$, $\partial\chi/\partial\nu$ would vanish all along the two boundaries, and thus be continuous, $\partial\chi/\partial x$ would be equal to $\frac{1}{2}\pi$ along Ox and 0 along Oy, $\partial\chi/\partial y$ would be equal to 0 along Ox and to 1 along Oy. There would be no tractions on the boundaries except at the origin, and the concentrated mean force applied at the origin would have components equal to 1 in the positive direction of the axis of x, and to $\frac{1}{2}\pi$ in the positive direction of the axis of y.

4.32. To determine χ from the boundary values of χ and $\partial\chi/\partial\nu$, we put

$$\chi = -(W/4h)(\omega_1+\omega_2), \tag{1}$$

where ω_1 and ω_2 are both biharmonic in the rectangle, and satisfy the

boundary conditions

$$\left.\begin{array}{l} \text{all along the boundary, } \omega_1 = 0\,; \\ \text{along } AB \text{ and } CD,\ \partial\omega_1/\partial\nu = 0\,;\ \text{along } BC \text{ and } DA,\ \partial\omega_1/\partial\nu = 1\,; \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l} \text{all along the boundary, } \partial\omega_2/\partial\nu = 0\,; \\ \text{along } PQ,\ \omega_2 = l\,;\ \text{along } QBCR,\ \omega_2 = x\,;\ \text{along } RDAP,\ \omega_2 = -x. \end{array}\right\} \tag{3}$$

At any point on the boundary

$$\frac{\partial\omega_1}{\partial\rho} = \left|\frac{dz}{d\zeta}\right|\frac{\partial\omega_1}{\partial\nu}, \tag{4}$$

and, therefore, on the arcs BC, DA of Fig. 1 in 2 . 1, we have, by 2 . 11 (9),

$$\left(\frac{\partial\omega_1}{\partial\rho}\right)_{\rho=1} = \frac{a}{K}\frac{1}{\sqrt{(\cos^2\alpha - \cos^2\phi)}}, \tag{5}$$

and on the arcs AB, CD, $\partial\omega_1/\partial\rho = 0$.

In the notation of 2 . 22 we have then

$$\left(\frac{\partial\omega_1}{\partial\rho}\right)_{\rho=1} = \frac{4a}{\pi K}\left\{\tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_{2n}\cos 2n\phi\right\}. \tag{6}$$

We have therefore to take for ω_1 a series of the form

$$\omega_1 = A_0\chi_0 + \sum_{n=1}^{\infty} A_{2n}\chi_{2n}^{(2n)}, \tag{7}$$

and the coefficients A_0, A_2, ... are determined by the equations

$$\left.\begin{array}{rrrrrr} A_0 b_{0,0} & & & & & = (2a/\pi K)\, b_0, \\ A_0 b_{0,2} & + A_2 b_{2,2}^{(2)} & & & & = (4a/\pi K)\, b_2, \\ A_0 b_{0,4} & + A_2 b_{2,4}^{(2)} & + A_4 b_{4,4}^{(4)} & & & = (4a/\pi K)\, b_4, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right\} \tag{8}$$

4 . 33. To determine ω_2 from the boundary conditions 4 . 32 (3) we write θ for the boundary value of ω_2, and begin by determining $\Phi(\theta)$. Since ω_2 is an even function of x, the Fourier series for θ in terms of ϕ will involve cosines only. So we put

$$\theta = \delta_0 + \sum_{n=1}^{\infty} \delta_n \cos n\phi\,; \tag{1}$$

then we shall have

$$\Phi(\theta) = \delta_0 + \sum_{n=1}^{\infty} \delta_n \rho^n \cos n\phi. \tag{2}$$

Omitting δ_0 for the present, we shall have

$$\tfrac{1}{2}\pi\delta_n = \int_0^{\pi} \theta \cos n\phi\, d\phi$$

$$= \int_0^{\phi_0} l \cos n\phi\, d\phi + \int_{\phi_0}^{\alpha} x \cos n\phi\, d\phi + \int_{\alpha}^{\pi-\alpha} a \cos n\phi\, d\phi + \int_{\pi-\alpha}^{\pi} x \cos n\phi\, d\phi, \tag{3}$$

where ϕ_0 is the value of ϕ at Q, and $0 < \phi_0 < \alpha$. The values of x at $\phi = \phi_0$, $\phi = \alpha$, $\phi = \pi - \alpha$, $\phi = \pi$ are l, a, a, 0. Also

$$\int_{\phi_0}^{\alpha} x \cos n\phi\, d\phi = \frac{1}{n}(a \sin n\alpha - l \sin n\phi_0) - \int_{\phi_0}^{\alpha} \frac{1}{n} \sin n\phi \frac{\partial x}{\partial \phi} d\phi, \tag{4}$$

and

$$\int_{\pi-\alpha}^{\pi} x \cos n\phi\, d\phi = \frac{1}{n}\{-a \sin n(\pi-\alpha)\} - \int_{\pi-\alpha}^{\pi} \frac{1}{n} \sin n\phi \frac{\partial x}{\partial \phi} d\phi; \tag{5}$$

and, furthermore, on the arc $\phi_0 < \phi < \alpha$,

$$\frac{\partial x}{\partial \phi} = \left| \frac{dz}{d\xi} \right| = \frac{a}{K} \frac{1}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}, \tag{6}$$

and, on the arc $\pi - \alpha < \phi < \pi$,

$$\frac{\partial x}{\partial \phi} = -\left| \frac{dz}{d\xi} \right| = -\frac{a}{K} \frac{1}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}. \tag{7}$$

Hence we find

$$\tfrac{1}{2}\pi\delta_n = -\frac{1}{n}\frac{a}{K}\int_{\phi_0}^{\alpha} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}} + \frac{1}{n}\frac{a}{K}\int_{\pi-\alpha}^{\pi} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}},$$

or

$$\delta_n = \frac{2a}{n\pi K}\left\{(-1)^{n+1}\int_0^{\alpha} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}} - \int_{\phi_0}^{\alpha} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\right\}. \tag{8}$$

Apart from the evaluation of the integrals in (8) and the determination of δ_0, $\Phi(\theta)$ has now been found. We put

$$\omega_2 = \Phi(\theta) + \omega'_2; \tag{9}$$

then ω'_2 is biharmonic in the rectangle, vanishes at the boundary, and satisfies the boundary condition

$$\left(\frac{\partial \omega'_2}{\partial \rho}\right)_{\rho=1} = -\sum_{n=1}^{\infty} n\delta_n \cos_n \phi, \tag{10}$$

and we therefore take for ω'_2 a series of the form

$$\omega'_2 = \sum_{n=1}' B_n \chi_n^{(n)}. \tag{11}$$

Then the coefficients B are determined by the two sets of equations

$$\left.\begin{aligned} B_1 b_{1,1} &= -\delta_1 \\ B_1 b_{1,3} + B_3 b^{(3)}_{3,3} &= -3\delta_3 \\ B_1 b_{1,5} + B_3 b^{(3)}_{3,5} + B_5 b^{(5)}_{5,5} &= -5\delta_5 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \end{aligned}\right\} \tag{12}$$

and

$$\left.\begin{aligned} B_2 b^{(2)}_{2,2} &= -2\delta_2 \\ B_2 b^{(2)}_{2,4} + B_4 b^{(4)}_{4,4} &= -4\delta_4 \\ B_2 b^{(2)}_{2,6} + B_4 b^{(4)}_{4,6} + B_6 b^{(6)}_{6,6} &= -6\delta_6 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \end{aligned}\right\} \tag{13}$$

As δ_0 only makes its appearance as a constant additive to ω_2, and a constant added to a stress-function does not affect the stress, it is not necessary to determine δ_0.

4 . 331. For the integrals in 4 . 33 (8) there is a formula of reduction :

$$\int \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}} = -2\cos(n-2)\phi \cdot \sqrt{(\sin^2\alpha - \sin^2\phi)} + \cos 2\alpha \int \frac{\sin(n-2)\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}\, d\phi. \tag{1}$$

When n is odd, we have

$$\delta_n = \frac{2a}{n\pi K} \int_0^{\phi_0} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}, \tag{2}$$

and the last integral to be determined is

$$\int_0^{\phi_0} \frac{\sin\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}}, \tag{3}$$

which is

$$\log \frac{1+\sin\alpha}{\cos\phi_0 + \sqrt{(\sin^2\alpha - \sin^2\phi_0)}}. \tag{4}$$

When n is even, we have

$$\delta_n = -\frac{2a}{n\pi K} \left\{ 2\int_0^{\alpha} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}} - \int_0^{\phi_0} \frac{\sin n\phi\, d\phi}{\sqrt{(\sin^2\alpha - \sin^2\phi)}} \right\}, \tag{5}$$

and all the integrals are determined by successive applications of (1).

The analytical solution of the problem is now completed, but, both in this question and in the one discussed in 4 . 1 – 4 . 12, a considerable amount of work remains to be done to render the solutions practically valuable. The question discussed in 4 . 2 – 4 . 22 is rather of academic than practical interest.

A CONTRIBUTION TO THE THEORY OF UNIQUENESS OF REPRESENTATION BY TRIGONOMETRICAL INTEGRALS

By M. E. GRIMSHAW.

[Received 23 March, 1928.—Read 26 April, 1928.]

1. The idea in what follows is to obtain in the theory of trigonometrical integrals some developments of methods which Rajchman and Zygmund have applied with considerable success to the theory of trigonometrical series. The methods themselves date back to Riemann and are extremely well known. Rajchman and Zygmund have shown, however, that they can be applied to series summable by the very general process due to Poisson and Abel, and not merely to convergent series*. The application of the methods to the case of integrals is of interest in that the formal manipulations assume an elegance and simplicity not so easily obtained in the case of series, and in some ways allow the underlying ideas to be seen more readily.

We shall assume throughout that integrals are to be interpreted in the general Denjoy sense, unless the contrary is definitely stated. The integral under consideration is

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds, \tag{1}$$

which is not necessarily convergent. What is eventually shown is that, under certain conditions, the constants a_s, b_s are determined, except possibly in a set of measure zero, by the Poisson values of the integral, *i.e.* the upper and lower limits as $\eta \to +0$ of

$$\int_0^\infty e^{-\eta s} (a_s \cos sx + b_s \sin sx)\, ds.$$

These Poisson values can be obtained either by means of the

* 1 and 2.

exponential factor $e^{-\eta s}$, or by means of an algebraic factor r^s. The advantage of the former is that it is reproduced by differentiation, and its employment considerably simplifies the functional equations which occur in the work.

If $-(a_s/s^2)$, $-(b_s/s^2)$ are the Fourier constants* of a function $F(x)$, which we express by writing

$$F(x) \sim \left[-\frac{a_s}{s^2},\ -\frac{b_s}{s^2}\right],$$

then the upper and lower Riemann values of (1) are the upper and lower limits as $t \to +0$ of

$$\frac{F(x+t)-2F(x)+F(x-t)}{t^2}.$$

The first section is concerned with the establishment of relations of inequality between the Poisson and Riemann values of (1). It is found that the intervals determined by the two sets of values overlap, provided that $F(x)$ satisfies simple conditions. In view of this, the uniqueness theory can be reduced to the application of well known properties of the symmetrical second derivative.

2. LEMMA I. *If $\phi(t)$ is integrable in $(0, c)$ for every positive c, $\phi(t)/t$ is absolutely integrable at ∞, and*

$$\Psi(s) = \frac{1}{\pi}\int_0^\infty \phi(t)\,\frac{\sin st}{t}\,dt\,;$$

then the Poisson values of $\displaystyle\int_0^\infty 2s\Psi(s)\,ds$

lie between the lower and upper limits, as $t \to +0$, of $\phi(t)/t^2$.

We say that $f(t)$ is absolutely integrable at ∞ if

$$\int_c^\infty |f(t)|\,dt < \infty$$

for sufficiently large values of c.

The lemma states that

$$(1)\quad \varliminf_{t\to+0}\frac{\phi(t)}{t^2} \leqslant \varliminf_{\eta\to+0}\int_0^\infty e^{-\eta s}\,2s\Psi(s)\,ds, \qquad \varlimsup_{\eta\to+0}\int_0^\infty e^{-\eta s}\,2s\Psi(s)\,ds \leqslant \varlimsup_{t\to+0}\frac{\phi(t)}{t^2}.$$

---

* *I.e.*, $-\dfrac{a_s}{s^2} = \lim\limits_{\lambda\to\infty}\dfrac{1}{\pi}\displaystyle\int_{-\lambda}^{\lambda} F(x)\cos sx\,dx$, $\quad -\dfrac{b_s}{s^2} = \lim\limits_{\lambda\to\infty}\dfrac{1}{\pi}\displaystyle\int_{-\lambda}^{\lambda} F(x)\sin sx\,dx$.

It is sufficient to prove one of the inequalities, since the other may be deduced by replacing $\phi(t)$ by $-\phi(t)$. We show that

$$\overline{\lim_{t\to+0}}\,\frac{\phi(t)}{t^2} < q$$

implies

$$\overline{\lim_{\eta\to+0}}\int_0^\infty e^{-\eta s}\,2s\Psi(s)\,ds \leqslant q$$

for every finite value of q. This covers every case of the second inequality except

$$\overline{\lim_{t\to+0}}\,\frac{\phi(t)}{t^2} = +\infty,$$

which is trivial.

Consider the special function

$$\phi(t) = t^2 \quad (0 \leqslant t \leqslant 1),$$
$$= 0 \quad (t > 1).$$

For this function

$$\pi\Psi(s) = -\frac{\cos s}{s} + \frac{\sin s}{s^2},$$

$$\int_0^\infty 2s\Psi(s)\,ds = 1 \quad (P),$$

and it is clear that (1) is satisfied.

By subtracting q times the above function from the general $\phi(t)$, the case of an arbitrary q can be reduced to that of $q = 0$. We have therefore only to show that

(2) $$\overline{\lim_{t\to+0}}\,\frac{\phi(t)}{t^2} < 0$$

implies

(3) $$\overline{\lim_{\eta\to+0}}\int_0^\infty e^{-\eta s}\,2s\Psi(s)\,ds \leqslant 0.$$

If (2) holds, there is a positive number δ such that

(4) $$\phi(t) < 0 \quad (0 \leqslant t < \delta).$$

Let η be a positive number, and consider the integrals

(5) $$\int_0^\infty ds\,2se^{-\eta s}\int_0^c \phi(t)\frac{\sin st}{t}\,dt, \quad \int_0^\infty ds\,2se^{-\eta s}\int_c^\infty \phi(t)\frac{\sin st}{t}\,dt,$$

where c is so chosen that

$$\int_c^\infty \left|\frac{\phi(t)}{t}\right| dt < \infty .$$

In both of these integrals the order of integration may be changed. The second offers no difficulty since, in view of the above,

$$\int_0^\infty ds\, 2se^{-\eta s} \int_c^\infty \left|\phi(t)\frac{\sin st}{t}\right| dt < \infty .$$

To consider the first, write

$$\int_0^c \phi(u)\, du = ca, \quad \theta(t) = \phi(t) - a, \quad \Theta(t) = \int_0^t \theta(u)\, du.$$

Then
$$\int_0^c \theta(t)\frac{\sin st}{t}\, dt = -\int_0^c \Theta(t)\frac{d}{dt}\left(\frac{\sin st}{t}\right) dt,$$

and

$$(6) \quad \int_0^\infty ds\, 2se^{-\eta s}\int_0^c \phi(t)\frac{\sin st}{t}\, dt$$
$$= -\int_0^\infty ds\, 2se^{-\eta s}\int_0^c \Theta(t)\frac{d}{dt}\left(\frac{\sin st}{t}\right) dt + \int_0^\infty ds\, 2se^{-\eta s}\int_0^c a\frac{\sin st}{t}\, dt.$$

Now
$$\left|\frac{d}{dt}\left(\frac{\sin st}{t}\right)\right| \leqslant s^2,$$

so that if, in the integrals on the right-hand side of (6), we replace the integrands by their absolute values the results are respectively not greater than

$$\int_0^\infty 2Mcs^3 e^{-\eta s}\, ds, \quad \int_0^\infty 2cas^2 e^{-\eta s}\, ds,$$

where M is the upper bound of $|\Theta(t)|$ in $(0, c)$. These are both finite. We may therefore change the order of integration and write

$$\int_0^\infty ds\, 2se^{-\eta s}\int_0^c \phi(t)\frac{\sin st}{t}\, dt$$
$$= -\int_0^c dt\Theta(t)\int_0^\infty 2se^{-\eta s}\frac{d}{dt}\left(\frac{\sin st}{t}\right) ds + \int_0^c dta\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\, ds$$
$$= -\left[\Theta(t)\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\, ds\right]_0^c + \int_0^c dt\{\theta(t)+a\}\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\, ds$$
$$= \int_0^c \phi(t)\, dt\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\, ds.$$

This justifies the change of order of integration in the first of the integrals in (5).

We now have

$$\int_0^\infty 2se^{-\eta s}\Psi(s)\,ds$$

$$= \frac{1}{\pi}\int_0^c \phi(t)\,dt\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\,ds + \frac{1}{\pi}\int_c^\infty \phi(t)\,dt\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \phi(t)\,dt\int_0^\infty 2se^{-\eta s}\frac{\sin st}{t}\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \phi(t)\frac{4\eta}{(\eta^2+t^2)^2}\,dt.$$

Divide the range of integration into the three parts $(0, \delta)$, (δ, c), (c, ∞), and consider the contributions of the separate parts. By (4), that of the first is definitely negative for positive values of η. That of the third tends to zero as $\eta\to+0$, since

$$\left|\int_c^\infty \phi(t)\frac{\eta}{(\eta^2+t^2)^2}\,dt\right| \leqslant \eta\int_c^\infty \left|\frac{\phi(t)}{t^4}\right| dt.$$

Finally, by the second mean value theorem,

$$\int_\delta^c \phi(t)\frac{\eta}{(\eta^2+t^2)^2}\,dt = \frac{\eta}{(\eta^2+\delta^2)^2}\int_\delta^\gamma \phi(t)\,dt \quad (\delta\leqslant\gamma\leqslant c)$$

$$\to 0$$

as $\eta\to+0$.

It is now clear that (3) holds, and the lemma is proved.

3. LEMMA II. *If $\psi(t)$ is integrable in $(0, s)$ for every positive value of s, and*

$$\Psi(s) = \int_0^s \psi(t)\,dt;$$

then, whenever

(1) $$\int_0^\infty \psi(t)\,dt = 0 \quad (P),$$

the Poisson values of $\int_0^\infty 2s\Psi(s)\,ds$ *lie between those of* $\int_0^\infty -s^2\psi(s)\,ds$.

We first show that, if (1) is satisfied, then, as $s\to\infty$, for all positive values of η,

(2) $$e^{-\eta s}\Psi(s)\to 0$$

By (1), for each positive η,

$$\int_0^u e^{-\eta t}\psi(t)\,dt$$

has a finite limit as $u\to\infty$, so that

$$\int_u^v e^{-\eta t}\psi(t)\,dt\to 0$$

as $u, v\to\infty$. If now $s>u$,

$$\Psi(s)=\Psi(u)+\int_u^s e^{\eta t}e^{-\eta t}\psi(t)\,dt$$

$$=\Psi(u)+e^{\eta s}\int_w^s e^{-\eta t}\psi(t)\,dt \quad (u\leqslant w\leqslant s)$$

by the second mean value theorem. Thus

$$e^{-\eta s}\Psi(s)=e^{-\eta s}\Psi(u)+\int_w^s e^{-\eta t}\psi(t)\,dt.$$

If u is sufficiently large,

$$\left|\int_w^s e^{-\eta t}\psi(t)\,dt\right|<\epsilon,$$

where ϵ is any prescribed positive number. Make $s\to\infty$ and keep u fixed. We have

$$\overline{\lim_{s\to\infty}}\,|e^{-\eta s}\Psi(s)|\leqslant\epsilon,$$

which gives (2).

It immediately follows that

$$\int_0^\infty e^{-\eta s}\psi(s)\,ds=\lim_{\lambda\to\infty}\int_0^\lambda e^{-\eta s}\psi(s)\,ds$$

$$=\lim_{\lambda\to\infty}\left\{\left[e^{-\eta s}\Psi(s)\right]_0^\lambda+\eta\int_0^\lambda e^{-\eta s}\Psi(s)\,ds\right\}$$

$$=\eta\int_0^\infty e^{-\eta s}\Psi(s)\,ds.$$

For the lemma it is sufficient to prove that

$$\overline{\lim_{\eta\to+0}}\int_0^\infty e^{-\eta s}\{-s^2\psi(s)\}\,ds<q$$

implies

$$\overline{\lim_{\eta\to+0}}\int_0^\infty e^{-\eta s}\,2s\Psi(s)\,ds\leqslant q$$

for every finite value of q. As before, we find that there is no loss of

generality in assuming q to be zero. A special form of $\psi(t)$ is defined by taking

$$\begin{aligned}\psi(t) &= \tfrac{1}{2} \quad (0 \leqslant t < 1),\\ &= -\tfrac{1}{2} \quad (1 \leqslant t < 2),\\ &= 0 \quad (t \geqslant 2).\end{aligned}$$

Clearly $\Psi(t) = 0$ for $t \geqslant 2$, and (1) is satisfied. Also

$$\lim_{\eta \to +0} \int_0^\infty e^{-\eta s} \{-s^2 \psi(s)\}\, ds = \lim_{\eta \to +0} \int_0^\infty e^{-\eta s} 2s\, \Psi(s)\, ds = 1,$$

so that the inequalities in question are not affected by subtracting from the general $\psi(t)$ q times the function just defined.

All we have now to prove is that

$$\overline{\lim_{\eta \to +0}} \int_0^\infty e^{-\eta s} \{-s^2 \psi(s)\}\, ds < 0$$

implies

(3) $$\overline{\lim_{\eta \to +0}} \int_0^\infty e^{-\eta s} 2s\, \Psi(s)\, ds \leqslant 0.$$

Write $$G(\eta) = \int_0^\infty e^{-\eta s} \psi(s)\, ds \quad (\eta > 0),$$

$$G(0) = \lim_{\eta \to +0} G(\eta) = 0,$$

by (1). We have shown that

(4) $$\int_0^\infty e^{-\eta s} \Psi(s)\, ds = \frac{G(\eta)}{\eta}.$$

Thus $$\int_0^\infty e^{-\eta s} s \Psi(s)\, ds = -\frac{d}{d\eta} \frac{G(\eta)}{\eta} = \frac{G(\eta)}{\eta^2} - \frac{G'(\eta)}{\eta}.$$

From (2) and (4) it is readily seen that

$$G'(\eta) = -\int_0^\infty e^{-\eta s} s \psi(s)\, ds, \quad G''(\eta) = \int_0^\infty e^{-\eta s} s^2 \psi(s)\, ds,$$

exist and are finite for $\eta > 0$. Moreover, $G(\eta)$ is continuous for $\eta \geqslant 0$, and so, by Taylor's theorem,

$$G(0) = G(\eta) - \eta G'(\eta) + \tfrac{1}{2}\eta^2 G''(\theta\eta) \quad (0 < \theta < 1),$$

$$\frac{G(\eta)}{\eta^2} - \frac{G'(\eta)}{\eta} = -\tfrac{1}{2} G''(\theta\eta).$$

But $-G''(\eta)$ is, by hypothesis, negative for all sufficiently small values

of η. Therefore

$$\overline{\lim_{\eta\to+0}}\int_0^\infty e^{-\eta s}s\Psi(s)\,ds = \overline{\lim_{\eta\to+0}}\left[\frac{G(\eta)}{\eta^2}-\frac{G'(\eta)}{\eta}\right]\leqslant 0,$$

which gives (3).

4. THEOREM I. *If $F(t)$ is integrable in every finite interval, absolutely integrable at $\pm\infty$, and*

$$F(t)\ \mathcal{V}\left[-\frac{a_s}{s^2},\ -\frac{b_s}{s^2}\right] \tag{1}$$

for almost all positive value of s, then

$$\begin{cases}\underline{\lim}_{t\to+0}\dfrac{F(x+t)-2F(x)+F(x-t)}{t^2}\leqslant\overline{\lim}_{\eta\to+0}\displaystyle\int_0^\infty e^{-\eta s}(a_s\cos sx+b_s\sin sx)\,ds,\\ \underline{\lim}_{\eta\to+0}\displaystyle\int_0^\infty e^{-\eta s}(a_s\cos sx+b_s\sin sx)\,ds\leqslant\overline{\lim}_{t\to+0}\dfrac{F(x+t)-2F(x)+F(x-t)}{t^2},\end{cases} \tag{2}$$

whenever x is a regular point of $F(t)$, i.e.

$$F(x+t)+F(x-t)\to 2F(x)$$

as $t\to 0$.

By (1) we mean

$$-\frac{a_s}{s^2}=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\lambda}^{\lambda}F(t)\cos st\,dt,\quad -\frac{b_s}{s^2}=\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\lambda}^{\lambda}F(t)\sin st\,dt.$$

Suppose first that $F(t)$ is continuous and has an absolutely integrable symmetrical second derivative $f(t)$, so that

$$F(t)=\int_0^t dv\int_0^v f(u)\,du+pt+q,$$

where p and q are constants. It may be shown* that

$$\int_{-\lambda}^{\lambda}F(t)\cos st\,dt$$
$$=\{F(\lambda)+F(-\lambda)\}\frac{\sin s\lambda}{s}+\{F'(\lambda)-F'(-\lambda)\}\frac{\cos s\lambda}{s^2}-\frac{1}{s^2}\int_{-\lambda}^{\lambda}f(t)\cos st\,dt.$$

If

$$F(t)\to 0,\quad F'(t)\to 0$$

as $t\to\pm\infty$, it immediately appears that

$$f(t)\ \mathcal{V}\ [a_s,\ b_s],$$

* For the calculations see **3**, 458.

for almost all positive values of s. If also $f(t)$ is continuous at x,

$$\lim_{\eta \to +0} \int_0^\infty e^{-\eta s}(a_s \cos sx + b_s \sin sx)\, ds = f(x),$$

and there is equality in the relations (2).

Any constant multiple of $e^{-|t|}$ satisfies the conditions just stated. By subtracting from $F(t)$ a function of this kind the general case may be reduced to the case $F(x) = 0$, which we now consider.

Write $$\phi(t) = F(x+t) + F(x-t),$$

so that $\phi(t)$ is continuous at the origin, absolutely integrable at infinity, and $\phi(0) = 0$. Let

$$\psi(s) = \frac{1}{\pi} \int_0^\infty \phi(t) \cos st\, dt,$$

$$\Psi(s) = \int_0^s \psi(u)\, du = \frac{1}{\pi} \int_0^\infty \phi(t) \frac{\sin st}{t}\, dt,$$

the change of the order of integration being justified by the methods of Lemma I. By the same methods it may be shown that, if η is positive,

$$\begin{aligned} \pi \int_0^\infty e^{-\eta s} \psi(s)\, ds &= \int_0^\infty ds\, e^{-\eta s} \int_0^\infty \phi(t) \cos st\, dt \\ &= \int_0^\infty dt\, \phi(t) \int_0^\infty e^{-\eta s} \cos st\, ds \\ &= \int_0^\infty \frac{\eta \phi(t)}{\eta^2 + t^2}\, dt \\ &= \left\{ \int_0^\delta + \int_\delta^c + \int_c^\infty \right\} \frac{\eta \phi(t)}{\eta^2 + t^2}\, dt, \end{aligned} \tag{3}$$

where δ and c are so chosen that $\phi(t)$ is absolutely integrable in $(0, \delta)$ and (c, ∞). The last two integrals on the right in (3) tend to zero with η. The first is in absolute value not greater than $\frac{1}{2} M\pi$, where M is the upper bound of $|\phi(t)|$ in $(0, \delta)$, and may be made as small as we please with δ. Thus

$$\int_0^\infty \psi(s)\, ds = 0 \quad (P),$$

and the conditions of Lemma II are satisfied. It follows that the Poisson values of

$$\int_0^\infty 2s \Psi(s)\, ds \tag{4}$$

lie between those of $\int_0^\infty -s^2\psi(s)\,ds.$

Now
$$-s^2\psi(s) = -\frac{s^2}{\pi}\int_0^\infty \phi(t)\cos st\,dt$$
$$= -\frac{s^2}{\pi}\int_{-\infty}^\infty \{F(x+t)+F(x-t)\}\cos st\,dt$$
$$= -\frac{s^2}{\pi}\int_{-\infty}^\infty F(t)\cos s(t-x)\,dt$$
$$= a_s\cos sx+b_s\sin sx,$$

for almost all positive values of s, so that the Poisson values of (4) lie between those of

$$\int_0^\infty (a_s\cos sx+b_s\sin sx)\,ds.$$

By Lemma I, the Poisson values of (4) lie between the lower and upper limits of $\phi(t)/t^2$ as $t\to+0$.

This will be found to yield the desired result.

5. We shall now restrict our integrals to the special Denjoy sense. It is known that, if $F(x)$ is continuous in (a, b) and

$$(1)\quad \underline{\lim_{t\to 0}}\frac{F(x+t)-2F(x)+F(x-t)}{t^2} \leqslant f(x) \leqslant \overline{\lim_{t\to 0}}\frac{F(x+t)-2F(x)+F(x-t)}{t^2},$$

where $f(x)$ is finite and has a finite integral in (a, b), then in (a, b)

$$(2)\quad F(x) = \int_a^x dv\int_a^v f(u)\,du+px+q$$

for suitable values of the constants p and q*. The function $f(x)$ may take infinite values in a set which does not contain a perfect set, provided that corresponding to each point of this set there can be found positive numbers h, k to make

$$\frac{F(x+h)-F(x)}{h} - \frac{F(x-k)-F(x)}{-k}$$

as small as we please.

6. LEMMA III. *If $F(x)$ is continuous for all finite values of x, and is such that*

$$\frac{1}{2u}\int_{-u}^u |F(x)|\,dx\to 0$$

* 4, 711–712.

as $u \to \infty$; *then, whenever* (5, 2) *is satisfied with* $f(x)$ *integrable in every finite interval, for all finite values of* s

$$(1) \qquad \begin{cases} \int_{-\infty}^{\infty} f(x) \cos sx\, dx = -s^2 \int_{-\infty}^{\infty} F(x) \cos sx\, dx & (C, 2) \\ \int_{-\infty}^{\infty} f(x) \sin sx\, dx = -s^2 \int_{-\infty}^{\infty} F(x) \sin sx\, dx & (C, 2) \end{cases}$$

subject to the existence $(C, 2)$ *of one of the integrals of each pair.*

For

$$\int_{-\nu}^{\nu} \{f(x) \cos sx + s^2 F(x) \cos sx\}\, dx$$
$$= \{F'(\nu) - F'(-\nu)\} \cos s\nu + \{F(\nu) + F(-\nu)\}\, s \sin s\nu,$$
$$\int_0^{\mu} [\{F'(\nu) - F'(-\nu)\} \cos s\nu + \{F(\nu) + F(-\nu)\}\, s \sin s\nu]\, d\nu$$
$$= \int_0^{\mu} \{F'(\nu) - F'(-\nu)\} \cos s\nu + o(\mu)$$
$$= \Big[\{F(\nu) + F(-\nu)\} \cos s\nu\Big]_0^{\mu} + \int_0^{\mu} \{F(\nu) + F(-\nu)\}\, s \sin s\nu\, d\nu + o(\mu)$$
$$= \{F(\mu) + F(-\mu)\} \cos s\mu + o(\mu) + o(\mu),$$
$$\int_0^{\lambda} \{F(\mu) + F(-\mu)\} \cos s\mu\, d\mu = o(\lambda).$$

These show that $f(x) \cos sx + s^2 F(x) \cos sx$

is annihilated by the operator

$$\lim_{\lambda \to \infty} \frac{2}{\lambda^2} \int_0^{\lambda} d\mu \int_0^{\mu} d\nu \int_{-\nu}^{\nu} dx,$$

and this gives the first of the required results. The proof of the other is exactly similar.

7. THEOREM II. *If the integral*

$$(1) \qquad \int_0^{\infty} (a_s \cos sx + b_s \sin sx)\, ds$$

is such that $-(a_s/s^2)$, $-(b_s/s^2)$ *are the Fourier constants of a continuous function* $F(x)$ *which is absolutely integrable in* $(-\infty, \infty)$, *and if it has Poisson values* $P_1(x)$, $P_2(x)$ *which are integrable in the special Denjoy sense in every finite interval, and are everywhere finite, then it is a Cesàro-Fourier integral.*

We say that (1) is the Cesàro-Fourier integral corresponding to $f(x)$ if

$$a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx\,dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx\,dx$$

in some positive Cesàro sense.

Theorem I shows that at each point one of the Poisson values is an intermediate Riemann value. Identify the function $f(x)$ of (5, 1) with this Poisson value, so that $f(x)$ is of the form

$$P_1(x)+\theta(x)\{P_2(x)-P_1(x)\},$$

where $\theta(x)$ takes the value 0 or 1. Clearly $f(x)$ is finite, and since

$$P_2(x) \geqslant P_1(x)$$

it is integrable in every finite interval. The conditions of Lemma III are satisfied and the integrals on the right of (6, 1) are convergent, whence it follows that (1) is the Cesàro-Fourier integral corresponding to $f(x)$.

In Theorem II the alteration of a_s, b_s at points of a set of values of s of measure zero can only mean that they cease to be the Cesàro-Fourier constants of $f(x)$ for those values. Thus, if (1) satisfies the conditions of Theorem II for almost all positive values of s, and if its Poisson values are almost everywhere zero, $f(x)$ is almost everywhere zero, and so are the constants a_s, b_s. From this we obtain immediately

THEOREM III. *Two trigonometrical integrals which satisfy the conditions of Theorem II for almost all positive values of s, and which are Poisson summable to the same values almost everywhere, have their constants equal almost everywhere.*

8. THEOREM IV. *If a_s, $b_s \to 0$ as $s \to \infty$, the results of Theorems I and II still hold, even when there are Poisson values infinite in some set, provided that this set does not contain a perfect set.*

By § 5 it will be sufficient to show that

$$\frac{F(x+2t)-2F(x)+F(x-2t)}{2t}$$

tends to zero with t.

Since $F(x)$ is continuous and absolutely integrable

$$F(x) = \int_0^{\infty}\left(-\frac{a_s}{s^2}\cos sx - \frac{b_s}{s^2}\sin sx\right)ds \qquad (P)$$

$$= \int_0^{\infty}\left(-\frac{a_s}{s^2}\cos sx - \frac{b_s}{s^2}\sin sx\right)ds,$$

the integral being convergent in view of the condition $a_s, b_s \to 0$. Thus

$$\frac{F(x+2t)-2F(x)+F(x-2t)}{2t} = 2t\int_0^\infty (a_s \cos sx + b_s \sin sx)\left(\frac{\sin st}{st}\right)^2 ds.$$

Write $$\int_0^u (a_s \cos sx + b_s \sin sx)\, ds = \Phi(u, x),$$

and let v be some positive constant. Now

$$\left|\frac{d}{ds}\left(\frac{\sin st}{st}\right)^2\right| \leqslant 2t,$$

so that, if M is the upper bound of $|\Phi(u, x)|$ in $(0, v)$,

$$2t\int_0^v (a_s \cos sx + b_s \sin sx)\left(\frac{\sin st}{st}\right)^2 ds$$

$$= 2t\left(\frac{\sin st}{st}\right)^2 \Phi(v, x) - 2t\int_0^v \Phi(s, x)\frac{d}{ds}\left(\frac{\sin st}{st}\right)^2 ds$$

which is not greater in absolute value than

$$2Mt + 4Mvt^2 \to 0$$

as $t \to 0$. If ϵ is some prescribed positive number we can so choose v that

$$|a_s|, \quad |b_s| < \frac{\epsilon}{2\pi} \quad (s > v).$$

Then

$$\left|2t\int_0^\infty (a_s \cos sx + b_s \sin sx)\left(\frac{\sin st}{st}\right)^2 ds\right| \leqslant \frac{2\epsilon t}{\pi}\int_0^\infty \left(\frac{\sin st}{st}\right)^2 ds = \epsilon.$$

Together with what has been proved, this shows that

$$\frac{F(x+2t)-2F(x)+F(x-2t)}{2t}$$

tends to zero with t, as required.

9. The following result has a formal connexion with the previous work, although it is not immediately concerned with the theory of uniqueness. It may be obtained as a corollary of Lemma I.

If $-(b_s/s)$, (a_s/s) are the Fourier constants of a function $f(x)$ which is integrable in the general Denjoy sense in every finite interval and absolutely integrable at $\pm\infty$, then the Poisson values of

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds$$

lie between the lower and upper limits of

$$\frac{f(x+t)-f(x-t)}{2t}$$

as $t \to +0$.

With the notation of Lemma I write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}\, t,$$

$$\Psi(s) = \tfrac{1}{2}\int_0^\infty \{f(x+t)-f(x-t)\} \sin st\, dt.$$

Then

$$2s\Psi(s) = \frac{s}{\pi}\int_{-\infty}^{\infty} f(x+t) \sin st\, dt$$

$$= \frac{s}{\pi}\int_{-\infty}^{\infty} f(t) \sin s(t-x)\, dt$$

$$= a_s \cos sx + b_s \sin sx.$$

This gives what was required.

It is of interest to observe that in this case the relation between the two sets of limits is simpler than the corresponding one which was obtained in Theorem I.

The analogous result for trigonometrical series is due to W. Grosz*.

References.

1. A. Rajchman and A. Zygmund, *Math. Zeitschrift*, 25 (1926), 261-273.
2. A. Zygmund, *Math. Zeitschrift*, 25 (1926), 274-290.
3. S. Pollard, *Proc. London Math. Soc.* (2), 25 (1926), 451-468.
4. C. J. de la Vallée Poussin, *Bull. de l'Acad. Royale de Belgique*, 1912, 702-718.
5. W. Grosz, *Wiener Sitzungsberichte*, 124 (1915), 1017-1037.

* 5, 1021.

THE COMPUTATION OF CHARACTERISTIC NUMBERS AND CHARACTERISTIC FUNCTIONS

By G. TEMPLE.

[Received 24 January, 1928.—Read 16 February, 1928.]

Summary.

The object of this paper is the computation of the characteristic numbers $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ and the characteristic functions

$$\phi_0(x), \ \phi_1(x), \ \ldots, \ \phi_n(x), \ \ldots$$

of a self-adjoint linear differential system of even order, consisting of a differential equation $L(y)+\lambda\sigma(x)y = 0$ $(a \leqslant x \leqslant b)$ and a set of linear homogeneous boundary conditions connecting the derivatives of $y(x)$ at $x = a$ and $x = b$. By the principles of the theory of integral equations, summarized in § 1, it is shown that an upper limit to the $(\kappa+1)$-th characteristic number λ_κ may be obtained from the inequality

$$\int_a^b \sigma(x) f(x) g(x)\, dx \geqslant \lambda_\kappa \int_a^b \sigma(x) g^2(x)\, dx \quad \text{(Theorem 2.1)},$$

where

$$g(x) = \int_a^b \sigma(s)\, G(x, s) f(s)\, ds$$

and $G(x, s)$ is the Green's function of the differential system, $f(x)$ being any integrable function orthogonal to the first κ characteristic functions (§ 2).

In § 3, some physical applications of this theorem are given, together with a number of methods of obtaining lower limits to the first characteristic number.

§ 4 is devoted to the sequence of functions $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ obtained from any integrable function $f_0(x)$ by means of the recurrence relation

$$f_n(x) = \int_a^b \sigma(s)\, G(x, s) f_{n-1}(s)\, ds \quad (n = 1, 2, \ldots).$$

The number μ_n being defined by the relation $\mu_n = (n-1)/(n)$, where

$$(n) = \int_a^b \sigma(x) f_0(x) f_n(x)\, dx,$$

the sequence (μ_n) is monotonic decreasing and converges to the limit λ_κ if $f_0(x)$ is orthogonal to the first κ characteristic functions. The sequence of functions $(\mu_n)^n f_n(x)$ converges uniformly to a multiple of the $(\kappa+1)$-th characteristic function $\phi_\kappa(x)$. The difference $\mu_n - \lambda_\kappa$ is less than

$$\frac{\mu_{n-1} - \mu_n}{\lambda_\kappa \lambda_{\kappa+1} \mu_n^{-2} - 1}$$

if this is positive.

In § 5 it is shown that, where the Green's function $G(x, s)$ is known, together with the first κ characteristic numbers, a generating function can always be constructed which is orthogonal to the first κ characteristic functions, and hence the first few characteristic numbers can be obtained in succession. A simple condition is given (Theorem 5 . 3), the fulfilment of which ensures that

$$\mu_n - \lambda_0 \leqslant \mu_{n-1} - \mu_n.$$

Finally the methods given above are applied to the calculation of the first zeros of Bessel's functions of fractional order.

1. *Definitions and principles.*

Every self-adjoint differential expression of even order $2N$ may be written in the form

$$L(y) \equiv \frac{d^N}{dx^N}\left\{P_N(x)\frac{d^N y}{dx^N}\right\} + \frac{d^{N-1}}{dx^{N-1}}\left\{P_{N-1}(x)\frac{d^{N-1}y}{dx^{N-1}}\right\} + \ldots + P_0(x)\, y, \quad (1\,.\,1)$$

which involves $N+1$ functions $P_n(x)$ $(n = 0, 1, 2, \ldots, N)$. It is assumed that there exists a real finite interval $a \leqslant x \leqslant b$, throughout which the functions $P_n(x)$ $(n = 0, 1, 2, \ldots, N)$ possess continuous derivatives of all orders up to and including the n-th. With this differential expression we associate a set of $2N$ homogeneous linear boundary conditions, which, together with the differential expression, form a self-adjoint system. This set of boundary conditions will be referred to as the conditions "M"*.

* It is explained later in this section that we shall suppose that the boundary conditions "M" are sufficient to ensure that the characteristic numbers are all positive and simple.

Henceforward differentiation with respect to the independent variable x will be indicated by accents.

If the boundary conditions "M" are incompatible with the differential equation $L(y) = 0$, then there will exist a Green's function $G(x, s)$ possessing the following properties :—

(1) $G(x, s)$ is a symmetric function of x and s ;

(2) $G(x, s)$ is finite and continuous, together with its first $2N-2$ derivatives with respect to x, throughout the closed interval $a \leqslant x \leqslant b$;

(3) $G(x, s)$ possesses finite and continuous derivatives of orders $2N-1$ and $2N$ throughout each of the open intervals

$$a \leqslant x < s < b, \quad a < s < x \leqslant b ;$$

(4) In these intervals $G(x, s)$ satisfies the differential equation

$$L(G) = 0 ;$$

(5) $G(x, s)$ also satisfies the boundary conditions "M" ;

(6) the $(2N-1)$-th derivative of $G(x, s)$, which we indicate by $H(x)$, satisfies the condition

$$\lim_{\epsilon \to 0} \{H(s-\epsilon) - H(s+\epsilon)\} = \{P_N(s)\}^{-1}.$$

We shall assume, moreover, that the functions $P_n(x)$ $(n=0, 1, 2, \ldots, N)$, occurring in the expression $L(y)$, are such that the Green's function $G(x, s)$ is a function of positive, definite type.

Let $\sigma(x)$ denote a function which is everywhere continuous and nowhere negative in the interval $a < x < b$, and let $f(x)$ denote a function which may have any finite number of discontinuities, but which is integrable together with its square in the interval $a \leqslant x \leqslant b$. [Such functions as $f(x)$ we shall briefly designate as "integrable functions".] Then the function $g(x)$, defined by the equation

$$g(x) = A(f) \equiv \int_a^b \sigma(s)\, G(x, s)\, f(s)\, ds,$$

possesses the following properties :—

(1) $g(x)$ is finite and continuous and possesses derivatives up to the $2N$-th order in the interval $a < x < b$;

(2) at every point in this interval $g(x)$ satisfies the equation

$$L(g) + \sigma(x) f(x) = 0 ;$$

(3) $g(x)$ satisfies the boundary conditions "M";

(4) $g(x)$ is not identically zero unless $f(x)$ is also identically zero.

Let $K(x, s)$ denote the symmetric function $G(x, s)\ \{\sigma(x)\,\sigma(s)\}^{\frac{1}{2}}$. It follows from the last-mentioned property of the integral $A(f)$ that there does not exist any "integrable function" $f(x)$ such that

$$\int_a^b K(x, s)\, f(s)\, ds = 0$$

for all values of x in the interval $a \leqslant x \leqslant b$, and hence that the function $K(x, s)$ is "closed". Since $\sigma(x)$ is positive, this function is also of "positive definite" type.

From this it follows that the values of λ, for which the homogeneous linear equation

$$\eta(x) = \lambda \int_a^b K(x, s)\, \eta(s)\, ds \tag{1.2}$$

possesses a real continuous solution other than $\eta(x) = 0$, form an infinite series of real positive numbers; and we shall assume that the boundary conditions "M" are such as to ensure that the solutions of the integral equation which correspond to any one of these "characteristic numbers" are all numerical multiples of the same standard solution or "characteristic function" $\psi(x)$, the specification of which is completed by the condition of normality

$$\int_a^b \psi^2(x)\, dx = 1.$$

The only values of λ for which the differential equation

$$L(y) + \lambda\sigma(x)\, y(x) = 0 \tag{1.3}$$

possesses a solution, valid in the interval $a \leqslant x \leqslant b$ and satisfying the boundary conditions "M", are the characteristic numbers of the integral equation (1.2). The only solutions of the differential equation (1.3) when λ is a characteristic number are numerical multiples of the standard solution

$$\phi(x) = \psi(x)\{\sigma(x)\}^{-\frac{1}{2}},$$

where $\psi(x)$ denotes the characteristic function associated with the characteristic number λ. If $\phi_m(x)$ and $\phi_n(x)$ are the standard solutions of the differential equation (1.3) associated with two characteristic numbers, then the integral

$$\int_a^b \sigma(x)\, \phi_m(x)\, \phi_n(x)\, dx$$

is equal to unity or zero according as the numbers are the same or different.

Let $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\kappa, \ldots$ denote the series of characteristic numbers arranged in ascending order of magnitude, and let

$$\phi_0(x), \ \phi_1(x), \ \phi_2(x), \ \ldots, \ \phi_\kappa(x), \ \ldots$$

denote the associated standard solutions of the equation (1 . 3). Let $f(x)$ and $g(x)$ denote any pair of "integrable functions". Let a_n and b_n denote the two series of numbers defined by the equations

$$a_n = \int_a^b \sigma(x) f(x) \phi_n(x) \, dx, \qquad b_n = \int_a^b \sigma(x) g(x) \phi_n(x) \, dx.$$

These numbers will be referred to as the "Fourier constants" of $f(x)$ and $g(x)$. Then, since the nucleus $K(x, s)$ is "closed", the series

$$\sum_{n=0}' a_n b_n$$

is absolutely convergent, and its sum is equal to the integral

$$\int_a^b \sigma(x) f(x) g(x) \, dx \quad \text{(Riesz' theorem).}$$

In the succeeding paragraphs no ambiguity will be caused if we transfer the title of "characteristic functions" to the standard solutions $\phi_0(x), \phi_1(x), \ldots, \phi_\kappa(x), \ldots$ of the differential equation (1 . 3).

(It will be noted that this method cannot be applied to differential equations whose order is odd. In fact, Birkhoff* has indicated that such equations possess, in general, only a finite set of characteristic numbers.)

2. *Upper limits to the characteristic numbers.*

We shall adopt the following notation for two types of definite integrals which frequently occur in our investigations :

$$A(f) = \int_a^b \sigma(s) G(x, s) f(s) \, ds$$

and

$$(f, g) = \int_a^b \sigma(s) f(s) g(s) \, ds.$$

* *Proc. Fifth International Congress of Mathematicians*, 1 (Cambridge 1912), 180.

THEOREM 2.1. *If $f(x)$ is any integrable function which is orthogonal to the first κ characteristic functions*

$$\phi_0(x),\ \phi_1(x),\ \ldots,\ \phi_{\kappa-1}(x),$$

while $g(x) = A(f)$, then the number μ, defined by the equation

$$(f, g) = \mu(g, g),$$

cannot be less than the $(\kappa+1)$-th characteristic number.

Let $R(x)$ be the bilinear concomitant of the functions $\phi_m(x)$ and $g(x)$, formed with regard to the self-adjoint expression $L(y)$ (Equation 1.1); then

$$R(b)-R(a) = \int_a^b \{\phi_m(x)\, L(g)-g(x)\, L(\phi_m)\}\, dx$$

$$= \lambda_m(\phi_m, g)-(\phi_m, f).$$

In virtue of the boundary conditions "M" satisfied by the functions $\phi_m(x)$ and $g(x)$, the left-hand side of this equation is zero for the function $R(x)$ considered. Therefore

$$(\phi_m, f) = \lambda_m(\phi_m, g).$$

Hence, if $a_m \equiv (\phi_m, g)$ is the $(m+1)$-th Fourier constant of the function $g(x)$, the corresponding Fourier constant of $f(x)$ will be $\lambda_m a_m$. Moreover, since $f(x)$ is orthogonal to the first κ characteristic functions, the first κ Fourier constants of $f(x)$ vanish. Therefore

$$a_0 = a_1 = \ldots = a_{\kappa-1} = 0.$$

It now follows from Riesz' theorem that the series whose general terms are respectively $(\lambda_m a_m^2)$ and (a_m^2), are both absolutely convergent and have sums equal to the integrals (f, g) and (g, g). Accordingly, the expression

$$(f, g)-\lambda_\kappa(g, g)$$

may be written in the form of the absolutely convergent series

$$\sum_{m=\kappa}^{\infty} (\lambda_m-\lambda_\kappa)\, a_m^2.$$

The first term of this series is zero, and, since

$$\lambda_m > \lambda_\kappa \quad \text{when} \quad m > \kappa,$$

none of the subsequent terms can be negative. Hence

$$(f, g) \geqslant \lambda_\kappa(g, g).$$

Therefore the number μ, defined by the equation

$$(f, g) = \mu(g, g),$$

cannot be less than λ_κ.

THEOREM 2.2. *If $f(x)$ is any integrable function which is orthogonal to the first κ characteristic functions, while $g(x) = A(f)$ and $h(x) = A(g)$, then the number ν, defined by the equation*

$$\{(f, g)(g, h)-(g, g)^2\} = \nu\{(g, g)(h, h)-(g, h)^2\},$$

cannot be less than the product $\lambda_\kappa\lambda_{\kappa+1}$.

As before, it may be shown that, if (a_m) is the $(m+1)$-th Fourier constant of $h(x)$, then the corresponding Fourier constants of $g(x)$ and $f(x)$ are $(\lambda_m a_m)$ and $(\lambda_m^2 a_m)$ respectively. Moreover, the first κ of these constants vanish, so that

$$a_0 = a_1 = \ldots = a_{\kappa-1} = 0.$$

It follows from Riesz' theorem that the five series

$$\sum_{m=\kappa}^{\infty} \lambda_m^n a_m^2 \quad (n = 0, 1, 2, 3, 4)$$

are all absolutely convergent, and have the sums

(h, h) when $n = 0$, (g, h) when $n = 1$, $(g, g) = (f, h)$ when $n = 2$,

(f, g) when $n = 3$, and (f, f) when $n = 4$.

The double series formed by taking the product of any pair of these series will also converge absolutely and may therefore have its terms arranged in any order. Accordingly, the double series

$$\sum_{m, n=\kappa}^{\infty} (\lambda_n^3\lambda_m-\lambda_n^2\lambda_m^2)\, a_m^2 a_n^2 \quad \text{and} \quad \sum_{m, n=\kappa}^{\infty} (\lambda_n^2-\lambda_m\lambda_n)\, a_m^2 a_n^2$$

are convergent and have the sums

$$I = (f, g)(g, h)-(g, g)^2 \quad \text{and} \quad J = (g, g)(h, h)-(g, h)^2$$

respectively.

Hence the expression $I-\lambda_\kappa\lambda_{\kappa+1}J$ may be written in the form of the absolutely convergent series

$$\sum_{m, n=\kappa}^{\infty} (\lambda_m\lambda_n-\lambda_\kappa\lambda_{\kappa+1})(\lambda_n^2-\lambda_n\lambda_m)\, a_m^2 a_n^2$$

or

$$\tfrac{1}{2} \sum_{m, n=\kappa}^{\infty} (\lambda_m\lambda_n-\lambda_\kappa\lambda_{\kappa+1})(\lambda_n-\lambda_m)^2\, a_m^2 a_n^2.$$

In the last series those terms vanish in which

$$m = n;$$

or $$n = \kappa, \quad m = \kappa+1;$$

or $$m = \kappa, \quad n = \kappa+1;$$

and none of the other terms are negative. Therefore

$$I \geqslant \lambda_\kappa \lambda_{\kappa+1} J,$$

and, as before, the number ν cannot be less than $\lambda_\kappa \lambda_{\kappa+1}$.

The method of proof adopted above is suggested by Richardson's analysis*. A special case of Theorem 2.2 ($\kappa = 0$) has been stated by van den Dungen†.

Theorem 2.1 is an immediate result of Hilbert's researches on the relation between integral equations and the calculus of variations. A more general form of the same theorem, established by other methods, is given by Courant and Hilbert‡.

3. *Physical applications. Rayleigh's method.*

The method of the preceding section receives certain notable simplifications when applied to such problems as the calculation of the frequency of an oscillating system or the determination of the stability of an elastic system. The differential equations involved are usually of the second or fourth order, so that our fundamental equation (1.1) may be written

$$\{P_2(x)\,y''\}'' + \{P_1(x)\,y'\}' + P_0(x)\,y(x) = L(y) = -\lambda\sigma(x)\,y(x). \quad (3.1)$$

Again, the boundary conditions "M" are usually of a peculiarly simple type, which may be defined precisely as follows :—

With any pair of functions $u(x)$ and $v(x)$, which possess derivations up to the fourth order in the integral $a < x < b$, we form the expression

$$S(u, v) = \{P_2(x)\,u''\}'\,v - P_2(x)\,u''\,v' + P_1(x)\,u'\,v.$$

The bilinear concomitant of these functions may then be written as

$$R(u, v) = S(u, v) - S(v, u)$$

---

* *Math. Annalen*, 68 (1910), 279.

† *Comptes rendus*, 176 (1923), 1864; 184 (1927), 1413.

‡ R. Courant u. D. Hilbert, *Methoden der mathematischen Physik* (1924), 116.

and Lagrange's identity will be

$$\{R(u, v)\}' = vL(u) - uL(v).$$

The special type of boundary conditions with which we are now concerned is such that the expressions $S(u, u)$ and $R(u, v)$, formed with any two solutions of the equation (3.1), vanish at both ends of the interval $x = a$ and $x = b$. This type of boundary conditions will be referred to as the conditions "N".

An immediate deduction is that any function $u(x)$ possessing derivations up to the fourth order and fulfilling the conditions "N" will satisfy the equation

$$\int_a^b u(x) L(u)\, dx = \int_a^b \{P_2(x) u''^2(x) - P_1(x) u'^2(x) + P_0(x) u^2(x)\}\, dx.$$

This integral, with its sign reversed, we shall denote by $V(u)$, while $T(u)$ will be used to denote the integral $\int_a^b \sigma(x) u^2(x)\, dx$.

Theorem (2.1) may now be restated as follows :—

THEOREM 3.1. *If $g(x)$ is any function which possesses continuous derivatives up to the fourth order in the interval $a < x < b$, and which satisfies the boundary conditions* "N", *then the number μ, defined by the equation*

$$V(g) = \mu T(g),$$

cannot be less than the smallest characteristic number λ_0.

This theorem furnished a proof of Rayleigh's method* of determining an upper limit to the frequency of the gravest mode of vibration of an oscillating system. If the general equation of motion of the system is

$$L_x(u) \equiv \frac{\partial^2}{\partial x^2}\left\{P_2(x) \frac{\partial^2 u}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{P_1(x) \frac{\partial u}{\partial x}\right\} + P_0(x)\, u(x, t) = \sigma(x) \frac{\partial^2 u}{\partial t^2},$$

the frequencies of the normal modes of vibration, obtained by writing

$$u(x, t) = y(x) \sin 2\pi q t,$$

are $$q_0, q_1, q_2, \ldots, q_n, \ldots,$$

where $$q_n = \frac{\sqrt{\lambda_n}}{2\pi}.$$

* *The theory of sound* (1st edition, 1877, *passim*).

If the system is constrained to vibrate according to the law

$$u(x, t) = g(x) \sin 2\pi qt,$$

the quantities $\frac{1}{4}V(g)$ and $\pi q^2 T(g)$ are respectively the mean potential and kinetic energies of the system. The value of q obtained by equating these quantities is $(2\pi)^{-1}\{V(g)/T(g)\}^{\frac{1}{2}}$, a number which Theorem 3.1 shows to be in excess of the frequency of the gravest mode, namely q_0.

The same theorem also establishes the energy test for the stability of a one-dimensional elastic structure. The equation which determines the deformation $y(x)$ of the structure under the action of the external forces is usually of the form

$$-L(y) = \lambda\sigma(x)\, y(x),$$

wherein the parameter λ depends upon the critical end-thrust, whirling speed, &c., producing instability. Let the structure pass from its state of equilibrium to the state of strain determined by the equation

$$y = g(x);$$

then the strain-potential energy acquired by the structure will be $\frac{1}{2}V(g)$, and the additional work done by the external forces will be $\frac{1}{2}\lambda T(g)$. By equating these two quantities we obtain a value for λ, namely $V(g)/T(g)$, which is necessarily in excess of the true critical value λ_0.

Method of decomposition of the function $\sigma(x)$.

It is well known that Rayleigh's method of estimating the frequency of the gravest mode of vibration of an oscillating system and the energy method of determining the stability of an elastic system furnish close superior limits to the quantities to be computed. Methods of obtaining close inferior limits are provided by the following theorems.

Suppose that the function $\sigma(x)$ may be expressed as the sum of ν functions, $\sigma_1(x)$, $\sigma_2(x)$, ..., $\sigma_\nu(x)$, each of which is everywhere continuous and nowhere negative in the interval $a < x < b$, while it is possible to compute the smallest characteristic numbers

$$\lambda_0(1),\ \lambda_0(2),\ \ldots,\ \lambda_0(\nu),$$

for each of the auxiliary equations

$$L(y) + \lambda\sigma_n(x)\, y(x) = 0 \quad (n = 1, 2, \ldots, \nu),$$

and the boundary conditions "M". Then a lower limit to λ_0 is provided by the inequality

THEOREM 3.2. $$\lambda_0^{-1} \leqslant \sum_{n=1}^{\nu} \lambda_0^{-1}(n).$$

To prove this theorem we note that, by Theorem 2.1, any function $g(x)$ which possesses continuous derivatives up to the $2N$-th order in the interval $a < x < b$, and which satisfies the boundary conditions "M", may be substituted in the inequality

$$-\int_a^b g(x)\, L(g)\, dx \geqslant \lambda_0(n) \int_a^b \sigma_n(x)\, g^2(x)\, dx. \tag{3.2}$$

Now write for $g(x)$ the first characteristic function of the equation

$$L(y) + \lambda\sigma(x)\, y(x) = 0,$$

i.e. $\phi_0(x)$. We thus obtain

$$\lambda_0^{-1}(n) \geqslant \lambda_0^{-1} \int_a^b \sigma_n(x)\, \phi_0^2(x)\, dx.$$

Now set n equal to 1, 2, ..., ν, and sum the ν inequalities obtained. Then

$$\sum_{n=1}^{\nu} \lambda_0^{-1}(n) \geqslant \lambda_0^{-1} \int_a^b \sigma(x)\, \phi_0^2(x)\, dx = \lambda_0^{-1},$$

whence the theorem follows.

An immediate generalization of this theorem which needs only to be stated is

THEOREM 3.3. *If there exists a function $\sigma(x, s)$, everywhere continuous with regard to x and nowhere negative in the interval $a < x < b$, and such that the integral*

$$\int_A^B \sigma(x, s)\, ds \quad (a \leqslant x \leqslant b)$$

exists and is equal to $\sigma(x)$, then

$$\lambda_0^{-1} \leqslant \int_A^B \lambda_0^{-1}(s)\, ds,$$

where $\lambda_0(s)$ is the smallest characteristic number of the equation

$$L(y) + \lambda\sigma(x, s)\, y(x) = 0.$$

A development of the preceding results, which is of some importance in physical applications, is obtained by considering the equation

$$L(y) + \Sigma(x)\, y(x) = 0, \tag{3.3}$$

where $$\Sigma(x) = \lambda(1)\sigma_1(x)+\lambda(2)\sigma_2(x)+\ldots+\lambda(\nu)\sigma_\nu(x), \qquad (3.4)$$

and the functions $\sigma_1(x)$, $\sigma_2(x)$, ..., $\sigma_\nu(x)$ are each everywhere continuous and nowhere negative in the interval $a < x < b$.

Let Λ_1, Λ_2, ..., Λ_ν be any set of ν positive numbers which, when substituted for $\lambda(1)$, $\lambda(2)$, ..., $\lambda(\nu)$ in the expression for $\Sigma(x)$ (3.4), make the differential equation (3.3) compatible with the boundary conditions "M". Such a set of numbers may be called a "characteristic set". Let $\phi(x)$ denote any solution of the equation

$$L(y)+\{\Lambda_1\sigma_1(x)+\Lambda_2\sigma_2(x)+\ldots+\Lambda_\nu\sigma_\nu(x)\}y(x) = 0,$$

which fulfils the boundary conditions "M".

THEOREM 3.4. *If $\lambda_0(n)$ is the smallest characteristic number of the equation*

$$L(y)+\lambda(n)\sigma_n(x)\,y(x) = 0 \quad (n = 1, 2, \ldots, \nu),$$

and Λ_1, Λ_2, ..., Λ_ν is a set of characteristic numbers for the equation (3.3), *then*

$$\sum_{n=1}^{\nu} \Lambda_n\lambda_0^{-1}(n) \geqslant 1.$$

By substituting $\phi(x)$ for $g(x)$ in the inequalities (3.2) we find that

$$-\Lambda_n\lambda_0^{-1}(n)\int_a^b \phi(x)\,L(\phi)\,dx \geqslant \Lambda_n\int_a^b \sigma_n(x)\,\phi^2(x)\,dx.$$

Let $n = 1, 2, \ldots, \nu$, and sum the ν inequalities obtained. Then

$$-\left\{\sum_{n=1}^{\nu}\Lambda_n\lambda_0^{-1}(n)\right\}\int_a^b \phi(x)\,L(\phi)\,dx \geqslant \int_a^b\left\{\sum_{n=1}^{\nu}\Lambda_n\sigma_n(x)\right\}\phi^2(x)\,dx,$$

and therefore $$\sum_{n=1}^{\nu}\Lambda_n\lambda_0^{-1}(n) \geqslant 1.$$

This principle may be illustrated from Prescott's investigations on the stability of a cantilever*. Suppose that the cantilever (supported in some way the details of which are irrelevant to our present considerations) is acted upon by a concentrated load P and a distributed load W. Let P_0 be the critical value of P when W is zero, and let W_0 be the critical value of W when P is zero. Then, when both loads operate, inferior limits to the values of P and W which just produce instability

* *Phil. Mag.* (6), 39 (1920), 194–219.

may be deduced from the inequality

$$1 \leqslant P/P_0 + W/W_0.$$

(*Cf.* equations 140, 156, and 173 of Prescott's paper.)

A similar line of argument may be developed in connexion with any elastic system in which the work done by the external forces is naturally divisible into two or more parts which may separately vanish. Consider, for example, Southwell's researches* on the stability of a boiler flue subjected to an end-thrust T and to a difference of hydrostatic pressure P between the outside and inside. Let P_0 be the critical pressure difference in the absence of the end-thrust and let T_0 be the critical end-thrust in the absence of pressure difference. Then, if either P or T is specified, an inferior limit to the critical value of the other may be obtained from the relation

$$1 \leqslant P/P_0 + T/T_0.$$

(*Cf.* equation 108 of Southwell's second paper†.)

Morrow's method.

A method of obtaining simultaneously both upper and lower limits to the smallest characteristic number has been given by Morrow‡. This method depends upon the following comparison theorem.

THEOREM 3.5. *If* Λ_1 *and* Λ_2 *are the smallest characteristic numbers of the equations*

$$L(y) + \lambda\sigma_1(x)y = 0, \quad L(y) + \lambda\sigma_2(x)y = 0,$$

for the boundary conditions "M", and if the functions $\sigma_1(x)$, $\sigma_2(x)$ *satisfy the inequality*

$$0 \leqslant \sigma_1(x) \leqslant \sigma_2(x)$$

throughout the interval $a \leqslant x \leqslant b$, *then*

$$\Lambda_2 \leqslant \Lambda_1.$$

Since any function $g(x)$ fulfilling the conditions of Theorem 3.2 satisfies the inequality

$$-\int_a^b g(x)\,L(g)\,dx \geqslant \Lambda_2 \int_a^b \sigma_2(x)\,g^2(x)\,dx,$$

we may replace $g(x)$ by the first characteristic function of the equation

* *Phil. Mag.* (6), 25 (1913), 687–695.

† *Phil. Trans. Royal Soc.*, A 213 (1914), 187–244.

‡ *Proc. Phys. Soc.*, 19 (1905), 588-602; 20 (1907), 107–127, 223–234; 21 (1909), 125–127.

involving $\sigma_1(x)$, say $\psi(x)$, thus obtaining the inequalities

$$\Lambda_1 = -\int_a^b \psi(x)\, L(\psi)\, dx \geqslant \Lambda_2 \int_a^b \sigma_2(x)\, \psi^2(x)\, dx$$

$$\geqslant \Lambda_2 \int_a^b \sigma_1(x)\, \psi^2(x)\, dx$$

$$= \Lambda_2,$$

which establish the theorem.

The preceding theorem leads to an extremely simple method of determining upper and lower limits to the value of the smallest characteristic number. Let $\theta(x)$ denote any function of x which satisfies the following conditions :—

(1) $\theta(x)$ possesses continuous derivatives up to the $2N$-th order throughout the interval $a \leqslant x \leqslant b$;

(2) $\theta(x)$ satisfies the boundary conditions "M" ;

(3) $\theta(x)$ is never negative at any point in the interval $a \leqslant x \leqslant b$.

Then, provided that $\sigma(x)$ is never zero, the function

$$F(x) = -L(\theta)/\{\theta(x)\,\sigma(x)\}$$

will be continuous in the interval $a \leqslant x \leqslant b$, and will possess an upper limit P and a lower limit Q. We add the further assumptions that Q is positive, without which the theorem to be proved becomes self-evident and trivial.

THEOREM 3 . 6. *The smallest characteristic number λ_0 of the equation*

$$L(y)+\lambda\sigma(x)\,y = 0$$

satisfies the inequalities $\quad Q \leqslant \lambda_0 \leqslant P.$

The proof of this theorem, which is stated in this section in view of its physical applications, depends upon the known result that the first characteristic function, and only the first characteristic function, is never negative in the interval $a \leqslant x \leqslant b$.

Consider now the differential equations

$$L(y)+\lambda\sigma_1(x)\,y(x) = 0 \quad \text{and} \quad L(y)+\lambda\sigma_2(x)\,y(x) = 0, \qquad (3\,.\,5)$$

where $$Q\sigma_1(x) = F(x)\,\sigma(x) = P\sigma_2(x).$$

From the hypotheses made concerning the functions $\theta(x)$ and $F(x)$, it follows that Q and P are characteristic numbers of these equations, while

the corresponding characteristic functions are numerical multiples of $\theta(x)$. But, since $\theta(x)$ is never negative in the interval $a < x < b$, it follows from the result quoted above that Q and P are the smallest characteristic numbers of the equations (3 . 5). But

$$\sigma_2(x) \leqslant \sigma(x) \leqslant \sigma_1(x).$$

Therefore, from Theorem 3 . 4, it follows that $Q \leqslant \lambda_0 \leqslant P$.

This theorem provides the explanation of the results obtained by Morrow (*loc. cit.*), whose approximate method of computing frequencies sometimes yielded a result in excess and sometimes a result in defect of the true value.

Southwell's method.

This is a method analogous to the method of decomposing the function $\sigma(x)$, which, indeed, was suggested by Southwell's investigations.

Suppose that the operator $L(y)$ may be expressed as the sum of ν operators $L_1(y)$, $L_2(y)$, ..., $L_\nu(y)$, each of which is self-adjoint and observes all the restrictions laid upon $L(y)$ in § 1.

THEOREM 3 . 7. *If Λ_n is the smallest characteristic number of the equation*

$$L_n(y) + \lambda\sigma(x)\, y(x) = 0,$$

then

$$\lambda_0 \geqslant \Lambda_1 + \Lambda_2 + \ldots + \Lambda_\nu.$$

Substitute $\phi_0(x)$ for $g(x)$ in the ν inequalities

$$-\int_a^b g(x)\, L_n(g)\, dx \geqslant \Lambda_n \int_a^b \sigma(x)\, g^2(x)\, dx \quad (n = 1, 2, \ldots, \nu),$$

and sum from $n = 1$ to $n = \nu$. We obtain

$$\lambda_0 = -\int_a^b \phi_0(x)\, L(\phi)\, dx \geqslant (\Lambda_1 + \Lambda_2 + \ldots + \Lambda_\nu) \int_a^b \sigma(x)\, \phi_0^2(x)\, dx$$

$$= (\Lambda_1 + \Lambda_2 + \ldots + \Lambda_\nu).$$

Lamb and Southwell* have applied this method to determine lower limits to the frequency of the gravest mode of vibration of certain elastic systems. Certain questions regarding stability can be treated in a similar manner. For example, suppose that an external force P acting upon an elastic structure produces a deformation the potential energy of which

* *Proc. Royal Soc.* (A), 99 (1921), 272–280, and (A), 101 (1922), 133–153.

may be, in general, expressed as the sum of a number of terms

$$A_1V_1+A_2V_2+\ldots+A_\nu V_\nu,$$

each of which is the product of some elastic constant of the structure (A) and a factor (V) determined by the type of deformation produced. If P_n is the critical value of P producing instability when the structure is so constrained that only the n-th type of deformation is produced, a lower limit to the critical value when the structure is unconstrained is provided by the relation

$$P_1+P_2+\ldots+P_\nu \leqslant P.$$

4. *Successive approximations to the characteristic numbers and functions.*

From any "integrable function" $f_0(x)$ we may derive an infinite sequence of functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, ... by means of the recurrence relation

$$f_{n+1}(x) = A(f_n) \quad (n = 0, 1, 2, \ldots). \qquad (4.1)$$

If $f_0(x)$ is orthogonal to the first κ characteristic functions, so also is each function belonging to this sequence. Moreover, each function, excluding $f_0(x)$, satisfies the boundary conditions "M". Hence, if $R(x)$ is the bilinear concomitant of the functions $f_{m+1}(x)$ and $f_{n+1}(x)$, then

$$\begin{aligned} R'(x) &= f_{n+1}(x)L(f_{m+1})-f_{m+1}(x)L(f_{n+1}) \\ &= \sigma(x)\{f_{m+1}(x)f_n(x)-f_m(x)f_{n+1}(x)\}. \end{aligned}$$

Hence we find, as in Theorem 2.1, that

$$(f_m, f_{n+1}) = (f_{m+1}, f_n) \quad (m, n = 0, 1, 2, \ldots).$$

By means of this reduction formula every integral of the type (f_m, f_{n+1}) can be reduced either to the form

$$(f_p, f_p) \quad \text{if } m+n+1 \text{ is even and equal to } 2p,$$

or to the form

$$(f_p, f_{p+1}) \quad \text{if } m+n \text{ is even and equal to } 2p.$$

In view of this result it is permissible to adopt the notation

$$(m+n+1) = (f_m, f_{n+1}) \quad (m, n = 0, 1, 2, \ldots)$$

and

$$[n] = (n)(n+2)-(n+1)^2.$$

By writing $g(x) = f_m(x)$ in Theorems 2 . 1 and 2 . 2 we obtain at once the following result on the supposition that $f_0(x)$, the generating function of the sequence $f_1(x)$, $f_2(x)$, ..., is orthogonal to the first κ characteristic functions.

THEOREM 4 . 1. *The numbers μ_{2m} and ν_{2m}, defined by the equations $(2m-1) = (2m)\mu_{2m}$, and $[2m-1] = [2m]\nu_{2m}$ $(m = 1, 2, \ldots)$ cannot be less than λ_κ and $\lambda_\kappa\lambda_{\kappa+1}$ respectively.*

Again, by expanding the integrals of

$$\sigma(x)\sigma(y)\begin{vmatrix} f_n(x) & f_n(y) \\ f_{n+1}(x) & f_{n+1}(y) \end{vmatrix}^2 \quad \text{and} \quad \sigma(x)\sigma(y)\sigma(z)\begin{vmatrix} f_n(x) & f_n(y) & f_n(z) \\ f_{n+1}(x) & f_{n+1}(y) & f_{n+1}(z) \\ f_{n+2}(x) & f_{n+2}(y) & f_{n+2}(z) \end{vmatrix}^2$$

taken over the regions $a \leqslant x,\ y \leqslant b$ and $a \leqslant x,\ y,\ z \leqslant b$, we obtain the inequalities

$$(2m)(2m+2) \geqslant (2m+1)^2$$

and

$$[2m][2m+2] \geqslant [2m+1]^2.$$

Hence we may deduce

THEOREM 4 . 2. *The numbers μ_{2m+1} and ν_{2m+1}, defined by the equations*

$$(2m) = (2m+1)\mu_{2m+1}, \quad [2m] = [2m+1]\nu_{2m+1} \quad (m = 0, 1, 2, \ldots),$$

satisfy the inequalities $\mu_{2m+1} \geqslant \mu_{2m+2}$, $\nu_{2m+1} \geqslant \nu_{2m+2}$, and hence cannot be less than λ_κ and $\lambda_\kappa\lambda_{\kappa+1}$ respectively.

In discussing the convergence of the sequence (μ_n) it is convenient to introduce an associated sequence (R_n) defined by the equation

$$R_n = \frac{\mu_{n-1} - \mu_n}{\mu_n - \mu_{n+1}} \quad (n = 1, 2, \ldots).$$

From the definitions of μ_n and ν_n it follows that

$$\nu_{n-1} = \frac{[n-2]}{[n-1]} = \frac{(n-2)(n) - (n-1)^2}{(n-1)(n+1) - (n)^2} = \mu_n\mu_{n+1}\frac{\mu_{n-1} - \mu_n}{\mu_n - \mu_{n+1}}.$$

Therefore

$$R_n = \nu_{n-1} / \{\mu_n\mu_{n+1}\}. \tag{4 . 2}$$

Neither in the preceding theorems of this section, nor in the theorems of § 2, have we yet made use of the fact that all the characteristic numbers are positive. But, in view of this fact, it now follows from

Theorems 4.1 and 4.2 that all the numbers μ_n and ν_n are positive. Hence R_n is always positive, and we deduce

THEOREM 4.3. *The sequence* (μ_n) *is monotonic and decreasing, and therefore converges to a unique finite limit* μ *as* $n \to \infty$ *.

This result requires to be completed by an investigation of the value of the limit and of the rapidity of the convergence.

THEOREM 4.4. *If the generating function* $f_0(x)$ *is orthogonal to the first* κ *characteristic functions, then the sequence of numbers* (μ_n) *converges to the* $(\kappa+1)$*-th characteristic number* λ_κ*, and the sequence of functions* $(\mu_n)^n f_n(x)$ *converges uniformly to a multiple of the* $(\kappa+1)$*-th characteristic function* $\phi_\kappa(x)$.*

Let

$$F(x, y, s) = \begin{vmatrix} (\mu_m)^m f_{m-1}(x) - (\mu_n)^n f_{n-1}(x) & (\mu_m)^m f_{m-1}(y) - (\mu_n)^n f_{n-1}(y) \\ G(x, s) & G(y, s) \end{vmatrix}.$$

Since the integral $$\int_a^b \int_a^b \sigma(x)\, \sigma(y)\, F^2(x, y, s)\, dx\, dy$$

cannot have a negative value, by expansion we obtain the inequality

$$\{(\mu_m)^m f_m(s) - (\mu_n)^n f_n(s)\}^2 \leqslant Q \cdot P(s), \tag{4.3}$$

where $$P(s) = \int_a^b \sigma(x)\, G^2(x, s)\, dx,$$

and $$Q = (\mu_m)^{2m}(2m-2) - 2(\mu_m)^m(\mu_n)^n(m+n-2) + (\mu_n)^{2n}(2n-2).$$

Let $a_\kappa = (f_0, \phi_\kappa)$. Then $(f_n, \phi_\kappa) = a_\kappa \lambda_\kappa^{-n}$. But, since the integral

$$\int_a^b \sigma(x) \{\lambda_\kappa^n f_n(x) - a_\kappa \phi_\kappa(x)\}^2\, dx$$

cannot be negative, we deduce that

$$\lambda_\kappa^{2n}(2n) \geqslant a_\kappa^2. \tag{4.4}$$

Now, if $\gamma_m = (\mu_m)^m(m)$,

$$\gamma_m - \gamma_{m+1} = (m+1)\, \mu_{m+1} [(\mu_m)^m - (\mu_{m+1})^m] \geqslant 0.$$

Also $$\gamma_{2m} = (\mu_m)^{2m}(2m) \geqslant \mu^{2m}(2m) \geqslant \lambda_\kappa^{2m}(2m) \geqslant a_\kappa^2,$$

* The fallacious proof of these results given by Picard, *Cours d'Analyse* 3, (1909), Chap. VI, has been rectified and the result extended by Kellogg, *Math. Annalen*, 86 (1922), 14–17.

by equation (4.4). Therefore, the sequence $\{\gamma_m\}$ steadily decreases to a unique, finite limit γ, not less than a_κ^2.

Since $$Q = \mu_m^2\gamma_{2m-2} - 2\mu_m\mu_n\gamma_{m+n-2} + \mu_n^2\gamma_{2n-2},$$

it follows that to any positive number ϵ there corresponds a positive integer p, such that, when $m, n > p$, then $Q < \epsilon$. Hence the sequence $\mu_n^n f_n(x)$ converges uniformly to a limit $f(x)$, which is not identically zero, since

$$(f, f) = \lim_{m\to\infty} \mu_m^{2m}(f_m, f_m) \geqslant a_\kappa^2.$$

Moreover, this limiting function must satisfy the equation

$$f(x) = \mu A(f).$$

Hence μ is a characteristic number and $f(x)$ is a numerical multiple of the corresponding characteristic function. Since $f_0(x)$, and therefore $f(x)$, is orthogonal to the first κ characteristic functions, μ must equal λ_κ.

An estimate of the rapidity of the convergence of the sequence (μ_n) is provided by the following theorem.

THEOREM 4.5. *If L is any number such that $\mu_n^2 < L < \lambda_\kappa\lambda_{\kappa+1}$, a lower bound to the limit λ is given by the inequality*

$$\mu_n - \lambda \leqslant \frac{\mu_{n-1} - \mu_n}{L\mu_n^{-2} - 1}.$$

From the definition of the sequence (R_n), given by equation (4.2), and from Theorem 4.2, it follows that

$$L\mu_n^{-2} \leqslant R_n$$

and $$L(\mu_n - \mu_{n+1}) \leqslant \mu_n^2(\mu_{n-1} - \mu_n).$$

Similarly
$$\begin{aligned} L(\mu_{n+1} - \mu_{n+2}) &\leqslant \mu_{n+1}^2(\mu_n - \mu_{n+1}) \\ &\leqslant \mu_n^2(\mu_n - \mu_{n+1}), \end{aligned}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

and $$L(\mu_{n+p-1} - \mu_{n+p}) \leqslant \mu_n^2(\mu_{n+p-2} - \mu_{n+p-1}) \quad (p = 2, 3, \ldots).$$

Therefore, by addition,

$$L(\mu_n - \mu_{n+p}) \leqslant \mu_n^2(\mu_{n-1} - \mu_{n+p-1}).$$

The limiting form of this relation as $p \to \infty$ is

$$L(\mu_n - \lambda) \leqslant \mu_n^2(\mu_{n-1} - \lambda).$$

Hence $$\mu_n - \lambda \leqslant \frac{\mu_{n-1} - \mu_n}{L\mu_n^{-2} - 1}.$$

It will be shown later (Theorem 5.2) that it is not necessary to

know λ_0 or λ_1 in applying this result to the determination of a lower limit to λ_0.

5. *Practical details of the computation.*

The choice of a generating function.

In order to complete the theoretical investigations given in the preceding section, it is necessary to show how it is possible to construct a generating function which is orthogonal to the first κ characteristic functions without having computed these functions. To do this, we suppose that $g(s)$ is a function such that $g(s)\,G(x, s)$ tends to a finite limit other than zero as $s \to a$ or b. Since $G(x, s)$ is symmetrical, it then follows from the relation $\phi_n(x) = \lambda A(\phi_n)$ that the corresponding limit $\lim_{x \to a,\, b} g(x)\,\phi_n(x)$ is also finite and distinct from zero for all values of n.

THEOREM 5.1. *If $\psi_1(x)$, $\psi_2(x)$, ..., $\psi_n(x)$, ... denote the sequence of functions derived from the generating functions $\psi_0(x) = g(s)\,G(x, s)$, then the function*

$$F_\kappa(x) = \begin{vmatrix} 1 & \lambda_0^{-1} & \lambda_0^{-2} & \dots & \lambda_0^{-\kappa} \\ 1 & \lambda_1^{-1} & \lambda_1^{-2} & \dots & \lambda_1^{-\kappa} \\ \dots & \dots & \dots & & \dots \\ 1 & \lambda_{\kappa-1}^{-1} & \lambda_{\kappa-1}^{-2} & \dots & \lambda_{\kappa-1}^{-\kappa} \\ \psi_0(x) & \psi_1(x) & \psi_2(x) & \dots & \psi_\kappa(x) \end{vmatrix}$$

is orthogonal to the first κ characteristic functions and to no others, provided that $s = a$ or b.

The Fourier constants of $\psi_0(x)$ are the numbers

$$a_n = \lambda_n^{-1} g(s)\,\phi_n(s) \quad (n = 1, 2, \dots),$$

and the Fourier constants of $\psi_n(x)$ are therefore the numbers

$$\lambda_n^{-m} a_n = \lambda_n^{-m-1} g(s)\,\phi_n(s).$$

Hence the integral (F_κ, ϕ_n), or the $(n+1)$-th Fourier constant of $F_\kappa(x)$ is equal to the product of $\lambda_n^{-1} g(s)\,\phi_n(s)$ by the determinant

$$\begin{vmatrix} 1 & \lambda_0^{-1} & \lambda_0^{-2} & \dots & \lambda_0^{-\kappa} \\ 1 & \lambda_1^{-1} & \lambda_1^{-2} & \dots & \lambda_1^{-\kappa} \\ \dots & \dots & \dots & & \dots \\ 1 & \lambda_{\kappa-1}^{-1} & \lambda_{\kappa-1}^{-2} & \dots & \lambda_{\kappa-1}^{-\kappa} \\ 1 & \lambda_n^{-1} & \lambda_n^{-2} & \dots & \lambda_n^{-\kappa} \end{vmatrix}$$

This determinant vanishes when $n = 0, 1, 2, \ldots, \kappa-1$. Hence $F_\kappa(x)$ is orthogonal to the first κ characteristic functions. When n is greater than $\kappa-1$ the determinant is numerically equal to the continued product of the $\frac{1}{2}\kappa(\kappa-1)$ differences $\lambda_\mu^{-1}-\lambda_\nu^{-1}$ and the factor

$$(\lambda_0\lambda_1\lambda_2 \ldots \lambda_{\kappa-1}\lambda_n)^{-1}.$$

Hence the determinant cannot vanish and $F_\kappa(x)$ will be orthogonal only to the first κ characteristic functions.

The construction of the sequence of approximating functions

Although the sequence of approximating functions is defined by the recurrence relation (4 . 1)

$$f_{n+1}(x) = A(f_n) \equiv \int_a^b \sigma(s)\, G(x, s)\, f_n(s)\, ds,$$

the discontinuity in the Green's function $G(x, s)$ makes it more convenient to compute the successive functions by the following method :

Let $H(x, s)$ denote Cauchy's function, which is finite and continuous, together with its first $2N$ derivatives in the interval $a < x < b$, and which satisfies in this interval the equation

$$L(H) = 0,$$

while at the point $x = s$,

$$H(x, s) = H'(x, s) = \ldots = d^{2N-2} H(x, s)/dx^{2N-2} = 0,$$

and

$$d^{2N-1} H(x, s)/dx^{2N-1} = -1.$$

Let $J(x)$ denote the general solution of the equation

$$L(H) = 0,$$

involving $2N$ independent arbitrary constants. Then the general solution of the equation

$$L(u)+\sigma(x) f_n(x) = 0$$

is

$$u(x) = J(x)+\int_a^x \sigma(s)\, H(x, s)\, f_n(s)\, ds$$

and the arbitrary constants in $J(x)$ can be adjusted until $u(x)$ satisfies the boundary conditions "M". Then $u(x) \equiv f_{n+1}(x)$.

The estimation of a lower limit to $\lambda_0\lambda_1$.

By Theorem 4 . 5 the difference between any term μ_n of the sequence (μ_n) and the limit of the sequence λ_κ does not exceed the difference be-

tween μ_n and the term μ_{n-1} immediately preceding, divided by $L\mu_n^{-2}-1$, where L is any number not greater than the product $\lambda_\kappa\lambda_{\kappa+1}$ and not less than μ_n^2. In the practical application of this theorem it is sufficient to obtain a number L such that

$$\lambda_\kappa\lambda_{\kappa+1} \geqslant L \geqslant 2\mu_n^2,$$

for then we may deduce that

$$\mu_n-\lambda_\kappa \leqslant \mu_{n-1}-\mu_n,$$

and hence $$2\mu_n-\mu_{n-1} \leqslant \lambda_\kappa \leqslant \mu_n.$$

This method is easily applied when it is the first characteristic number λ_0 which is to be computed. Since the integral

$$\int_a^b\int_a^b \sigma(x)\,\sigma(y)\,\{G(x,y)-\lambda_0^{-1}\,\phi_0(x)\,\phi_0(y)-\lambda_1^{-1}\phi_1(x)\,\phi_1(y)\}^2\,dx\,dy$$

is necessarily positive, we find on expansion that

$$\lambda_0^{-2}+\lambda_1^{-2} < I = \int_a^b\int_a^b \sigma(x)\,\sigma(y)\,G^2(x,y)\,dx\,dy$$

$$= 2\int_a^b \sigma(y)\,dy\int_a^y \sigma(x)\,G^2(x,y)\,dx\,dy.$$

This comparatively crude inequality furnishes the following test.

THEOREM 5.2. *The condition* $2\mu_n^2 I < 1+\sqrt{2}$ *is sufficient to ensure that*

$$\mu_n-\lambda_0 \leqslant \mu_{n-1}-\mu_n.$$

For $$I > \lambda_0^{-2}+\lambda_1^{-2} \quad \text{and} \quad \lambda_0 < \mu_n,$$

hence $$I > \lambda_0^{-2} \quad \text{and} \quad I-\mu_n^{-2} > \lambda_1^{-2},$$

and therefore $$I(I-\mu_n^{-2}) > \lambda_0^{-2}\lambda_1^{-2}$$

or $$\{I-(2\mu_n^2)^{-1}\}^2-(4\mu_n^4)^{-1} > \lambda_0^{-2}\lambda_1^{-2}.$$

If the condition $2\mu_n^2 I < 1+\sqrt{2}$ is fulfilled, it follows that

$$\lambda_0\lambda_1 > 2\mu_n^2.$$

Since $$\mu_n-\lambda_0 \leqslant \frac{\mu_{n-1}-\mu_n}{L\mu_n^{-2}-1},$$

where L is any number less than $\lambda_0\lambda_1$, and greater than μ_n^2, we may write

$$L = 2\mu_n^2,$$

and then $$\mu_n - \lambda \leqslant \mu_{n-1} - \mu_n.$$

EXAMPLE. *The computation of the first zero of Bessel functions of fractional order* $(-\frac{1}{2} \leqslant \nu \leqslant 1)$.

The solution of the self-adjoint equation

$$\frac{d}{dx}\left\{x^{2\nu+1}\frac{dy}{dx}\right\} + \lambda x^{2\nu+1} y = 0 \quad (0 \leqslant x \leqslant 1)$$

which remains finite when $x = 0$ is $x^{-\nu} J_\nu(x\sqrt{\lambda})$. In order that this solution should satisfy the additional requirement of vanishing when $x = 1$, it is necessary and sufficient that $\sqrt{\lambda}$ should be a zero of the Bessel's function of order ν. These two boundary conditions may be written in the self-adjoint form

$$dy/dx = 0 \quad \text{when } x = 0,$$

and $$y = 0 \quad \text{when } x = 1.$$

The Green's function for the equation $(x^{2\nu+1} y')' = 0$ and these conditions is given by the formulae

$$\left.\begin{aligned} G(x, s) &= \frac{s}{2\nu}(1 - s^{2\nu}) && \text{when } 0 \leqslant x \leqslant s \\ &= \frac{s}{2\nu}\left\{\left(\frac{s}{x}\right)^{2\nu} - s^{2\nu}\right\} && \text{when } s \leqslant x \leqslant 1 \end{aligned}\right\} \quad \text{if } \nu \neq 0,$$

or $$\left.\begin{aligned} G(x, s) &= -s \log s && \text{when } 0 \leqslant x \leqslant s \\ &\quad -s \log x && \text{when } s \leqslant x \leqslant 1 \end{aligned}\right\} \quad \text{if } \nu = 0.$$

In accordance with Theorem 5 . 1 we take $f_0(x) = 1$ as the generating function. The first few members of the sequence of approximating functions are then found to be

$$f_1(x) = \frac{1 - x^2}{4(\nu+1)},$$

$$f_2(x) = \frac{(\nu+3) - 2(\nu+2)\, x^2 + (\nu+1)\, x^4}{32(\nu+1)^2(\nu+2)},$$

$$f_3(x) = \frac{(\nu^2+8\nu+19) - 3(\nu+3)^2 x^2 + 3(\nu+1)(\nu+3)\, x^4 - (\nu+1)^2 x^6}{384(\nu+1)^3(\nu+2)(\nu+3)}.$$

The first few terms in the sequence (μ_n) which converges to the first

characteristic number λ_0 are

$$\mu_1 = 4(\nu+1)(\nu+2),$$
$$\mu_2 = 2(\nu+1)(\nu+3),$$
$$\mu_3 = 8(\nu+1)(\nu+2)(\nu+4)(5\nu+11)^{-1},$$
$$\mu_4 = 2(\nu+1)(\nu+5)(5\nu+11)(7\nu+19)^{-1}.$$

By Theorem 5 . 2 the inequality

$$\mu_4 - \lambda_0 < \mu_3 - \mu_4$$

will hold provided that $2\mu_4^2 I < 1+\sqrt{2}$,

where
$$I = 2\int_0^1 s^{2\nu+1}\,ds \int_0^s x^{2\nu+1}\,G^2(x, s)\,dx$$
$$= \{4(\nu+1)(2\nu+3)(3\nu+3)(4\nu+3)\}^{-1}.$$

It is easily verified that this inequality is always satisfied if $-\frac{1}{2} \leqq \nu \leqq 1$.

Hence
$$\frac{\sqrt{\mu_4} - \sqrt{\lambda_0}}{\sqrt{\mu_4}} < \frac{\mu_3 - \mu_4}{\mu_4 + \sqrt{(\mu_4\lambda_0)}},$$

or the proportionate error in $\sqrt{\mu_4}$, regarded as an approximation to the first zero of $J_\nu(x)$, is about $(\mu_3-\mu_4)/(2\mu_4)$. When ν has the values $-\frac{1}{2}$, 0, $\frac{1}{2}$, and 1, this quantity has the values 1/400, 1/400, 1/200, and 3/400 respectively.

SOME THEOREMS ON DIRICHLET SERIES WITH POSITIVE COEFFICIENTS AND RELATED INTEGRALS

By H. P. MULHOLLAND.

[Received and read 26 April, 1928.]

1. *Introduction.*

In a recent paper* Hardy and Littlewood have given certain theorems of which the following are the principal:

$$(1.1) \qquad \int_0^1 A^r(x)(1-x)^{(r/q)-1}\,dx \leqslant K\left(\sum_1^\infty a_n^p n^{-(p+q-pq)/q}\right)^{r/p},$$

where $$A(x) = \Sigma\, a_n x^n \quad (a_n \geqslant 0);$$

$$(1.2) \qquad \sum_1^\infty A_n^r n^{(r/q)-1} \leqslant K\left(\int_0^1 a^p(x)(1-x)^{-(p+q-pq)/q}\,dx\right)^{r/p},$$

where $$A_n = \int_0^1 a(x)\,x^n\,dx \quad [a(x) \geqslant 0],$$

the quantities A_n being in fact the moment constants of the function $a(x)$. These theorems were established subject to the conditions $r \geqslant p > 1$ and $q > 0$, in them $K = K(p, q, r)$, and depends only on these parameters, and the inequalities are understood to assert *inter alia* that the convergence of the right-hand side implies that of the left.

Such theorems were suggested by the (more difficult) inequalities of similar form concerning Fourier constants. If, for instance, we put

$$r = q = p' = \frac{p}{p-1},$$

* "Power series and moment constants", *Journal für Math.*, 157 (1927), 141–158.

with $1 < p \leqslant 2$, in (1.1) and (1.2), we get

$$\int_0^1 A^{p/(p-1)}(x)\,dx \leqslant K\left(\sum_1^\infty a_n^p\right)^{1/(p-1)}, \tag{1.3}$$

$$\sum_1^\infty A^{p/(p-1)} \leqslant K\left(\int_0^1 a^p(x)\,dx\right)^{1/(p-1)}. \tag{1.4}$$

These results are clearly analogous to the W. H. Young-Hausdorff extensions of the Riesz-Fischer and Parseval Theorems.

The theorems in the case of Fourier constants have a greater unity, since it is the Fourier coefficients themselves which correspond to the moment constants A_n of the function $a(x)$. On the other hand, with Fourier series $p = 2$ appears as a border-line case dividing two distinct groups of theorems, whose analogues, however, are incorporated, as two special cases, in Theorems (1.1), (1.2), etc.

The present paper is concerned with the corresponding group of theorems for Dirichlet series and the related integrals

$$A_n = \int_0^\infty a(x)\,n^{-x}\,dx.$$

It turns out that there is a surprisingly close parallel with the case of power series, to which the treatment here employed applies equally well and for which it provides an alternative and direct proof. This shows incidentally that, in the special case $r = p$, the best possible value of K in (1.1) and (1.2) is $[\Gamma(1/q)]^p$.

It may be remarked that all quantities occurring are non-negative, unless the contrary is stated.

2. *Theorems with* $p \geqslant 1$.

2.0. Two preliminary lemmas are needed. Here and elsewhere K will be used indifferently for some positive constant depending only on the parameters, such as c, p, q or r, occurring; it may stand for several such constants in the same context.

LEMMA 1. *If* $c > 0$ *and* $x > 0$, *then*

$$\sum_2^\infty n^{-1-x}(\log n)^{c-1} \leqslant X(c)\,x^{-c}, \tag{2.011}$$

where $X(c)$ *depends only on* c *and is equal to* $\Gamma(c)$ *if* $c \leqslant 1$; *while*

$$\sum_2^\infty n^{-1-x}(\log n)^{c-1} \geqslant Kx^{-c}, \tag{2.012}$$

provided that $0 < x \leqslant 1$.

If $c \leqslant 1$, $t^{-1-x}(\log t)^{c-1}$ is decreasing, and so

$$\int_2^\infty t^{-1-x}(\log t)^{c-1}\,dt \leqslant \sum_2^\infty n^{-1-x}(\log n)^{c-1} \leqslant \int_1^\infty t^{-1-x}(\log t)^{c-1}\,dt.$$

The integral on the right is equal to

$$\int_0^\infty \tau^{c-1}e^{-x\tau}\,d\tau = x^{-c}\int_0^\infty s^{c-1}e^{-s}\,ds = \Gamma(c)\,x^{-c},$$

while the integral on the left is equal to

$$\int_{\log 2}^\infty \tau^{c-1}e^{-x\tau}\,d\tau = x^{-c}\int_{x\log 2}^\infty s^{c-1}e^{-s}\,ds \geqslant Kx^{-c}$$

for x in (0, 1); and the inequalities have been established for $c \leqslant 1$. If $c > 1$, $t^{-1-x}(\log t)^{c-1}$ has a maximum at $t = \exp[(c-1)/(1+x)]$ of value

$$e^{1-c}[(c-1)/(1+x)]^{c-1} = M, \quad \text{say.}$$

Then

$$\sum_2^\infty n^{-1-x}(\log n)^{c-1} \leqslant \int_1^\infty t^{-1-x}(\log t)^{c-1}\,dt + \operatorname*{Max}_n [n^{-1-x}(\log n)^{c-1}]$$
$$\leqslant \Gamma(c)\,x^{-c} + M.$$

Now M is bounded as $x \to 0$, while for x large the series is plainly $O(2^{-x})$; hence (2.011) holds with a suitable $X(c)$. Similarly

$$\sum_2^\infty n^{-1-x}(\log n)^{c-1} \geqslant \Gamma(c)\,x^{-c} - M, \tag{2.013}$$

and, since the series does not vanish in (0, 1), it must clearly satisfy (2.012).

LEMMA 2. *If $c > 0$ and $\lambda > 1$*

$$\int_0^\infty x^{c-1}\lambda^{-x}\,dx = (\log\lambda)^{-c}\,.\,\Gamma(c). \tag{2.02}$$

This is immediate.

2.1. THEOREM 1. *If $a_n \geqslant 0$, $r \geqslant p \geqslant 1$, $q > 0$ and*

$$A(x) = \sum_2^\infty a_n n^{-1-x}, \tag{2.11}$$

then

$$\int_0^\infty A^r(x)\,x^{(r/q)-1}\,dx \leqslant K\left\{\sum_2^\infty a_n^p n^{-1}(\log n)^{-(p+q-pq)/q}\right\}^{r/p}, \tag{2.12}$$

where K depends only on p, q, r, and, if $q \geqslant 1$,

$$K \leqslant C(p, q, r) = \left[\Gamma\left(\frac{1}{q}\right)\right]^{r(p-1)/p} \Gamma\left(\frac{r}{pq}\right). \tag{2.13}$$

When also $r = p$, $K = C(p, q, p)$ and is best possible.

We write $S = \Sigma u_n$ for the sum on the right of (2.12) and regard the u_n as fixed, while p, q, and r vary. The triple form* of Hölder's (or Schwarz') inequality is then used to reduce the case (p, q, r) to the case $(1, pq/r, 1)$, which is settled by Lemma 2. Let

$$p' = \frac{p}{p-1}, \quad r' = \frac{r}{r-1}, \quad Q = \frac{pq}{r}.$$

We have

$$\begin{aligned} A(x) = A_{p,q}(x) &= \Sigma u_n^{1/p} n^{-(1/p')\cdot x} (\log n)^{(1/p)+(1/q)-1} \\ &= \Sigma u_n^{1/r} n^{-x/r} (\log n)^{1/pq} \cdot u_n^{(1/p)-(1/r)} n^{-(x/p)+(x/r)} \cdot n^{-(1+x)/p'} (\log n)^{(1/p'q)-(1/p')} \\ &\leqslant \left\{\Sigma u_n n^{-x} (\log n)^{r/pq}\right\}^{1/r} \left\{\Sigma u_n n^{-x}\right\}^{(1/p)-(1/r)} \left\{\Sigma n^{-1-x} (\log n)^{(1/q)-1}\right\}^{1/p'} \\ &\leqslant \left\{A_{1,Q}(x)\right\}^{1/r} S^{(1/p)-(1/r)} \left\{X\left(\frac{1}{q}\right) x^{-1/q}\right\}^{1/p'}, \end{aligned}$$

using (2.011) of Lemma 1. Therefore

$$\int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx \leqslant \left[X\left(\frac{1}{q}\right)\right]^{r/p'} S^{(r/p)-1} \int_0^\infty A_{1,Q}(x)\, x^{(1/Q)-1}\, dx,$$

Since $X\left(\frac{1}{q}\right) = \Gamma\left(\frac{1}{q}\right)$ when $q \geqslant 1$, it only remains to establish the case $(1, Q, 1)$. Here we have an equality, for

$$\int_0^\infty A_{1,Q}(x)\, x^{(1/Q)-1}\, dx = \int_0^\infty \sum_2^\infty u_n (\log n)^{1/Q} n^{-x} \cdot x^{(1/Q)-1}\, dx,$$

and the series can easily be shown to converge uniformly in any interval (δ, ∞); hence, since everything is positive, the integral last written is convergent and it may be evaluated term by term, provided that the result is convergent†. But we find that

$$\sum_2^\infty u_n (\log n)^{1/Q} \int_0^\infty n^{-x} x^{(1/Q)-1}\, dx = \Gamma\left(\frac{1}{Q}\right) \sum_2^\infty u_n$$

by Lemma 2; so the equality has been proved.

* It will be seen, however, that the second bracket will be absent if $r = p$, the third if $p = 1$.

† Cf. Bromwich, *Infinite series*, §§175 and 176.

To show now that $[\Gamma(1/q)]^p$ is best possible when $r = p$ and $q \geqslant 1$, consider the example $a_n = (\log n)^{(1/q)-1-(\epsilon/p)}$; then

$$S = \Sigma n^{-1}(\log n)^{-1-\epsilon} \sim \epsilon^{-1} \quad \text{as} \quad \epsilon \to 0,$$

and

$$A(x) = \sum_2^\infty n^{-1-x}(\log n)^{(1/q)-(\epsilon/p)-1}$$

$$\geqslant \Gamma\left(\frac{1}{q} - \frac{\epsilon}{p}\right) x^{-(1/q)+(\epsilon/p)} - M,$$

where M is bounded, as $x \to 0$, by (2.013). Therefore

$$\int_0^\infty A^p(x)\, x^{(p/q)-1}\, dx \geqslant \int_0^1 \geqslant \left[\Gamma\left(\frac{1}{q} - \frac{\epsilon}{p}\right)\right]^p \int_0^1 x^{\epsilon-1}\left[1 - Mx^{(1/q)-(\epsilon/p)}\right]^p dx$$

$$\sim \left[\Gamma\left(\frac{1}{q}\right)\right]^p \epsilon^{-1},$$

and this can only happen if $K \geqslant [\Gamma(1/q)]^p$.

2.2. Theorem 1 has a sort of reciprocal theorem involving a non-negative function $a(x)$ and constants A_n connected with it by the relation

$$A_n = \int_0^\infty a(x)\, n^{-x}\, dx.$$

This is Theorem 2 below. Either of Theorems 1 and 2 may be obtained from the other by interchanging (i) Σ and $\int$, (ii) a_n and $a(x)$, $A(x)$ and A_n, (iii) x^α and $n^{-1}(\log n)^\alpha$, α representing the various indices appearing, provided that we regard A_n as having a factor x^0 in its integrand and $A(x)$ a factor $(\log n)^0$ in its general term. The same process transforms the proof of one into that of the other, the rôles of Lemmas 1 and 2 being interchanged; the proof of Theorem 2 is consequently omitted below.

Like their analogues in the paper* by Hardy and Littlewood, they may be deduced from one another (with the exception of the value of K) when $p \neq 1$ by the converse of Hölder's inequality.

2.2. THEOREM 2. *If* $a(x) \geqslant 0$, $r \geqslant p \geqslant 1$, $q > 0$ *and*

$$A_n = \int_0^\infty a(x)\, n^{-x}\, dx, \tag{2.21}$$

* Theorems 3 and 4; *loc. cit.*

then

$$(2.22)\qquad \sum_{2}^{\infty} A_n^r n^{-1}(\log n)^{(r/q)-1} \leqslant K \left\{ \int_0^\infty a^p(x)\, x^{-(p+q-pq)/q}\, dx \right\}^{r/p},$$

where, in the case $pq \geqslant r$, $K \leqslant C(p, q, r)$ *as defined in* (2.13), *and, when also* $r = p$, $K = [\Gamma(1/q)]^r$ *and is best possible.*

2.3. It appears to be more convenient to omit the term $n = 1$ from the series occurring. Its inclusion would need a convention replacing $\log n$ by 1, say, in the first term, while the upper limits of the integrals on the left would have to be finite to avoid divergence; moreover, the inequality (2.13) for K is upset if we make these changes.

2.4. When $r > p$, by slightly sharpening the method of § 2.1, we can get a better upper bound for K, namely

$$\operatorname*{Min}_{0<\beta<1} \left\{ \left[\Gamma\left(\frac{p'(1-\beta)}{q}\right) \right]^{r/p'} \Gamma\left(\frac{r\beta}{q}\right) \left(\frac{p}{r}\right)^{r\beta/q} \right\}$$

obtained by reducing the case (p, q, r) to the case $(1, q/r\beta, 1)$. But it seems likely that this is not the best possible value, to get which a more delicate argument would probably be required.

2.5. The inequalities (2.12) and (2.22) take a simpler form when either (i) $q = p'$ or (ii) $q = r$. In case (i) the factor $(\log n)^{-(p+q-pq)/q}$ or $x^{-(p+q-pq)/q}$ on the right is absent, in (ii) $x^{(r/q)-1}$ or $(\log n)^{(r/q)-1}$ on the left. It will suffice* to give the special case of Theorem 1, in which both (i) and (ii) hold:

THEOREM 3.

$$(2.51)\qquad \int_0^\infty A^{p'}(x)\, dx \leqslant \frac{\pi}{\sin(\pi/p)} \left(\sum_{2}^{\infty} \frac{a_n^p}{n} \right)^{1/(p-1)}.$$

This case can only arise when $p' \geqslant p$ and so $p \leqslant 2$, in which respect (as well as in form) it resembles the extension of the Riesz-Fischer theorem for Fourier series.

3. *The case* $0 < p \leqslant 1$.

3.1. In this case the theorems appear with their inequalities reversed,

* The special cases of the analogous theorems for power series, etc., are set out in full in §3.4 of Hardy and Littlewood's paper, *loc. cit.*

owing to the reversal in Hölder's inequality, which now appears as

$$\Sigma ab \geqslant (\Sigma a^{k_1})^{1/k_1}(\Sigma b^{k_2})^{1/k_2},$$

where $0 < k_1 < 1$ and $k_1^{-1}+k_2^{-1} = 1$.

Repeated application of this gives the triple form

$$\Sigma abc \geqslant (\Sigma a^{k_1})^{1/k_1}(\Sigma b^{k_2})^{1/k_2}(\Sigma c^{k_3})^{1/k_3},$$

where $0 < k_1 < 1$, k_2 and k_3 are negative, and

$$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3} = 1.$$

In the present instance we have

$$k_1 = r, \qquad k_2 = \frac{rp}{r-p}, \qquad k_3 = p' = \frac{p}{p-1};$$

p' is negative when $p < 1$ and k_2 will be negative if we take $r < p$. Since the exponents of the last two brackets are negative, it is still an *upper bound* for their contents which is needed, and so the first stage of the proofs proceeds exactly as in the case $p > 1$. Lemma 2 is an equality and the proof of Theorem 4 below may thus be completed in the same way as that of Theorem 1, and it yields the same value of K when $q \geqslant 1$; Theorem 5, however, needs special treatment and no attempt is made to get a definite value for K.

THEOREM 4. *If* $0 < r \leqslant p \leqslant 1$, $q > 0$, *then*

$$\sum_2^\infty a_n^p n^{-1}(\log n)^{-(p+q-pq)/q} \leqslant K\left\{\int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx\right\}^{p/r}, \tag{3.11}$$

where, in the case $q \geqslant 1$,

$$K = C'(p, q, r) = [C(p, q, r)]^{-(p/r)} = \left[\Gamma\left(\frac{1}{q}\right)\right]^{1-p}\left[\Gamma\left(\frac{r}{pq}\right)\right]^{-p/r}, \tag{3.12}$$

and, when also $r = p$, $K = [\Gamma(1/q)]^{-p}$ *and is best possible.*

The last part may clearly be established by the example used in proving Theorem 1, namely $a_n = (\log n)^{-(1/q)-1-(\epsilon/p)}$.

3.2. THEOREM 5. *If* $0 < r \leqslant p \leqslant 1$, $q > 0$, *then*

$$\int_0^\infty a^p(x)\, x^{-(p+q-p)/q}\, dx \leqslant K\left\{A_1^r + \sum_2^\infty A_n^r n^{-1}(\log n)^{(r/q)-1}\right\}^{p/r} \tag{3.21}$$

Writing J for the integral and $u(x)$ for its integrand, we have

$$A_n = A_n(p, q) = \int_0^\infty u^{1/p}(x)\, n^{-x} x^{(1/p)+(1/q)-1}\, dx$$

$$\geqslant K\{A_n(1, pq/r)\}^{1/r} J^{(1/p)-(1/r)} (\log n)^{-1/p'q},$$

while

$$A_1 = A_1(p, q) = \int_0^\infty u^{1/p}(x)\, x^{(1/p)+(1/q)-1}\, dx = \int_0^1 + \int_1^\infty$$

$$\geqslant \left\{\int_0^1 u(x)\, x^{r/pq}\, dx\right\}^{1/r} \left\{\int_0^1 u(x)\, dx\right\}^{(1/p)-(1/r)} \left\{\int_0^1 x^{(1/q)-1}\, dx\right\}^{1/p'}$$

$$+ \left\{\int_1^\infty u(x)\, dx\right\}^{1/p} \left\{\int_1^\infty x^{(r'/q)-1}\, dx\right\}^{1/p'}$$

$$= q^{1/p'} \left\{\int_0^1 u(x)\, x^{r/pq}\, dx\right\}^{1/r} J_1^{(1/p)-(1/r)} + \left\{\frac{q}{-p'}\right\}^{1/p'} J_2^{1/p}, \quad \text{say.}$$

Therefore, taking the r-th power and summing

$$A_1^r + \sum_2^\infty A_n^r n^{-1} (\log n)^{(r/q)-1}$$

$$\geqslant K\left[J_2^{r/p} + J_1^{(r/p)-1} \int_0^1 u(x)\, x^{r/pq}\, dx\right.$$

$$\left. + J^{(r/p)-1} \sum_2^\infty n^{-1} (\log n)^{(r/pq)-1} \int_0^\infty u(x)\, x^{r/pq}\, n^{-x}\, dx\right]$$

$$\geqslant K\left[J_2^{r/p} + J_1^{(r/p)-1} \int_0^1 u(x)\, x^{r/pq} \left\{1 + \sum_2^\infty n^{-1-x} (\log n)^{(r/pq)-1}\right\} dx\right],$$

since it is permissible to evaluate this integral term by term on the same grounds* as in the proof of Theorem 1; hence, using Lemma 1, (2.012), the last expression is greater than or equal to

$$K(J_2^{r/p} + J_1^{r/p}) \geqslant KJ^{r/p}, \quad \text{as required.}$$

This theorem is not a strict analogue of Theorem 2; the reason is clearly that, when $n > 1$, the factor n^{-x} in A_n makes the values of $a(x)$ for large x of little effect, whereas these values may contribute largely to the integral on the left of (3.21). The theorem is false without the term A_1; an example is provided by the function $a(x) = x^{-1}$ for $x \geqslant X$ and $a(x) = 0$ for $x < X$, X being large.

* The present case is actually simpler as the range of integration is finite.

3.3. We get simple special cases by taking $q = r$, though $q = p'$ is not now permissible, since p' is negative.

3.4. The restrictions imposed in Theorems 1 to 5, namely $r \geqslant p \geqslant 1$ (or $0 < r \leqslant p \leqslant 1$) and $q > 0$, may easily be seen to be necessary.

The example $a_n = (\log n)^{(1/q)-1-(\epsilon/p)}$ (with ϵ small) used in the proof of Theorem 1 shows that it would be false with $r < p$ and that Theorem 4 would be false with $r > p$. Similarly for Theorems 2 and 5.

If, however, we had $q < 0$,

$$\int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx, \quad A_n = \int_0^\infty a(x)\, n^{-x}\, dx$$

would diverge, the former always, the latter in cases such as $a(x) = x^{(1/q)-1-(\epsilon/p)}$ $(x \leqslant 1)$, although this makes

$$\int_0^\infty a^p(x)\, x^{-(p+q-pq)/q}\, dx = \int_0^\infty x^{\epsilon-1}\, dx$$

convergent.

4. *Derivative theorems.*

4.1. Like the analogous theorems on power series and moment constants, the theorems in § 2 yield sundry derivative theorems, as follows:

THEOREM 6. *If* $p > 1$, $q > 1$, $\mu = (1/p)+(1/q)-1 \geqslant 0$, *then*

$$\int_0^\infty A(x)\, B(x) x^{-\mu}\, dx \leqslant K \left(\sum_2^\infty \frac{a_n^p}{n}\right)^{1/p} \left(\sum_2^\infty \frac{b_n^q}{n}\right)^{1/q}. \tag{4.11}$$

THEOREM 7. *Under the same conditions*

$$\sum_2^\infty A_n B_n n^{-1} (\log n)^{-\mu} \leqslant K \left(\int_0^\infty a^p(x)\, dx\right)^{1/p} \left(\int_0^\infty b^q(x)\, dx\right)^{1/q}. \tag{4.12}$$

COROLLARY. *When* $\mu = 0$ *and so* $q = p'$, *the best possible value of* K *in the above is* $\pi/\sin(\pi/p)$.

To prove (4.11) choose $r \geqslant p$, $s \geqslant q$ so that $r^{-1}+s^{-1} = 1$, this being possible whenever $\mu \geqslant 0$. Then we apply Theorem 1 with p' for q and get

$$\int_0^\infty A^r(x)\, x^{(r/p')-1}\, dx \leqslant \left[\Gamma\left(\frac{1}{p'}\right)\right]^{r/p'} \Gamma\left(\frac{r}{pp'}\right) \left(\sum_2^\infty \frac{a_n^p}{n}\right)^{r/p}.$$

Similarly

$$\int_0^\infty B^s(x)x^{(s/q)-1}\,dx \leqslant \left[\Gamma\left(\frac{1}{q'}\right)\right]^{s/q'}\Gamma\left(\frac{s}{qq'}\right)\left(\sum_2^\infty \frac{b_n''}{n}\right)^{s/q},$$

and (4.11) follows by Hölder's inequality.

If $\mu = 0$, $r = p$, and $s = q$, and the constant appears as

$$\Gamma\left(\frac{1}{p'}\right)\Gamma\left(\frac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)};$$

that this is best possible is shown by the example $a_n = (\log n)^{-(1+\epsilon)/p}$, $b_n = (\log n)^{-(1+\epsilon)/p'}$ (the proof goes as with Theorem 1). (4.12) may be dealt with in similar fashion.

4.2. We may recast Theorem 6 as follows:

$$\int_0^\infty A(x)B(x)x^{-\mu}\,dx = \Sigma\Sigma a_m b_n \int_0^\infty x^{-\mu}(mn)^{-1-x}\,dx$$

$$= \Gamma(1-\mu)\Sigma\Sigma a_m b_n[\log(mn)]^{1-\mu}.$$

THEOREM 8.

$$\sum_2^\infty\sum_2^\infty \frac{a_m b_n}{mn[\log(mn)]^{1-\mu}} \leqslant K\left(\sum_2^\infty \frac{a_n^p}{n}\right)^{1/p}\left(\sum_2^\infty \frac{b_n''}{n}\right)^{1/q},$$

and when $\mu = 0$, $K = \pi/\sin(\pi/p)$, *and this is the smallest* K *for which the inequality holds.*

The rem 7 may be treated similarly if we take the integrals over (0.1) and yields the same result as appears in Theorem 19 of Hardy and Littlewood's paper; we do not, however, get a convenient value for K.

5. *Generalizations.*

Let $\lambda_\nu(x) = x\log x\log_2 x \ldots \log_\nu x$, where

$$\log_2 x = \log(\log x), \quad \ldots, \quad \log_\nu x = \log(\log_{\nu-1} x), \quad \ldots,$$

and let $N = N(\nu)$ be the smallest integer n such that $\lambda_\nu(n) > 0$.

With the aid of $\lambda_\nu(x)$ we may generalize the theorems of §§2, 3, and 4. It will suffice to give as a specimen the generalization of Theorem 1.

THEOREM 9. *If* $a_n \geqslant 0$, $r \geqslant p \geqslant 1$, $q > 0$, *and*

$$A(x) = \sum_N a_n \lambda_\nu^{-1}(n) (\log_\nu n)^{-x}, \tag{5.11}$$

then

$$\int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx \leqslant K \left\{ \sum_N a_n^p \lambda_\nu^{-1}(n) (\log_{\nu+1} n)^{-(p+q-pr)/q} \right\}^{r/p}. \tag{5.12}$$

The proof is modelled on that of Theorem 1, Lemma 1 being replaced by

LEMMA 3. *If* $c > 0$, $x > 0$, *then*

$$\sum_N \lambda_\nu^{-1}(n) (\log_\nu n)^{-x} (\log_{\nu+1} n)^{c-1} \leqslant K x^{-c}, \tag{5.01}$$

and, if $0 < x \leqslant 1$,

$$\sum_N \lambda_\nu^{-1}(n) (\log_\nu n)^{-x} (\log_{\nu+1} n)^{c-1} \geqslant K x^{-c}. \tag{5.02}$$

This can be established by the method used for Lemma 1.

6. *The case of power series, etc.*

The theorems in the paper by Hardy and Littlewood were proved there by approximating to the integrals by means of series and then invoking certain inequalities. Direct proofs are possible, however, since it may readily be verified that the methods of the present paper will apply. Lemma 1 has to be replaced by

$$K[(1-x)^{-c} - 1] \leqslant \sum_1^\infty n^{c-1} x^n \leqslant \Gamma(c)(1-x)^{-c}, \tag{6.1}$$

which is trivial except for the constant on the right, and this may be obtained by comparing the series with the integral from $t = 0$ to $t = \infty$ of $t^{c-1} x^t$ when $c \leqslant 1$ and with the binomial expansion of $(1-x)^{-c}$ when $c > 1$. Lemma 2 is replaced by

$$K n^{-c} \leqslant \int_0^1 (1-x)^{c-1} x^n\, dx \leqslant \Gamma(c)\, n^{-c}, \tag{6.2}$$

which follows easily by means of the known expression of the Eulerian integral in terms of the Gamma-function. It is clear that we shall obtain

the $C(p, q, r)$ of (2.13) as an upper bound for the constant factor K in (1.1) and (1.2), though without any restriction such as $q \geqslant 1$; and it can be shown without trouble that $[\Gamma(1/q)]^p$ is the best possible K when $r = p$ [*e.g.* by taking $a_n = n^{(1/q)-1-(\epsilon/p)}$ in (1.1) and letting $\epsilon \to 0$]. We shall also get the same values for K in the derivative theorems as were obtained for their analogues in § 4 above; and it may be remarked that one of the former leads to a simple proof* of Hilbert's theorem on double series.

In conclusion I should like to thank Prof. Littlewood for his suggestions and advice during the preparation of the paper.

* See *Journal*, 3 (1928) 197–199.

THEOREMS STATED BY RAMANUJAN (V): APPROXIMATIONS CONNECTED WITH e^x

By G. N. WATSON.

[Received 17 April, 1928.—Read 10 May, 1928.]

1. Among the theorems stated by Ramanujan* in his letter of January 16, 1913, to Hardy is the following:

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^x}{x!}\,y=\frac{e^x}{2},$$

where $y=\dfrac{1}{3}+\dfrac{4}{135(x+k)}$ *where* k *lies between* $\dfrac{8}{45}$ *and* $\dfrac{2}{21}$.

This is an extension of a question proposed earlier by Ramanujan†, namely to prove that y lies between $\frac{1}{2}$ and $\frac{1}{3}$. A "partial solution" of this question was published by him‡, in the course of which he showed that $y=\frac{1}{2}$ when $x=0$ and that $y\to\frac{1}{3}$ when $x\to\infty$; and he remarked that "it is difficult to prove that y lies between $\frac{1}{2}$ and $\frac{1}{3}$".

With the help of analysis which resembles the analysis used by Ramanujan, I shall now give a complete proof§ that y actually lies between $\frac{1}{2}$ and $\frac{1}{3}$; and I shall also give reasons, which seem to me to be fairly convincing, for believing that k lies between 8/45 and 2/21.

In the preceding statements, x is naturally a positive integer or zero; it is, however, possible to define y in a simple manner as a

* *Collected papers of Srinivasa Ramanujan* (1927), xxvi.

† *Journal Indian Math. Soc.*, 3 (1911), 128.

‡ *Ibid.*, 4 (1911), 151–152.

§ [*Added*, 11 February, 1929.—Since this paper was written, a rather different proof that $\frac{1}{3}\leqslant y\leqslant\frac{1}{2}$ has been published by Szegö, *Journal London Math. Soc.*, 3 (1928), 225-232; I decided, however, not to alter my proof because it occupies only a small part of this paper, and it would not be very convenient to base my investigation of k on the analysis contained in Szegö's paper.]

function of a continuous positive variable x. When this is done, $\frac{1}{2}$ and $\frac{1}{3}$ are still the upper and lower bounds for y; but, for a certain range of values of x, namely $0<x<0{\cdot}0316$, *k is slightly greater than* $8/45$.

2. It is easy to see that, when x is a positive integer,

$$\begin{aligned} y &= \frac{e^x \cdot x!}{2x^x}+1-\frac{x!}{x^x}\left\{1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^x}{x!}\right\} \\ &= \frac{e^x}{2x^x}\int_0^\infty v^x e^{-v}\,dv+1-\frac{1}{x^x}\sum_{m=0}^{x} {}_xC_m\, x^m \int_0^\infty v^{x-m}\, e^{-v}\,dv \\ &= 1+\frac{x}{2}\int_0^\infty u^x e^{x(1-u)}\,du-x\int_0^\infty \left(1+\frac{v}{x}\right)^x e^{-v}\,dv \\ &= 1+\frac{x}{2}\left[\int_0^1 (ue^{1-u})^x\,du-\int_1^\infty (Ue^{1-U})^x\,dU\right], \end{aligned}$$

by writing ux for v in the first integral and $(U-1)x$ for v in the second integral.

We shall adopt the last expression as the definition of y when x is a continuous positive variable, *so that, whenever* $x>0$,

$$y=1+\frac{x}{2}\left[\int_0^1 (ue^{1-u})^x\,du-\int_1^\infty (Ue^{1-U})^x\,dU\right].$$

Our next step is to combine the two integrals which occur in this expression. As u increases from 0 to 1, ue^{1-u} increases steadily from 0 to 1; and as U increases from 1 to ∞, Ue^{1-U} decreases steadily from 1 to 0. Hence, if in *both* integrals we take a new variable t, defined by the equations

$$ue^{1-u}=e^{-t}, \quad Ue^{1-U}=e^{-t},$$

it is evident that

$$y=1-\frac{x}{2}\int_0^\infty e^{-xt}\left(\frac{dU}{dt}+\frac{du}{dt}\right)dt.$$

We shall henceforward use the symbols t, u, U in the sense just specified, so that, for any given positive value of t, u and U are functions of t which are determined uniquely by the conditions

$$ue^{1-u}=e^{-t}, \quad Ue^{1-U}=e^{-t}, \quad 0\leqslant u\leqslant 1\leqslant U.$$

Since I have not infrequently felt the need for Tables of u and U, *qua* functions of t, I have considered it desirable to compute the Tables of u and U which are given at the end of this paper.

3. We shall now study the properties of u and U, *qua* functions of t. It is easy to see that

$$\frac{dU}{dt} = \frac{U}{U-1}, \quad \frac{du}{dt} = -\frac{u}{1-u}, \quad \frac{d^2U}{dt^2} = -\frac{U}{(U-1)^3}, \quad \frac{d^2u}{dt^2} = \frac{u}{(1-u)^3}.$$

It is possible to express u, U and t as explicit functions of a parameter v by writing $U = ue^{2v}$, and then

$$u = v(\coth v - 1), \quad U = v(\coth v + 1),$$

$$t = v \coth v - 1 + \log \frac{\sinh v}{v},$$

but the complexity of the values of t and dt/dv in terms of v makes this representation comparatively useless for practical purposes.

It may also be remarked that, as a consequence of Legendre's deduction* from Bürmann's theorem,

$$e^{\alpha u} = 1 + \frac{\alpha u e^{-\beta u}}{1!} + \frac{\alpha(\alpha+2\beta)\,u^2 e^{-2\beta u}}{2!} + \frac{\alpha(\alpha+3\beta)^2 u^3 e^{-3\beta u}}{3!} + \ldots$$

(which is certainly valid when $0 \leqslant u < 1$), we have

$$u = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1} e^{-(n+1)(t+1)}}{n!},$$

by taking $\alpha = \beta = 1$; and, in fact, the latter part of the appended Table of values of u is indistinguishable from a Table of the function $e^{-(t+1)}$ which is the first term of this expansion for u.

We next examine the behaviour of u and U when t is small; it may be proved that the expansion of U in ascending powers of t is

$$U = 1 + (2t)^{\frac{1}{2}} + \frac{2t}{3} + \frac{(2t)^{\frac{3}{2}}}{36} - \frac{2t^2}{135} + \frac{(2t)^{\frac{5}{2}}}{4320} + \frac{4t^3}{8505}$$

$$- \frac{139(2t)^{\frac{7}{2}}}{5443200} + \frac{2t^4}{25515} - \frac{571(2t)^{\frac{9}{2}}}{2351462400} - \ldots,$$

and the corresponding expansion of u is derived from this series by changing the signs of the terms with fractional indices.

The order of magnitude of the coefficients in this expansion is of some importance. To investigate it, we put $t = \frac{1}{2}z^2$, and then

$$U = 1 + z + \tfrac{1}{3}z^2 + \ldots,$$

* A. M. Legendre, *Exercices de calcul intégral*, 2 (1816), 234.

while, if z is now regarded as a complex variable, U is one of the branches of w the many-valued function of z defined by the equation

$$w-1-\log w = \tfrac{1}{2}z^2.$$

The only branch points of w, *qua* function of z, are points at which dw/dz is zero or infinite, and, at all such points, $w=1$ so that $z^2 = 4n\pi i$ $(n = 1, 2, 3, \ldots)$.

Hence the expansions of U and u are convergent when

$$|z| < 2\sqrt{\pi}.$$

The only possible singularities of w on the circle of convergence $|z| = 2\sqrt{\pi}$ are consequently

$$2e^{\pm\frac{1}{4}\pi i}\sqrt{\pi}, \quad 2e^{\pm\frac{3}{4}\pi i}\sqrt{\pi};$$

now a simple computation shows that

$$1+z+\tfrac{1}{3}z^2+\ldots$$

is not equal to 1 when $z = 2e^{\pm\frac{1}{4}\pi i}\sqrt{\pi}$, so that, in point of fact, if z approaches the circle of convergence from inside, the points $2e^{\pm\frac{3}{4}\pi i}\sqrt{\pi}$ are the only singularities* of the branch U.

Near $z = 2e^{\frac{3}{4}\pi i}\sqrt{\pi}$, we have

$$U = 1-(2t+4\pi i)^{\frac{1}{2}}+\frac{2t+4\pi i}{3}-\ldots$$

$$= 1-2e^{\frac{1}{4}\pi i}\sqrt{\pi}\,.\left(1+\frac{t}{2\pi i}\right)^{\frac{1}{2}}+\ldots$$

$$= 1-2e^{\frac{1}{4}\pi i}\sqrt{(2\pi)}\,.\left(1-\frac{z}{2e^{\frac{3}{4}\pi i}\sqrt{\pi}}\right)^{\frac{1}{2}}+\ldots,$$

with a similar expansion (obtained by writing $-i$ for i throughout) near $z = 2e^{-\frac{3}{4}\pi i}\sqrt{\pi}$.

Hence, by the theory of functions of large numbers, due to Darboux†, the coefficient of z^n in the expansion of U is approximately equal to the coefficient of z^n in the expansion of

$$-2e^{\frac{1}{4}\pi i}\sqrt{(2\pi)}\,.\left(1-\frac{z}{2e^{\frac{3}{4}\pi i}\sqrt{\pi}}\right)^{\frac{1}{2}}-2e^{-\frac{1}{4}\pi i}\sqrt{(2\pi)}\,.\left(1-\frac{z}{2e^{-\frac{3}{4}\pi i}\sqrt{\pi}}\right)^{\frac{1}{2}},$$

* The other two points are, of course, branch points of $u = 1-z+\frac{1}{3}z^2-\ldots$.

† *Journal de Math.* (3), 4 (1878), 5–36, 377–416.

when n is large; this coefficient is equal to

$$(-)^n \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n)} \frac{\cos\frac{1}{4}(n+1)\pi}{2^{n-\frac{3}{2}}\pi^{\frac{1}{2}(n-1)}} \sim (-)^n \frac{\cos\frac{1}{4}(n+1)\pi}{2^{n-\frac{3}{2}}\pi^{\frac{1}{2}n}n\sqrt{n}},$$

by Stirling's theorem. Hence the coefficient of $t^{\frac{1}{2}n}$ in the expansion of U is approximately equal to

$$(-)^n \frac{\cos\frac{1}{4}(n+1)\pi}{2^{\frac{1}{2}(n-3)}\pi^{\frac{1}{2}n}n\sqrt{n}}.$$

4. Now that the expansions of u and U in ascending powers of t have been obtained, it is possible to investigate the asymptotic expansion of y. On integration by parts we have

$$y = 1+\tfrac{1}{2}\left[e^{-xt}\left(\frac{dU}{dt}+\frac{du}{dt}\right)\right]_0^\infty - \tfrac{1}{2}\int_0^\infty e^{-xt}\left(\frac{d^2U}{dt^2}+\frac{d^2u}{dt^2}\right)dt$$

$$= \tfrac{1}{3} - \tfrac{1}{2}\int_0^\infty e^{-xt}\left(\frac{d^2U}{dt^2}+\frac{d^2u}{dt^2}\right)dt.$$

The last integral is convergent when $x \geqslant 0$, whereas previous integrals have been convergent only when $x > 0$.

Now $d^2(U+u)/dt^2$ is bounded when $t \geqslant 0$; for, when t is small, it is expansible in a series of positive integral powers of t, convergent when $0 \leqslant t < 2\pi$; and, except when $t = 0$, it is expressible in the form

$$-\left\{\frac{U}{(U-1)^3}-\frac{u}{(1-u)^3}\right\}$$

which is obviously bounded when u and U are not nearly equal to 1.

It follows that the asymptotic expansion of y, when x is large, is obtained by substituting the expansion of $d^2(U+u)/dt^2$, namely

$$-\frac{8}{135}+\frac{16t}{2835}+\frac{16t^2}{8505}-\ldots,$$

in the last integral, and integrating term-by-term.

Hence, when x is large and positive,

$$y \sim \frac{1}{3}+\frac{4}{135x}-\frac{8}{2835x^2}-\frac{16}{8505x^3}+\ldots,$$

so that

$$\frac{1}{x+k} \sim \frac{1}{x}-\frac{2}{21x^2}-\frac{4}{63x^3}+\ldots,$$

and

$$k = \frac{2}{21}+\frac{32}{441x}+O\left(\frac{1}{x^2}\right).$$

It is evident from this result that k is greater than $2/21$ for all sufficiently large values of x.

5. In order to obtain definite bounds for y and k, as opposed to approximations, we need inequalities satisfied by $d^2(U+u)/dt^2$.

An inequality which is effective in dealing with y is

$$\frac{d^2U}{dt^2}+\frac{d^2u}{dt^2}\leqslant 0,$$

but, before proving it, we shall establish the more elementary inequality

$$\frac{dU}{dt}+\frac{du}{dt}\geqslant 0,$$

since the method required to prove it indicates the method which is appropriate for the former inequality.

(I) From the formulae of § 3 it is evident that the inequality

$$\frac{dU}{dt}+\frac{du}{dt}\geqslant 0$$

is true if $u+U-2uU\geqslant 0$, *i.e.* if

$$u\leqslant \frac{U}{2U-1}.$$

To prove that this is the case, define v by the equation

$$v=\frac{U}{2U-1}$$

and then we have to prove that $u\leqslant v$.

It is evident that

$$\log(ve^{1-v})-\log(ue^{1-u})=U-\frac{U}{2U-1}-\log(2U-1).$$

The function on the right vanishes when $U=1$; and, when $U>1$, its derivate, which is $8(U-1)^2/(2U-1)^2$, is positive. Hence the function itself is positive when $U>1$, so that

$$\log(ve^{1-v})\geqslant \log(ue^{1-u}).$$

Since ue^{1-u} is an increasing function of u, it follows from this inequality that $u\leqslant v$, and therefore we have proved that

$$\frac{dU}{dt}+\frac{du}{dt}\geqslant 0.$$

(II) The inequality involving second differential coefficients is rather more troublesome; it is evidently adequate to prove that

$$\frac{U}{(U-1)^3} \geqslant \frac{u}{(1-u)^3},$$

and for this purpose we shall prove that the function

$$\log \frac{U}{(U-1)^3} - \log \frac{u}{(1-u)^3}$$

(which vanishes when $t=0$) has a positive differential coefficient with respect to t when $t>0$.

It is easy to prove that

$$\frac{d}{dt}\left[\log \frac{U}{(U-1)^3} - \log \frac{u}{(1-u)^3}\right] = \frac{U}{U-1}\left\{\frac{1}{U} - \frac{3}{U-1}\right\} + \frac{u}{1-u}\left\{\frac{1}{u} + \frac{3}{1-u}\right\}$$

$$= \frac{(U-u)(2uU+u+U-4)}{(1-u)^2(U-1)^2},$$

and, to prove that this is positive, we have to show that

$$U > \frac{4-u}{2u+1}.$$

Let

$$V = \frac{4-u}{2u+1},$$

so that

$$\log (Ue^{1-U}) - \log (Ve^{1-V}) = \frac{4-u}{2u+1} - u + \log \frac{u(2u+1)}{4-u}.$$

The function on the right vanishes when $u=1$; and, when $u<1$, its derivate, which is

$$\frac{4(1-u)^4}{u(2u+1)^2(4-u)},$$

is positive. Hence the function itself is negative when $u<1$, that is to say

$$\log (Ue^{1-U}) < \log (Ve^{1-V}).$$

Since Ue^{1-U} is a decreasing function of U, it follows from this inequality that $V<U$, and consequently that

$$\frac{d}{dt}\left[\log \frac{U}{(U-1)^3} - \log \frac{u}{(1-u)^3}\right] > 0$$

when $t>0$; and this is a sufficient condition for the truth of the inequality

$$\frac{d^2U}{dt^2} + \frac{d^2u}{dt^2} \leqslant 0$$

when $t \geqslant 0$.

It does not seem to be practicable to treat higher differential coefficients of $U+u$ in this manner.

6. We now return to the expression for y; we define the function $\phi(t)$ by the equation

$$\phi(t) = \frac{135}{8}\left\{\frac{U}{(U-1)^3} - \frac{u}{(1-u)^3}\right\} = 1 - \frac{2t}{21} - \ldots,$$

so that

$$y = \frac{1}{3} + \frac{4}{135}\int_0^\infty e^{-xt}\phi(t)\,dt.$$

Since, as was proved in § 5, $\phi(t) \geqslant 0$, it is evident from this expression that y decreases steadily as x increases. Hence the value of y, for any positive value of x, lies between the limits to which y tends when $x \to 0$ and $x \to \infty$. Since these limits are $\frac{1}{2}$ and $\frac{1}{3}$, we have now established Ramanujan's preliminary theorem that $\frac{1}{2} \geqslant y \geqslant \frac{1}{3}$.

We next have to examine the value of k which is now given by the formula

$$\frac{1}{x+k} = \int_0^\infty e^{-xt}\phi(t)\,dt.$$

If the inequalities

$$\exp\left(-\frac{2t}{21}\right) \geqslant \phi(t) \geqslant \exp\left(-\frac{8t}{45}\right)$$

were true—or, what comes to the same thing, if the inequalities

$$\frac{2}{21} \leqslant -\frac{\log\phi(t)}{t} \leqslant \frac{8}{45}$$

were true—we should at once obtain the result that k lies between $2/21$ and $8/45$. But, in point of fact, *neither* of these inequalities holds for all positive values of t; for, as t increases from zero, $-t^{-1}\log\phi(t)$ increases from $2/21$, and it appears from a Table that the function attains a maximum value (when t is about 7·2) which is nearly equal to 0·2239, and this maximum is *greater* than $8/45$; and, as $t \to \infty$, $-t^{-1}\log\phi(t)$ behaves like $t^{-1}\log(8t^2/135)$ which is *less* than $2/21$.

Since the function $-t^{-1}\log\phi(t)$ seemed somewhat complicated to investigate analytically, it appeared preferable to tabulate it. For this purpose $\phi(t)$ is given to sufficient accuracy by the expression

$$\frac{135}{8}\left\{\frac{U}{(U-1)^3} - \frac{u}{(1-u)^3}\right\},$$

for all except small values of t; for values of t less than about 0·50 it was

found preferable to use the expansion of $\phi(t)$ in ascending powers of t. This expansion to a large number of terms was obtained by taking the formulae of § 3 :

$$u+U = 2v \coth v, \qquad t = v \coth v - 1 + \log \frac{\sinh v}{v},$$

expanding the expressions on the right in powers of v^2, reverting the second expansion so as to get v^2 in series of powers of t and substituting in the first expansion. The procedure, which involved considerable labour, seemed to be the simplest method of determining the series which, after two differentiations, gave the series for $\phi(t)$. It is adequate to quote here the first six terms, namely

$$\phi(t) = 1 - \frac{2t}{21} - \frac{2t^2}{63} + \frac{1124t^3}{280665} + \frac{10442t^4}{10945935} - \frac{21836t^5}{164189025} - \ldots.$$

It seems evident from the Table that $-t^{-1}\log\phi(t)$ has only one maximum in the range of the Table, and it is easy to prove that it has no maxima for values of t beyond the range of the Table; and it will appear in § 7 that the hypothesis that $-t^{-1}\log\phi(t)$ has only a single maximum is adequate for the purpose of obtaining the properties of k which we require. We now have to collect the available evidence in support of the hypothesis.

In the first place, we can prove that $-t^{-1}\log\phi(t)$ is certainly a decreasing function of t when t exceeds 8·4; for we have

$$-\frac{\log\phi(t)}{t} = \frac{1}{U-1-\log U}\log\left\{\frac{8(U-1)^3}{135U}\right\} + \frac{1}{t}\log\left\{1-\frac{u(U-1)^3}{U(1-u)^3}\right\}^{-1}.$$

Now
$$\frac{d}{dt}\left\{\frac{u(U-1)^3}{U(1-u)^3}\right\} = \frac{u(U-1)}{U(1-u)^5}(u-U)(2uU+u+U-4)$$
$$\leqslant 0,$$

by a result obtained in § 5, and so

$$\frac{1}{t}\log\left\{1-\frac{u(U-1)^3}{U(1-u)^3}\right\}^{-1}$$

is *always* a decreasing function of t. Moreover, the function

$$\frac{1}{U-1-\log U}\log\left\{\frac{8(U-1)^3}{135U}\right\}$$

is a decreasing function of U (and therefore also of t) when

$$\frac{2U+1}{(U-1)} + \frac{U^2-4U}{(U-1)^2}\log U - 3\log(U-1) + \log\frac{135}{8} < 0.$$

But the differential coefficient with respect to U of the function on the left has the negative value

$$-\frac{2U+4}{(U-1)^3}\{U-1-\log U\}$$

and so the function itself is negative when U exceeds the value for which the function is zero. It is found by trial that

$$\frac{2U+1}{U-1}+\frac{U^2-4U}{(U-1)^2}\log U-3\log(U-1)+\log\frac{135}{8}$$

vanishes when U is about 11·8 and t is about 8·4. Hence it is negative when $t>8{\cdot}4$, and so

$$\frac{1}{U-1-\log U}\log\left\{\frac{8(U-1)^3}{135U}\right\}$$

is a decreasing function of t when $t>8{\cdot}4$.

Hence $-t^{-1}\log\phi(t)$ is expressible as the sum of two functions which are both decreasing when $t>8{\cdot}4$, and therefore $-t^{-1}\log\phi(t)$ is certainly a decreasing function of t when $t>8{\cdot}4$.

When $0\leqslant t\leqslant 8{\cdot}4$, the values of $-t^{-1}\log\phi(t)$ shown in the Table indicate that the function varies continuously with t, the first and second differences being reasonably steady throughout the range of the Table; and there is no singularity of this function in the complex domain of t near this range of values of t to cause any abrupt variation of the function between successive entries. A singularity which could cause the function to have a maximum concealed between successive entries without producing any visible effect on the differences of the first four orders would be a singularity of a kind which could not be expected to occur in a function of so simple a type of function as the logarithmic-exponential type of which $-t^{-1}\log\phi(t)$ is a specimen.

It will therefore be *assumed* that the only maximum of $-t^{-1}\log\phi(t)$ in the interval $0\leqslant t\leqslant 8{\cdot}4$ is the maximum near $t=7{\cdot}2$, whose presence is evident from the Table.

With this assumption, the equation

$$-\frac{\log\phi(t)}{t}=\frac{2}{21}$$

has only one positive root (the root $t=0$ being, of course, excepted). This root is beyond the range of the Table; its value is about 56·3, and it will be called a.

Also the equation $$-\frac{\log \phi(t)}{t} = \frac{8}{45}$$

has only two positive roots b and c say $(b < c)$. The value of b is about 2·6 and the value of c, which is beyond the range of the Table, is about 18·4.

7. It is now possible to investigate the manner in which k varies as x increases from 0 to ∞.

Write $$\exp\left(-\frac{2}{21}t\right) - \phi(t) = f_1(t),$$

and $$\frac{1}{x+\frac{2}{21}} - \frac{1}{x+k} = \int_0^\infty e^{-xt} f_1(t)\, dt = \chi_1(x),$$

so that

$$f_1(t) \geqslant 0 \quad (t \leqslant a); \qquad f_1(t) \leqslant 0 \quad (t \geqslant a).$$

Then $$a\chi_1(x) + \frac{d\chi_1(x)}{dx} = \int_0^\infty e^{-xt}(a-t) f_1(t)\, dt \geqslant 0,$$

since the integrand is never negative.

Consequently $e^{ax}\chi_1(x)$ is an increasing function of x; and therefore, since its value when $x = 0$ is $\frac{21}{2} - \frac{45}{8}$, it is always positive.

Hence, on the hypothesis that $f_1(t) = 0$ has only one positive root, it has been proved that $k \geqslant 2/21$.

Next write $$\phi(t) - \exp\left(-\frac{8}{45}t\right) = f_2(t),$$

and $$\frac{1}{x+k} - \frac{1}{x+\frac{8}{45}} = \int_0^\infty e^{-xt} f_2(t)\, dt = \chi_2(x),$$

so that

$$f_2(t) \geqslant 0 \quad (t \leqslant b); \qquad f_2(t) \leqslant 0 \quad (b \leqslant t \leqslant c); \qquad f_2(t) \geqslant 0 \quad (t \geqslant c)$$

Then

$$bc\chi_2(x) + (b+c)\frac{d\chi_2(x)}{dx} + \frac{d^2\chi_2(x)}{dx^2} = \int_0^\infty e^{-xt}(b-t)(c-t) f_2(t)\, dt \geqslant 0,$$

since the integrand is never negative.

That is to say

$$\frac{d}{dx}\left[e^{bx}\left\{c\chi_2(x) + \frac{d\chi_2(x)}{dx}\right\}\right] \geqslant 0,$$

and so $$e^{bx}\left\{c\chi_2(x) + \frac{d\chi_2(x)}{dx}\right\}$$

is an increasing function of x; it is easy to see that it tends to $-\infty$ as $x \to 0$, and it tends to $+\infty$ as $x \to \infty$; it therefore has one (and only one) zero, say $x = x_0$.

Then

$$c\chi_2(x) + \frac{d\chi_2(x)}{dx} \leqslant 0 \quad (x \leqslant x_0); \qquad c\chi_2(x) + \frac{d\chi_2(x)}{dx} \geqslant 0 \quad (x \geqslant x_0);$$

that is to say

$$\frac{d}{dx}[e^{cx}\chi_2(x)] \leqslant 0 \quad (x \leqslant x_0); \qquad \frac{d}{dx}[e^{cx}\chi_2(x)] \geqslant 0 \quad (x \geqslant x_0);$$

Hence, as x increases from 0 to ∞, $e^{cx}\chi_2(x)$ decreases to a minimum at x_0 and then increases up to $+\infty$. Since $\chi_2(0) = 0$, this minimum value of $e^{cx}\chi_2(x)$ is negative, and therefore $e^{cx}\chi_2(x)$ has one (and only one) zero, say $x = x_1$, other than $x = 0$; and $x_1 > x_0$.

It is now evident that

$$\chi_2(x) < 0 \quad (0 < x < x_1); \qquad \chi_2(x) > 0 \quad (x > x_1).$$

Hence, on the hypothesis that $f_2(t) = 0$ has only two positive roots, it has been proved that

$$k > \frac{8}{45} \quad (0 < x < x_1); \qquad k < \frac{8}{45} \quad (x > x_1).$$

The only problem of interest which is now outstanding is the investigation of the magnitude of x_1. The values of k for integral values of x are given in the following Table :

x	k
0	0·1777778
1	0·1480979
2	0·1269317
3	0·1175621
4	0·1123994
5	0·1091546

x	k
6	0·1069334
7	0·1053200
8	0·1040960
9	0·1031360
10	0·1023631
∞	0·0952381

Since $k < 8/45$ when $x = 1$, it is evident that $x_1 < 1$ and that $k < 8/45$ for all values of x (integral or otherwise) which are greater than 1; and this establishes Ramanujan's theorem that k lies between $8/45$ and $2/21$ in the restricted case when x is an integer, on the hypothesis that $-t^{-1}\log\phi(t)$ has only one maximum.

In view of the fact that $x_1 < 1$, we proceed to investigate an asymptotic expansion of k which is valid for *small* values of x.

8. The modification of the original integral (§ 2) for y which is appropriate when x is small is

$$y = 1+x\int_0^1 e^{-xt}\,du - \frac{e^x\,\Gamma(x+1)}{2x^x}.$$

In this integral write

$$e^{-xt} = 1-\frac{xt}{1!}+\frac{x^2t^2}{2!}-\ldots+\frac{(-)^m x^m t^m}{m!}+\frac{(-)^{m+1}\theta x^{m+1}t^{m+1}}{(m+1)!},$$

where $0<\theta<1$, and we have the asymptotic expansion

$$\int_0^1 e^{-xt}\,du \sim \sum_{m=0}^{\infty}\frac{(-)^m x^m}{m!}\int_0^1 t^m\,du$$

$$\sim 1-\tfrac{1}{2}x+\tfrac{5}{12}x^2-\tfrac{11}{36}x^3+\ldots,$$

the error due to stopping at any term being of the same sign as, and numerically less than, the first term neglected.

Hence

$$y \sim 1-\frac{e^x\,\Gamma(x+1)}{2x^x}+x-\tfrac{1}{2}x^2+\tfrac{5}{12}x^3-\tfrac{11}{36}x^4+\ldots.$$

By expanding $\Gamma(x+1)$ and $e^{-x\log x}$ into series which converge when $0\leqslant x<1$, we deduce that

$$y \sim \tfrac{1}{2}+\tfrac{1}{2}x\log x+\tfrac{1}{2}(1+\gamma)x-\tfrac{1}{4}(x\log x)^2+\tfrac{1}{2}(1-\gamma)x^2\log x$$

$$+x^2(\tfrac{1}{2}\gamma-\tfrac{1}{4}\gamma^2-\tfrac{3}{4}-\tfrac{1}{24}\pi^2)+\ldots,$$

where γ is Euler's constant, and hence that

$$k \sim \frac{8}{45}-\frac{8x\log x}{15}-\frac{x(8\gamma+23)}{15}+\ldots,$$

the first term omitted being of the order of $(x\log x)^2$.

Now consider the expression

$$\frac{8x\log(1/x)-(8\gamma+23)x}{15}.$$

It vanishes when $x=0$; as x increases up to 0·01165 it increases up to a maximum equal to 0·006215, and then it decreases steadily for all larger values of x, vanishing when $x=0{\cdot}0316$. When x is as small as 0·0316, or smaller, the terms in k which are of order $(x\log x)^2$ are small compared with $(8\gamma+23)x/15$. Hence we infer that x_1 is in the neighbourhood of 0·03, and that k has one maximum value (and no more) which it attains when x is in the neighbourhood of 0·01, the maximum value of the excess of k over 8/45 being about 0·006.

Solutions of $ue^{1-u} = Ue^{1-U} = e^{-t}$, with values of $\phi(t) = \frac{135}{8}\left\{\frac{U}{(U-1)^5} - \frac{u}{(1-u)^3}\right\}$.

t	u	U	$\phi(t)$	$-t^{-1}\log_e\phi(t)$	t
0	1·0000000	1·0000000	1·0000000	0·0952381	0
0·02	0·8131051	1·2135497	0·9980826	0·0959634	0·02
0·04	0·7431713	1·3101147	0·9961399	0·0966882	0·04
0·06	0·6923808	1·3875128	0·9941723	0·0974124	0·06
0·08	0·6514587	1·4550189	0·9921799	0·0981359	0·08
0·10	0·6168168	1·5162212	0·9901628	0·0988588	0·10
0·12	0·5866172	1·5729578	0·9881214	0·0995809	0·12
0·14	0·5597691	1·6263195	0·9860558	0·1003024	0·14
0·16	0·5355627	1·6770161	0·9839662	0·1010232	0·16
0·18	0·5135055	1·7255401	0·9818529	0·1017432	0·18
0·20	0·4932394	1·7722498	0·9797161	0·1024624	0·20
0·22	0·4744943	1·8174154	0·9775559	0·1031808	0·22
0·24	0·4570605	1·8612463	0·9753727	0·1038984	0·24
0·26	0·4407716	1·9039093	0·9731666	0·1046152	0·26
0·28	0·4254928	1·9455391	0·9709380	0·1053310	0·28
0·30	0·4111131	1·9862469	0·9686869	0·1060460	0·30
0·32	0·3975398	2·0261252	0·9664138	0·1067600	0·32
0·34	0·3846947	2·0652525	0·9641187	0·1074731	0·34
0·36	0·3725108	2·1036956	0·9618020	0·1081852	0·36
0·38	0·3609307	2·1415122	0·9594639	0·1088963	0·38
0·40	0·3499042	2·1787524	0·9571047	0·1096064	0·40
0·42	0·3393876	2·2154601	0·9547245	0·1103154	0·42
0·44	0·3293423	2·2516739	0·9523237	0·1110234	0·44
0·46	0·3197339	2·2874281	0·9499026	0·1117302	0·46
0·48	0·3105321	2·3227532	0·9474613	0·1124359	0·48
0·50	0·3017096	2·3576767	0·9450001	0·1131405	0·50
0·52	0·2932416	2·3922231	0·9425194	0·1138439	0·52
0·54	0·2851061	2·4264148	0·9400193	0·1145461	0·54
0·56	0·2772829	2·4602719	0·9375001	0·1152471	0·56
0·58	0·2697536	2·4938128	0·9349622	0·1159469	0·58
0·60	0·2625015	2·5270542	0·9324057	0·1166454	0·60
0·62	0·2555113	2·5600119	0·9298310	0·1173426	0·62
0·64	0·2487688	2·5926997	0·9272383	0·1180385	0·64
0·66	0·2422612	2·6251307	0·9246280	0·1187331	0·66
0·68	0·2359764	2·6573169	0·9220002	0·1194263	0·68
0·70	0·2299032	2·6892697	0·9193552	0·1201181	0·70
0·72	0·2240315	2·7209991	0·9166934	0·1208086	0·72
0·74	0·2183517	2·7525151	0·9140151	0·1214976	0·74
0·76	0·2128548	2·7838264	0·9113204	0·1221851	0·76
0·78	0·2075325	2·8149415	0·9086098	0·1228712	0·78
0·80	0·2023770	2·8458682	0·9058834	0·1235559	0·80
0·82	0·1973811	2·8766139	0·9031416	0·1242390	0·82
0·84	0·1925379	2·9071854	0·9003846	0·1249205	0·84
0·86	0·1878411	2·9375893	0·8976128	0·1256005	0·86
0·88	0·1832845	2·9678316	0·8948264	0·1262790	0·88
0·90	0·1788626	2·9979181	0·8920258	0·1269558	0·90
0·92	0·1745699	3·0278542	0·8892112	0·1276310	0·92
0·94	0·1704014	3·0576450	0·8863828	0·1283046	0·94
0·96	0·1663523	3·0872954	0·8835411	0·1289765	0·96
0·98	0·1624180	3·1168101	0·8806863	0·1296468	0·98
1·00	0·1585943	3·1461932	0·8778186	0·1303153	1·00

Solutions of $ue^{1-u} = Ue^{1-U} = e^{-t}$, with values of $\phi(t) = \frac{135}{8}\left\{\frac{U}{(U-1)^3} - \frac{u}{(1-u)^3}\right\}$.

t	u	U	$\phi(t)$	$-t^{-1}\log_e \phi(t)$	t
1·0	0·1585943	3·1461932	0·8778186	0·1303153	1·0
1·1	0·1409992	3·2912750	0·8632985	0·1336316	1·1
1·2	0·1256364	3·4336131	0·8485020	0·1369024	1·2
1·3	0·1121586	3·5735632	0·8334657	0·1401252	1·3
1·4	0·1002877	3·7114125	0·8182259	0·1432977	1·4
1·5	0·0897971	3·8473908	0·8028183	0·1464179	1·5
1·6	0·0804998	3·9817118	0·7872781	0·1494836	1·6
1·7	0·0722401	4·1145229	0·7716393	0·1524930	1·7
1·8	0·0648867	4·2459704	0·7559352	0·1554442	1·8
1·9	0·0583281	4·3761751	0·7401977	0·1583358	1·9
2·0	0·0524691	4·5052415	0·7244575	0·1611661	2·0
2·1	0·0472278	4·6332609	0·7087439	0·1639338	2·1
2·2	0·0425334	4·7603135	0·6930847	0·1666378	2·2
2·3	0·0383241	4·8864702	0·6775062	0·1692768	2·3
2·4	0·0345463	5·0117939	0·6620329	0·1718500	2·4
2·5	0·0311529	5·1363409	0·6466880	0·1743565	2·5
2·6	0·0281025	5·2601618	0·6314927	0·1767957	2·6
2·7	0·0253585	5·3833019	0·6164666	0·1791671	2·7
2·8	0·0228887	5·5058025	0·6016278	0·1814701	2·8
2·9	0·0206646	5·6277010	0·5869926	0·1837045	2·9
3·0	0·0186606	5·7490314	0·5725756	0·1858702	3·0
3·1	0·0168544	5·8698248	0·5583900	0·1879670	3·1
3·2	0·0152256	5·9901097	0·5444474	0·1899950	3·2
3·3	0·0137565	6·1099124	0·5307578	0·1919544	3·3
3·4	0·0124309	6·2292571	0·5173299	0·1938454	3·4
3·5	0·0112345	6·3481659	0·5041711	0·1956685	3·5
3·6	0·0101544	6·4666597	0·4912873	0·1974240	3·6
3·7	0·0091791	6·5847575	0·4786833	0·1991125	3·7
3·8	0·0082983	6·7024772	0·4663628	0·2007343	3·8
3·9	0·0075027	6·8198353	0·4543285	0·2022910	3·9
4·0	0·0067838	6·9368474	0·4425818	0·2037825	4·0
4·1	0·0061343	7·0535279	0·4311235	0·2052099	4·1
4·2	0·0055473	7·1698904	0·4199533	0·2065742	4·2
4·3	0·0050167	7·2859475	0·4090703	0·2078763	4·3
4·4	0·0045371	7·4017112	0·3984729	0·2091172	4·4
4·5	0·0041036	7·5171928	0·3881588	0·2102980	4·5
4·6	0·0037116	7·6324027	0·3781249	0·2114197	4·6
4·7	0·0033572	7·7473510	0·3683681	0·2124835	4·7
4·8	0·0030368	7·8620470	0·3588844	0·2134906	4·8
4·9	0·0027470	7·9764997	0·3496695	0·2144422	4·9
5·0	0·0024849	8·0907174	0·3407189	0·2153395	5·0
5·1	0·0022479	8·2047032	0·3320276	0·2161837	5·1
5·2	0·0020336	8·3184795	0·3235906	0·2169762	5·2
5·3	0·0018397	8·4320386	0·3154024	0·2177181	5·3
5·4	0·0016643	8·5453922	0·3074576	0·2184108	5·4
5·5	0·0015057	8·6585469	0·2997503	0·2190555	5·5
5·6	0·0013622	8·7715088	0·2922750	0·2196536	5·6
5·7	0·0012324	8·8842839	0·2850256	0·2202063	5·7
5·8	0·0011150	8·9968776	0·2779964	0·2207150	5·8
5·9	0·0010088	9·1092954	0·2711814	0·2211809	5·9
6·0	0·0009127	9·2215423	0·2645747	0·2216053	6·0

Solutions of $ue^{1-u} = Ue^{1-U} = e^{-t}$, with values of $\varphi(t) = \frac{135}{8}\left\{\frac{U}{(U-1)^3} - \frac{u}{(1-u)^3}\right\}$.

t	u	U	$\varphi(t)$	$-t^{-1}\log_e \varphi(t)$	t
6·0	0·00091271	9·2215423	0·2645747	0·2216053	6.0
6·1	0·00082579	9·3336233	0·2581703	0·2219895	6·1
6·2	0·00074714	9·4455430	0·2519625	0·2223347	6·2
6·3	0·00067600	9·5573059	0·2459454	0·2226422	6·3
6·4	0·00061163	9·6689162	0·2401134	0·2229131	6·4
6·5	0·00055339	9·7803781	0·2344606	0·2231488	6·5
6·6	0 00050070	9·8916956	0·2289817	0·2233505	6·6
6·7	0·00045303	10·0028723	0·2236711	0·2235192	6·7
6·8	0·00040990	10·1139119	0·2185235	0·2236562	6·8
6·9	0·00037088	10·2248179	0·2135335	0·2237625	6·9
7·0	0·00033558	10·3355936	0·2086962	0·2238394	7·0
7·1	0·00030363	10·4462423	0·2040065	0·2238878	7·1
7·2	0·00027473	10·5567671	0·1994595	0·2239089	7·2
7·3	0·00024858	10·6671709	0·1950505	0·2239037	7·3
7·4	0·00022492	10·7774566	0·1907749	0·2238731	7·4
7·5	0·00020351	10·8876270	0·1866281	0·2238183	7·5
7·6	0·00018414	10·9976848	0·1826058	0·2237402	7·6
7·7	0·00016661	11·1076325	0·1787038	0·2236397	7·7
7·8	0·00015076	11·2174726	0·1749181	0·2235176	7·8
7·9	0·00013641	11·3272076	0·1712445	0·2233750	7·9
8·0	0·00012343	11·4368397	0·1676793	0·2232127	8·0
8·1	0·00011168	11·5463712	0·1642188	0·2230315	8·1
8·2	0·00010105	11·6558043	0·1608594	0·2228323	8·2
8·3	0·00009143	11·7651410	0·1575975	0·2226158	8·3
8·4	0·00008273	11·8743834	0·1544299	0·2223827	8·4
8·5	0·00007486	11·9835335	0·1513533	0·2221339	8·5
8·6	0·00006773	12·0925931	0·1483646	0·2218701	8·6
8·7	0·00006129	12·2015642	0·1454607	0·2215918	8·7
8·8	0·00005545	12·5104484	0·1426388	0·2212999	8·8
8·9	0·00005018	12·4192475	0·1398961	0·2209950	8·9
9·0	0·00004540	12·5279632	0·1372297	0·2206776	9·0
9·1	0·00004108	12·6365971	0·1346372	0·2203485	9·1
9·2	0·00003717	12·7451509	0·1321161	0·2200081	9·2
9·3	0·00003363	12·8536259	0·1296638	0·2196570	9·3
9·4	0·00003043	12·9620238	0·1272781	0·2192959	9·4
9·5	0·00002754	13·0703460	0·1249567	0·2189251	9·5
9·6	0·00002492	13·1785938	0·1226975	0·2185452	9·6
9·7	0·00002255	13·2867687	0·1204983	0·2181567	9·7
9·8	0·00002040	13·3948719	0·1183572	0·2177600	9·8
9·9	0·00001846	13·5029048	0·1162723	0·2173556	9·9
10·0	0·10001670	13·6108686	0·1142416	0·2169439	10·0
10·1	0·00001511	13·7187646	0·1122635	0·2165254	10·1
10·2	0·00001367	13·8265938	0·1103361	0·2161004	10·2
10·3	0·00001237	13·9343576	0·1084578	0·2156693	10·3
10·4	0·09001120	14·0420569	0·1066271	0·2152325	10·4
10·5	0·00001013	14·1496929	0·1048423	0·2147903	10·5
10·6	0·00000917	14·2572667	0·1031020	0·2143431	10·6
10·7	0·00000829	14·3647793	0·1014048	0·2138911	10·7
10·8	0·00000750	14·4722318	0·0997492	0·2134348	10·8
10·9	0·00000679	14·5796250	0·0981341	0·2129743	10·9
11·0	0·00000614	14·6869600	0·0965581	0·2125100	11·0

A PROBLEM IN COMBINATORIAL ANALYSIS GIVING THE DISTRIBUTION OF CERTAIN MOMENT STATISTICS

By J. WISHART.

[Received 15 December, 1927.—Read 12 January, 1928.]

1. *Statement of the problem.*

It is natural, in studying problems of statistical theory, to regard any experience or group of events as only a sample out of a larger population of events. This population we may assume (*a*) to be infinite, (*b*) to follow a known mathematical law, such as the normal. It may be specified by citing the appropriate moment coefficients; thus, if normal and univariate, the mean and the standard deviation are sufficient to specify a particular distribution. With a number of variates it is necessary to know, not only all the means and standard deviations, but also the coefficients of correlation between any two variates. Speaking generally, we require all the product moment coefficients of the second degree. Our only knowledge of such a population comes from the sample, which is of finite size and is subject to random sampling errors. From the sample we can calculate, in any agreed manner, estimates of the population moment coefficients, which will be more or less inexact. One naturally finds that more reliance can be placed on a large than on a small sample. But whatever its size it is unsafe to take a sample estimate as furnishing the required population value without determining the errors to which the estimates are subject. A complete knowledge regarding these can be obtained if the frequency distribution, in samples, of the various moment coefficients can be determined theoretically. Much modern research has lain in this direction, and the distributions of the mean, standard deviation, and correlation and regression coefficients are known. A very general result of this nature has been put forward in a recent paper*, and is as follows: Let σ_p, σ_q and ρ_{pq} be typical standard deviations and coefficient

* *Biometrika*, 20A (1928), 38.

of correlation of an n-variate population. Let Δ be the determinant

$$|\rho_{pq}| \quad (p, q = 1, 2, \ldots, n),$$

and let Δ_{pq} be the minor of ρ_{pq} in Δ. Let

$$a_{pq} = s_p s_q r_{pq} = \sum_1^N (x_p x_q)/N$$

be the product moment statistic of the p-th and q-th variates in a sample of size N from this population, x being the observed deviation from the mean. Then if we write

$$A_{pq} = A_{qp} = N \cdot \Delta_{pq}/2\sigma_p \sigma_q \Delta,$$

we find that the simultaneous distribution, in random samples of N, of the n variances (squared standard deviations) and the $\frac{1}{2}n(n-1)$ product moment coefficients, when the population sampled is normal, follows the law given by the differential relation

$$dp = \frac{|A_{11}, A_{22}, \ldots, A_{nn}|^{\frac{1}{2}(N-1)}}{(\sqrt{\pi})^{\frac{1}{2}n(n-1)}\,\Gamma(\frac{1}{2}N-\frac{1}{2})\,\Gamma(\frac{1}{2}N-1)\ldots\Gamma(\frac{1}{2}N-\frac{1}{2}n)}$$

$$\times \exp\left\{-\sum_{p,q=1}^{n}\sum A_{pq}a_{pq}\right\} |a_{11}, a_{22}, \ldots, a_{nn}|^{\frac{1}{2}(N-n-2)}\, da_{11}\, da_{12}\ldots da_{nn}, \quad (1)$$

in which the determinants are of the n-th order in both cases, with A_{pq} and a_{pq} respectively as typical elements.

A very great deal of information is contained within this general result. Thus the distribution, for one variate, of the mean and standard deviation, and, for two variates, of the correlation and regression coefficients, can be deduced directly from particular cases of (1)*. It seems likely that new distributions for three or more variates will in time follow. Meantime, as a preparatory step in the direction of elucidating the required information, all the independent moment coefficients, up to the fourth order and eight variates, were worked out. The method first used was a generalization of a formula of Romanovsky†, and will be illustrated for the special case of three variates. Let

$$\mu'\begin{pmatrix} j & q & p \\ & k & m \\ & & l \end{pmatrix}$$

represent the general product moment coefficient, about an arbitrary origin, of the j-th, k-th and l-th powers of the three variances, and the

* For historical references see the paper already cited.

† *Comptes rendus*, 180 (1925), 1898.

m-th, p-th and q-th powers of the three product moment statistics. Then

$$\mu'\begin{pmatrix} j & q & p \\ k & m & \\ l & & \end{pmatrix} = \left[\frac{\partial^{j+k+l+m+p+q}\,\phi(\alpha\beta\gamma\lambda\mu\nu)}{\partial\alpha^j\partial\beta^k\partial\gamma^l\partial\lambda^m\partial\mu^p\partial\nu^q}\right]_{\alpha=\beta=\gamma=\lambda=\mu=\nu=0} \tag{2}$$

and

$$\phi(\alpha\beta\gamma\lambda\mu\nu) = \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \\ A_{21}, & A_{22}, & A_{23} \\ A_{31}, & A_{32}, & A_{33} \end{vmatrix}^{\frac{1}{2}(N-1)} \cdot \begin{vmatrix} A_{11}-\alpha, & A_{12}-\tfrac{1}{2}\nu, & A_{13}-\tfrac{1}{2}\mu \\ A_{21}-\tfrac{1}{2}\nu, & A_{22}-\beta, & A_{23}-\tfrac{1}{2}\lambda \\ A_{31}-\tfrac{1}{2}\mu, & A_{32}-\tfrac{1}{2}\lambda, & A_{33}-\gamma \end{vmatrix}^{-\frac{1}{2}(N-1)}. \tag{3}$$

The general moment coefficient is thus expressed in terms of N, the size of the sample, and the following moment coefficients of the sampled population

$$\mu_{200} = \sigma_1^2, \qquad \mu_{020} = \sigma_2^2, \qquad \mu_{002} = \sigma_3^2,$$

$$\mu_{110} = \sigma_1\sigma_2\rho_{12}, \qquad \mu_{101} = \sigma_1\sigma_3\rho_{13}, \qquad \mu_{011} = \sigma_2\sigma_3\rho_{23}.$$

The corresponding general moment coefficients, when taken about their own mean value in samples as origin, will be designated by

$$\mu\begin{pmatrix} j & q & p \\ k & m & \\ l & & \end{pmatrix}.$$

It was soon evident, however, that, in all but the simplest cases, the labour attendant on the direct calculation by means of (2) of the 111 independent moments up to the fourth order would be very great. Moreover, the logical order of procedure is to calculate the general 4, 6 and 8 variate results

$$\mu\begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ & \cdot & \end{pmatrix}, \qquad \mu\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & 1 \\ & & \cdot & & \end{pmatrix}, \qquad \mu\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & \cdot & & 1 \\ & & & & & & \cdot \end{pmatrix},$$

from which, by associating the numbers 1 to 8 in all the possible ways, all the 2nd order, 3rd order, and 4th order formulae respectively can be reached. Such calculations would, however, be exceedingly difficult by the procedure outlined. At this stage the author was prompted, on the suggestion of Dr. R. A. Fisher, to consider his problem as a special case of the application, to general sampling theory, of the methods of combinatorial analysis. In what follows the correspondence between the two theories will be sketched, and the importance of the study of the partitions of unipartite and multipartite numbers in the theory of statistics will be indicated.

2. *Cumulative moment functions.*

The expressions which can be determined by an application of combinatorial analysis are not, however, the moments themselves, but the corresponding cumulative moment functions. These functions are identical with the semi-invariants of Thiele, and the applications to be discussed, together with the simplicity and elegance of the resulting formulae, combine to give an added importance to functions originally introduced for quite a different reason.

If the probability that a single sample value falls within the infinitesimal range dx is $\phi(x)dx$, so that

$$\int_{-\infty}^{\infty} \phi(x)\,dx = 1,$$

then the "characteristic function", or moment generating function, M, is defined by the relation

$$M = \int_{-\infty}^{\infty} e^{tx}\,\phi(x)\,dx.$$

This function is always well defined for all pure imaginary values of t*. Even should certain of the moments of the distribution, *i.e.*

$$\int_{-\infty}^{\infty} x^r\,\phi(x)\,dx,$$

become infinite, M is still well defined and $|M| \leqslant 1$ (since $|e^{tx}| = 1$). M is therefore absolutely convergent : it is, in fact, an integral of Stieltjes, which, when the moments as defined below have a meaning, can be expanded at the origin in the form

$$M = 1+\mu_1' t+\mu_2' \frac{t^2}{2!}+\mu_3' \frac{t^3}{3!}+\ldots$$

$$= e^{\mu_1' t}\left[1+\mu_2 \frac{t^2}{2!}+\mu_3 \frac{t^3}{3!}+\ldots\right],$$

where we have written, in the usual notation, μ_r' for

$$\int_{-\infty}^{\infty} x^r\,\phi(x)\,dx,$$

and μ_r for the corresponding moment about the mean of the distribution. If the logarithm of M can now be expanded in powers of t, we have

$$K = \log M = \kappa_1 t+\kappa_2 \frac{t^2}{2!}+\kappa_3 \frac{t^3}{3!}+\ldots$$

* See P. Lévy, *Calcul des probabilités* (Gautier-Villars, 1925), 161, 162, 172, *et seq.*

as the generating function of the cumulative moment functions κ. The κ's are determinate functions of the μ's, and can be obtained in terms of these by equating coefficients of like powers of t. It is sufficient for our present purpose to note that

$$\kappa_1 = \mu'_1, \qquad \kappa_2 = \mu_2, \qquad \kappa_3 = \mu_3, \qquad \kappa_4 = \mu_4 - 3\mu_2^2. \tag{4}$$

formulae which enable us to convert μ's into κ's and *vice versa*. Since the relations are invariant and independent of the nature of the distribution, the case where the logarithm of M cannot be so expanded need not be considered.

Similar relations exist when two or more variates fall to be considered. Thus, for a bi-variate system, corresponding to the *moment generating function*

$$\begin{aligned} M &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xt_1+yt_2}\,\phi(xy)\,dx\,dy \\ &= 1+\mu'_{10}t_1+\mu'_{01}t_2+\mu'_{20}\frac{t_1^2}{2!}+2\mu'_{11}\frac{t_1t_2}{2!}+\mu'_{02}\frac{t_2^2}{2!}+\mu'_{30}\frac{t_1^3}{3!}+3\mu'_{21}\frac{t_1^2t_2}{3!}+\dots \\ &= e^{\mu'_{10}t_1+\mu'_{01}t_2}\left[1+\mu_{20}\frac{t_1^2}{2!}+2\mu_{11}\frac{t_1t_2}{2!}+\mu_{02}\frac{t_2^2}{2!}+\mu_{30}\frac{t_1^3}{3!}+3\mu_{21}\frac{t_1^2t_2}{3!}+\dots\right], \end{aligned}$$

we have a *kappa generating function*

$$K = \log M = \kappa_{10}t_1+\kappa_{01}t_2+\kappa_{20}\frac{t_1^2}{2!}+2\kappa_{11}\frac{t_1t_2}{2!}+\kappa_{02}\frac{t_2^2}{2!}+\dots.$$

The μ's and the κ's agree up to the third order, by virtue of the origin being at the mean, while, for the fourth order,

$$\left.\begin{aligned} \kappa_{40} &= \mu_{40}-3\mu_{20}^2, & \kappa_{04} &= \mu_{04}-3\mu_{02}^2 \\ \kappa_{31} &= \mu_{31}-3\mu_{20}\mu_{11}, & \kappa_{13} &= \mu_{13}-3\mu_{02}\mu_{11} \\ & & \kappa_{22} &= \mu_{22}-\mu_{20}\mu_{02}-2\mu_{11}^2 \end{aligned}\right\}. \tag{5}$$

These formulae are, in fact, particular cases of the following one, which is all that need be remembered :

$$\kappa_{1111} = \mu_{1111}-\mu_{1100}\mu_{0011}-\mu_{1010}\mu_{0101}-\mu_{1001}\mu_{0110}. \tag{6}$$

Any number of noughts can be inserted in the suffixes of this formula, while by the association of two or more of the one's, all the special cases that are likely to arise, such as equations (5), can be deduced. Similar formulae give the μ's in terms of the κ's, but in this case all the terms are positive. Thus the reverse of (6) is

$$\mu_{1111} = \kappa_{1111}+\kappa_{1100}\kappa_{0011}+\kappa_{1010}\kappa_{0101}+\kappa_{1001}\kappa_{0110}. \tag{7}$$

The generating functions, introduced in this section, play a very useful rôle in the study of frequency distributions, since they present all the moments in an orderly scheme to the eye, and capable of transformation as required. Further, for the particular case of the normal distribution the kappa generating function degenerates into its first two terms, since all κ's beyond κ_2 vanish.

3. *Determination of sampling moments—two variates.*

Turning now to the determination of the cumulative moment functions corresponding to the moments given by (2), we may conveniently begin by writing down the well known results for the cumulative moment functions of the variance

$$\kappa(1) = (N-1)\,\kappa_2/N = (N-1)\,\sigma^2/N, \qquad \kappa(2) = 2(N-1)\,\kappa_2^2/N^2,$$
$$\kappa(3) = 8(N-1)\,\kappa_2^3/N^3, \qquad \kappa(4) = 48(N-1)\,\kappa_2^4/N^4,$$

the general result being

$$\kappa(s) = 2^{s-1}(s-1)!\,(N-1)\,\kappa_2^s/N^s,$$

which is directly derivable from the exact distribution of the variance from a normal sample

$$dp = A^{\frac{1}{2}(N-1)}e^{-Aa}a^{\frac{1}{2}(N-3)}\,da/\Gamma(\tfrac{1}{2}N-\tfrac{1}{2}),$$

in the notation of the early part of this paper.

A study of the rules* which govern the numerical coefficients in the case of moment statistics of any order derived in general from non-normal populations shows that the above coefficients should be regarded as the number of ways in which the multipartite numbers (2), (2^2), (2^3), ..., (2^s) can be partitioned into s parts of two units each; the restriction that we shall consider only those partitions in which the elements of the multipartite number are linked together to form a complete ring guarantees the correctness of the numerical coefficients. It is only in the case of the cumulative moment functions that these simple coefficients, both the numerical ones and those in terms of N, occur; while the actual form of the general coefficient indicates that it is these functions, and these alone, which are determined by the partitional problem about to be described. In this representation the problem of finding, for two variates, the coefficient of

* I owe the use of certain empirical conclusions which are made at this stage to information derived from Dr. R. A. Fisher, who felt justified in drawing them in consequence of some investigations which he has made but which are as yet unpublished. A general demonstration of these rules has since been obtained by Dr. Fisher, and will appear shortly in these *Proceedings*.

$(N-1)\,\kappa_{20}^{p}\kappa_{11}^{q}\kappa_{02}^{r}/N^{s}$ in $\kappa\begin{pmatrix} l & m \\ & n \end{pmatrix}$ will be equivalent to the problem of finding the number of ways of arranging in a ring l rods, of which both ends are black, n, of which both are white, and m with one black and one white end, in such a way that there are p junctions black to black, r junctions white to white, and q junctions black to white. Evidently $p+q+r=l+m+n=s$, and also $p-r=l-n$, so that the complete expression for $\kappa\begin{pmatrix} l & m \\ & n \end{pmatrix}$ will consist of a single series of terms with coefficients determined by the combinatorial problem, their sum being $2^{s-1}(s-1)!$.

To give two simple examples, $\kappa\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ will be found by considering the possible arrangements in a ring of three rods, black-black, white-white, and black-white respectively; only two essentially different arrangements are possible, as shown below:

(a) (b)

Each of these can be set up in four ways; arrangement (a) has only one unlike junction and supplies the term

$$4\,\frac{N-1}{N^3}\,\kappa_{20}\kappa_{11}\kappa_{02};$$

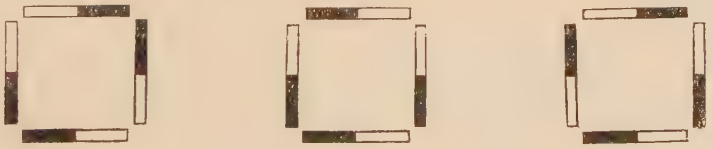
arrangement (b) has three unlike junctions and supplies the term

$$4\,\frac{N-1}{N^3}\,\kappa_{11}^{3}.$$

The intuitive solution of the combinatorial problem thus leads to the complete expression

$$\kappa\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = 4\,\frac{N-1}{N^3}\,\kappa_{11}(\kappa_{20}\kappa_{02}+\kappa_{11}^{2}) = 4\,\frac{N-1}{N^3}\,\sigma_1^3\sigma_2^3\rho(1+\rho^2).$$

As an example with four rods, consider $\kappa\begin{pmatrix} \cdot & 4 \\ & \cdot \end{pmatrix}$; here the rods are all black-white, and three essentially different arrangements are possible:



(a) (b) (c)

Of these (*a*) may be set up in six ways, (*b*) in thirty-six ways, and (*c*) in six ways, giving

$$\kappa\begin{pmatrix}\cdot & 4\\ \cdot & \end{pmatrix} = 6\,\frac{N-1}{N^4}\,(\kappa_{20}^2\kappa_{02}^2+6\kappa_{20}\kappa_{02}\kappa_{11}^2+\kappa_{11}^4)$$

$$= 6\,\frac{N-1}{N^4}\,\sigma_1^4\sigma_2^4(1+6\rho^2+\rho^4).$$

These two results are identical with those found from a direct application of (2), correcting in the usual way for the mean.

4. *Operational solution.*

The foregoing rules enable us to determine the simplest of the required formulae very quickly. It then becomes evident that higher order results can be deduced from those already calculated simply by considering the number of ways in which an additional rod can be inserted into the ring. Suppose that we desire to increase l by unity, leaving all else unchanged in a moment of the s-th order. A rod can be introduced in two ways at each of the s junctions, showing that the sum of the numerical coefficients is multiplied by $2s$. The effect it has upon the character of the result depends on the particular junction. Inserted between two blacks, it has the effect of wiping out the black-black junction and adding two new ones. If placed between two whites, the white-white junction is eliminated and two black-white junctions are substituted. Finally, if the old junction is black-white this is replaced by a black-black and a black-white. Remembering that the new rod may be turned round, we have the whole change described symbolically by means of the following operator :

(*a*) *To increase l by unity* :

$$\Omega\begin{pmatrix}1 & \cdot\\ \cdot & \end{pmatrix} = 2\kappa_{20}^2\,\frac{d}{d\kappa_{20}}+2\kappa_{20}\kappa_{11}\,\frac{d}{d\kappa_{11}}+2\kappa_{11}^2\,\frac{d}{d\kappa_{02}}.$$

(*b*) Similarly, *to increase n by unity* :

$$\Omega\begin{pmatrix}\cdot & \cdot\\ 1 & \end{pmatrix} = 2\kappa_{11}^2\,\frac{d}{d\kappa_{20}}+2\kappa_{02}\kappa_{11}\,\frac{d}{d\kappa_{11}}+2\kappa_{02}^2\,\frac{d}{d\kappa_{02}}.$$

Finally, if the rod to be inserted is black-white, a double-black junction is replaced by a double-black and a black-white, a double-white junction becomes a double-white plus a black-white; but for the black-white junction there are two possibilities, according to the way in which the rod is inserted. In one case black-white is eliminated and

black-black, white-white substituted; and in the other two white-blacks are formed from one. This corresponds to the operator

(*c*) *To increase m by unity*:

$$\Omega\begin{pmatrix}\cdot & 1\\ \cdot & \end{pmatrix} = (\kappa_{20}\kappa_{02}+\kappa_{11}^2)\frac{d}{d\kappa_{11}}+2\kappa_{20}\kappa_{11}\frac{d}{d\kappa_{20}}+2\kappa_{02}\kappa_{11}\frac{d}{d\kappa_{02}}$$

$$= (\kappa_{20}\kappa_{02}+\kappa_{11}^2)\frac{d}{d\kappa_{11}}+2\kappa_{11}\sqrt{(\kappa_{20}\kappa_{02})}\,\frac{d}{d\sqrt{(\kappa_{20}\kappa_{02})}}.$$

The operators (*a*), (*b*) and (*c*) enable us without difficulty to write down quite complex results. The arithmetical procedure is simple, and is best exhibited as in the following illustrative scheme, where $\kappa\begin{pmatrix}4 & 5\\ & 3\end{pmatrix}$ is evaluated from the simple result

$$\kappa\begin{pmatrix}4 & \cdot\\ 1 & \end{pmatrix} = 2^4.4!\,\frac{N-1}{N^5}\,\kappa_{20}^3\kappa_{11}^2 = 2^4.4!\,\frac{N-1}{N^5}\,\sigma_1^8\sigma_2^2\rho^2.$$

κ	Coefft.		Index of κ_{11}.											
			0	1	2	3	4	5	6	7	8	9	10	11
$\begin{pmatrix}4 & 0\\ 1 & \end{pmatrix}$	$2^7.3$	(i)			1									
$\begin{pmatrix}4 & 0\\ 2 & \end{pmatrix}$	$2^8.3$				2		3							
$\begin{pmatrix}4 & 0\\ 3 & \end{pmatrix}$	$2^{10}.3^2$				1		3		1					
$\begin{pmatrix}4 & 1\\ 3 & \end{pmatrix}$	$2^{11}.3^2$	(ii)		1		12		18		4				
$\begin{pmatrix}4 & 2\\ 3 & \end{pmatrix}$	$2^{11}.3^2$		1		51		246		226		36			
$\begin{pmatrix}4 & 3\\ 3 & \end{pmatrix}$	$2^{14}.3^3.5$			1		15		40		25		3		
$\begin{pmatrix}4 & 4\\ 3 & \end{pmatrix}$	$2^{14}.3^3.5$		1		64		455		775		352		33	
$\begin{pmatrix}4 & 5\\ 3 & \end{pmatrix}$	$2^{15}.3^3.5$			75		1550		6420		7608		2629		198

$$\kappa\begin{pmatrix}4 & 5\\ 3 & \end{pmatrix} = 2^{15}.3^3.5\,\frac{N-1}{N^{12}}\,\kappa_{20}\kappa_{11}(75\kappa_{20}^5\kappa_{02}^5+1550\kappa_{20}^4\kappa_{02}^4\kappa_{11}^2+6420\kappa_{20}^3\kappa_{02}^3\kappa_{11}^4$$
$$+7608\kappa_{20}^2\kappa_{02}^2\kappa_{11}^6+2629\kappa_{20}\kappa_{02}\kappa_{11}^8+198\kappa_{11}^{10})$$
$$= 2^{15}.3^3.5\,\frac{N-1}{N^{12}}\,\sigma_1^{13}\sigma_2^{11}\rho(75+1550\rho^2+6420\rho^4+7608\rho^6$$
$$+2629\rho^8+198\rho^{10}.$$

(i) Here we apply formula (*b*). The first stage produces two terms, with coefficients 2×2 and 2×3. These are doubled in the next stage and added to terms with coefficients 2×2, $3(2\times 4)$, and $3(2\times 2)$. The nature of each term is indicated by the column in which it is situated.

(ii) The application of the operator (*c*) is equivalent to multiplying diagonally to the left by the index of κ_{11}, and then diagonally to the right by a complementary series, such that the sum of the multipliers of any one term is equal to twice the sum of l, m, and n in that line. A common factor is divided out wherever it occurs.

It is evident that the entire scheme of product moments may be built up in the way illustrated, with the minimum of labour, where the direct application of formula (2) would be tedious. In fact, the labour attendant on the methods hitherto employed has, of necessity, confined attention solely to the direct moments and to the simplest of the product moments. In the case of the more general results for non-normal populations, too, it has been difficult in the past to check the results reached and detect errors*. On the other hand, the use of the equivalent partitional problem is an immense addition to our practical knowledge of sampling moments, since it makes available any required moment of the distribution (1).

5. *Extension to three or more variates.*

When the sampled population has three variates, the ring arrangement will be built up of rods whose ends may be of three different colours, *e.g.* black, white, red. The *ensemble* of colours in any calculation will be determined by the particular sample moment which it is desired to evaluate, and, as before, any arrangement will specify, by means of the paired colours at the junctions, a particular combination of powers of the six population moment coefficients, thus determining one term of the required κ. The sum of the coefficients of all the terms that can arise, from consideration of all the possible arrangements, will again be $2^{s-1}(s-1)!$, where there are s rods. To give one example

$$\kappa\begin{pmatrix} 1 \cdot 1 \\ \cdot 1 \end{pmatrix} = 2\,\frac{N-1}{N^3}\,(\kappa_{200}\kappa_{002}\kappa_{110}+\kappa_{200}\kappa_{101}\kappa_{011}+2\kappa_{110}\kappa_{101}^2)$$

$$= 2\,\frac{N-1}{N^3}\,\sigma_1^3\sigma_2\sigma_3^2(\rho_{12}+\rho_{13}\rho_{23}+2\rho_{12}\rho_{13}^2),$$

* Tchouproff, *Biometrika*, 12 (1919), 193-4; Church, *Ibid.*, 17 (1925), 79.

the three possible arrangements of the rods being given below:

(*a*) (*b*) (*c*)

(*a*) may be set up in two ways, (*b*) in two ways, and (*c*) in four ways.

Consideration of the possible number of ways of introducing a new rod leads to the following operators, which can be used to deduce higher order results.

(*a*) *To increase double-black by unity.*

$$\Omega\begin{pmatrix}1 & \cdot & \cdot\\ \cdot & & \\ \cdot & & \end{pmatrix} = 2\kappa_{200}^2\frac{d}{d\kappa_{200}} + 2\kappa_{200}\kappa_{110}\frac{d}{d\kappa_{110}} + 2\kappa_{200}\kappa_{101}\frac{d}{d\kappa_{101}} + 2\kappa_{110}^2\frac{d}{d\kappa_{020}} + 2\kappa_{101}^2\frac{d}{d\kappa_{002}} + 2\kappa_{110}\kappa_{101}\frac{d}{d\kappa_{011}}.$$

(*b*) *and* (*c*) *To increase double white or double red by unity.*

Similar to (*a*), and got from it by permuting the three suffixes.

(*d*) *To increase black-white by unity.*

$$\Omega\begin{pmatrix}\cdot & 1 & \cdot\\ \cdot & & \\ \cdot & & \end{pmatrix} = (\kappa_{200}\kappa_{020}+\kappa_{110}^2)\frac{d}{d\kappa_{110}} + (\kappa_{200}\kappa_{011}+\kappa_{110}\kappa_{101})\frac{d}{d\kappa_{101}} + (\kappa_{020}\kappa_{101}+\kappa_{110}\kappa_{011})\frac{d}{d\kappa_{011}} + 2\kappa_{200}\kappa_{110}\frac{d}{d\kappa_{200}} + 2\kappa_{020}\kappa_{110}\frac{d}{d\kappa_{020}} + 2\kappa_{101}\kappa_{011}\frac{d}{d\kappa_{002}}.$$

(*e*) and (*f*) *To increase white-red or red-black by unity.*

Obtained from (*d*) by permuting the three suffixes.

Generally, we may deal with the multi-variate case by supposing there to be as many different colours as there are variates. We are then enabled to present the general second, third, and fourth order cumulative moment functions by going up to a maximum of eight variates. It will make the exact correspondence between the two problems clear if we

write down these general results :

$$\kappa\begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \frac{N-1}{N^2}(\kappa_{1010}\kappa_{0101}+\kappa_{1001}\kappa_{0110}) = \frac{N-1}{N^2}\sigma_1\sigma_2\sigma_3\sigma_4(\rho_{13}\rho_{24}+\rho_{14}\rho_{23}), \tag{8}$$

$$\kappa\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \frac{N-1}{N^3}\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6[\rho_{13}(\rho_{25}\rho_{46}+\rho_{26}\rho_{45})+\rho_{14}(\rho_{25}\rho_{36}+\rho_{26}\rho_{35})$$
$$+\rho_{15}(\rho_{23}\rho_{46}+\rho_{24}\rho_{36})+\rho_{16}(\rho_{23}\rho_{45}+\rho_{24}\rho_{35})], \tag{9}$$

$$\kappa\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \frac{N-1}{N^4}\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8$$

$$\times[(\rho_{13}\rho_{25}+\rho_{15}\rho_{23})(\rho_{47}\rho_{68}+\rho_{48}\rho_{67})+(\rho_{13}\rho_{26}+\rho_{16}\rho_{23})(\rho_{47}\rho_{58}+\rho_{48}\rho_{57})$$
$$+(\rho_{13}\rho_{27}+\rho_{17}\rho_{23})(\rho_{45}\rho_{68}+\rho_{46}\rho_{58})+(\rho_{13}\rho_{28}+\rho_{18}\rho_{23})(\rho_{45}\rho_{67}+\rho_{46}\rho_{57})$$
$$+(\rho_{14}\rho_{25}+\rho_{15}\rho_{24})(\rho_{37}\rho_{68}+\rho_{38}\rho_{67})+(\rho_{14}\rho_{26}+\rho_{16}\rho_{24})(\rho_{37}\rho_{58}+\rho_{38}\rho_{57})$$
$$+(\rho_{14}\rho_{27}+\rho_{17}\rho_{24})(\rho_{35}\rho_{68}+\rho_{36}\rho_{58})+(\rho_{14}\rho_{28}+\rho_{18}\rho_{24})(\rho_{35}\rho_{67}+\rho_{36}\rho_{57})$$
$$+(\rho_{15}\rho_{27}+\rho_{17}\rho_{25})(\rho_{36}\rho_{48}+\rho_{38}\rho_{46})+(\rho_{15}\rho_{28}+\rho_{18}\rho_{25})(\rho_{36}\rho_{47}+\rho_{37}\rho_{46})$$
$$+(\rho_{16}\rho_{27}+\rho_{17}\rho_{26})(\rho_{35}\rho_{48}+\rho_{38}\rho_{45})+(\rho_{16}\rho_{28}+\rho_{18}\rho_{26})(\rho_{35}\rho_{47}+\rho_{37}\rho_{45})]. \tag{10}$$

Results (8) and (9) are equal to their corresponding μ's, while (10) can be converted, using (7), by adding the term

$$\frac{(N-1)^2}{N^4}\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8$$
$$\times[(\rho_{13}\rho_{24}+\rho_{14}\rho_{23})(\rho_{57}\rho_{68}+\rho_{58}\rho_{67})+(\rho_{15}\rho_{26}+\rho_{16}\rho_{25})(\rho_{37}\rho_{48}+\rho_{38}\rho_{47})$$
$$+(\rho_{17}\rho_{28}+\rho_{18}\rho_{27})(\rho_{35}\rho_{46}+\rho_{36}\rho_{45})].$$

These results, particularly the higher order ones, are so complex that no practical statistician would dream of computing them. They are given here, as certain analogous results were once given by Isserlis*, in order that the simpler results may be deduced by associating the numbers 1 to 8 in all possible ways. Thus, to select a third order result at

* *Biometrika*, 12 (1918), 183.

random, if we let $5 = 6 = 1$, in (9), we get

$$\kappa\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & 1 & \\ \cdot & & & \end{pmatrix} = 2\,\frac{N-1}{N^3}\,\sigma_1^3\sigma_2\sigma_3\sigma_4(\rho_{13}\rho_{24}+\rho_{14}\rho_{23}+2\rho_{12}\rho_{13}\rho_{14}).$$

Again, the product moment coefficient of the four variances, got from (10) by putting $5 = 1$, $6 = 2$, $7 = 3$, and $8 = 4$, and adjusting by (7), is

$$\mu\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ & 1 & \cdot & \cdot \\ & & 1 & \cdot \\ & & & 1 \end{pmatrix} = 4\,\frac{N-1}{N^4}\,\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\,[4(\rho_{12}\rho_{13}\rho_{24}\rho_{34}+\rho_{12}\rho_{14}\rho_{23}\rho_{34}+\rho_{13}\rho_{14}\rho_{23}\rho_{24}) + (N-1)(\rho_{12}^2\rho_{34}^2+\rho_{13}^2\rho_{24}^2+\rho_{14}^2\rho_{23}^2)].$$

But consideration of the partitional method obviates this indirect method of attack. For instead of working out the general result for all colours different and then associating two or more of these, it is easier to associate the colours first and then consider the simpler ring arrangement that now awaits solution. The utility of the method obviously lies along these lines.

6. *General considerations.*

The foregoing has dealt with one specific problem only, namely, the distribution of the second order moment statistics from a normal population. The correspondence with equivalent problems in the domain of combinatorial analysis has been indicated. It has since been established that the general problem for non-normal populations, and for moments higher than the second in the case of normal populations, is also intimately connected with combinatorial theory, and the special arrangement of rods in a ring has been of service by analogy in suggesting the more complicated forms of the correspondence between the two theories.

CONDITIONALLY CONVERGENT DOUBLE SERIES

By W. L. FERRAR.

[Received 20 March, 1928.—Read 26 April, 1928.]

1. *Introduction.*

IT has recently been proved by J. M. Whittaker* that "the necessary and sufficient condition that the series

$$\sum_{r=0}^{\infty} \frac{a_r}{x-r}$$

should converge uniformly over any finite domain of the variable x which excludes points $x = r$, is that the series $\Sigma a_r/r$ should converge".

The theorem is of some importance in the theory of the cardinal function of interpolation over a set of points $x = r$. The present paper provides the first step in the convergence theory of the corresponding method of interpolating over a set of points $z = m\omega_1 + n\omega_2$, where ω_1, ω_2 are given complex numbers. Apart from this context, the principal result obtained is of some interest as a theorem on the convergence of double series.

By "bounded convergence" of the double series $\Sigma u_{m,n}$, we mean (i) the convergence, in Pringsheim's sense, of the double series, together with (ii) the boundedness of its partial sums, *i.e.* $|s_{p,q}| < K$.

By "regular convergence" of the double series $\Sigma u_{m,n}$, we mean (i) the convergence, in Pringsheim's sense, of the double series, together with (ii) the separate convergence of each of its rows and columns. It may be noted that regular† convergence implies

"sums by rows = sums by columns = sum in Pringsheim's sense"

The theorems to be proved may now be stated thus:

THEOREM I. *If ω_1, ω_2 are complex numbers whose ratio is not real, then the necessary and sufficient condition for the bounded convergence, uniformly over any finite domain of the complex variable z which ex-*

* *Proc. Edinburgh Math. Soc.* (2), 1 (1927), 42. The theorem there given includes summability (C, k).

† The term is due to Hardy, *Proc. Camb. Phil. Soc.*, 19 (1917), 86–95.

cludes points $z = -m\omega_1 - n\omega_2$, *of the series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{m,n}}{z+m\omega_1+n\omega_2}, \tag{1.1}$$

is the uniform bounded convergence of the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{m,n}}{m\omega_1+n\omega_2}. \tag{1.2}$$

THEOREM II. *A similar theorem with regular convergence replacing bounded convergence, but with the additional proviso that the* $a_{m,n}$ *are either independent of* z *or are continuous functions of* z *in the domain considered.*

The interest of the theorems lies in the two facts :

(1) The condition is necessary and sufficient, so that, with a few changes of wording, the proofs establish that "any series of type (1.1), if it converge regularly (boundedly) at all must converge regularly (boundedly) over any finite domain which excludes points

$$z = -m\omega_1 - n\omega_2".$$

(2) An analytic function is here defined by a double series which need not be absolutely convergent.

We first find (§ 2) for double series of complex terms, "bounded convergence" tests which correspond to the well known Abel and Dirichlet tests for single series. We then show (§§ 3, 4) that these tests are sufficient to establish Theorem I, and that Hardy's tests for regular convergence are sufficient to establish Theorem II.

2. *Abel and Dirichlet tests for bounded convergence.*

2.1. THEOREM III. ABEL'S TEST. *If the convergence of* $\Sigma a_{m,n}$ *is bounded, and* $v_{m,n}$ *are numbers such that, if*

$$_q\Delta_m \equiv v_{m,q} - v_{m+1,q},$$

$$_p\Delta_n \equiv v_{p,n} - v_{p,n+1},$$

$$\Delta_{m,n} \equiv v_{m,n} - v_{m+1,n} - v_{m,n+1} + v_{m+1,n+1},$$

(i) $|v_{m,n}| < K$ *for all* m *and* n,

(ii) $\sum_{m,n}|\Delta_{m,n}|$ *is convergent,*

(iii) $\sum_{m=1}^{\infty}|{}_q\Delta_m| = w_q \to 0$ *as* $q \to \infty$,

(iv) $\sum_{n=1}^{\infty}|{}_p\Delta_n| = W_p \to 0$ *as* $p \to \infty$,

then the series $\Sigma a_{m,n}v_{m,n}$ *is of bounded convergence.*

Now, the terms $a_{m,n}v_{m,n}$ being understood in each sum, the first Σ referring to m, the second to n,

$$(2.11)\qquad \sum_{1}^{p+h}\sum_{1}^{q+k} - \sum_{1}^{p}\sum_{1}^{q} = \sum_{1}^{p}\sum_{q+1}^{q+k} + \sum_{p+1}^{p+h}\sum_{1}^{q} + \sum_{p+1}^{p+h}\sum_{q+1}^{q+k} = A+B+C, \text{ say,}$$

and the convergence of the series $\Sigma a_{m,n}v_{m,n}$ is established if the hypotheses ensure that $|A+B+C|$ is arbitrarily small for sufficiently large values of p and q. Our procedure is to deal separately with A, B, C by means of an extended form of Abel's identity, first given by Hardy*, namely,

$$(2.12)\qquad \sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n}v_{m,n} = \sum_{m=1}^{p-1}\sum_{n=1}^{q-1}\Delta_{m,n}s_{m,n} + \sum_{m=1}^{p-1}{}_{q}\Delta_{m}s_{m,q} + \sum_{n=1}^{q-1}{}_{p}\Delta_{n}s_{p,n} + v_{p,q}s_{p,q}.$$

We shall use $s_{p,q}$, $\sigma_{p,q}$ to denote

$$\sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n}, \qquad \sum_{m=1}^{p}\sum_{n=1}^{q}|\Delta_{m,n}|,$$

respectively; s, σ will denote the sums of the corresponding infinite series.

2.2. *The sum A.*

By hypothesis, there is a finite constant† K, such that

$$(2.21)\qquad |s_{p,q}| < K \text{ for all } p \text{ and } q,$$

$$(2.22)\qquad \Big|\sum_{\mu+1}^{\lambda}\sum_{\nu+1}^{\pi} a_{m,n}\Big| = |s_{\lambda,\pi} - s_{\mu,\pi} - s_{\lambda,\nu} + s_{\mu,\nu}| < 4K.$$

Also each of the numbers

$$w_q, \quad W_p, \quad |s_{p+h,q+k} - s_{p,q}|, \quad |\sigma_{p+h,q+k} - \sigma_{p,q}|,$$

is less than a given positive ϵ if both p and q are greater than some $N(\epsilon)$.

We shall, from now on, suppose $p-1$, $q-1$ each to be greater than $N(\epsilon)$. Let $a_{m,q+n} = a_{m,n}$ and $v_{m,q+n} = \xi_{m,n}$. Then, using (2.12), we have

$$A = \sum_{m=1}^{p}\sum_{n=1}^{k} a_{m,n}\xi_{m,n}$$

$$= \sum_{m=1}^{p-1}\sum_{n=1}^{q-1}\Delta'_{m,n}s'_{m,n} + \sum_{m=1}^{p-1}{}_{k}\Delta'_{m}s'_{m,k} + \sum_{n=1}^{q-1}{}_{p}\Delta'_{n}s'_{p,n} + \xi_{p,k}s'_{p,k},$$

where Δ' and s' are derived from ξ and a respectively.

* *Proc. London Math. Soc.* (2), 1 (1903), 124–8.

† We use K to denote any finite constant, independent of z, m, n, ϵ. It is not, of course, the same constant in each place where it occurs.

But $\Delta'_{m,n} = \Delta_{m,q+n}$, etc., and, using (2.21), (2.22), we have

$$(2.23)\quad |A| < 4K\left\{\sum_{m=1}^{p-1}\sum_{n=q+1}^{q+k-1}|\Delta_{m,n}| + \sum_{m=1}^{p-1}|{}_{q+k}\Delta_m| + \sum_{n=q+1}^{q+k-1}|{}_p\Delta_n|\right\} + |v_{p,q+k}|.|s_{p,q+k} - s_{p,q}|$$

$$< 4K\{\sigma_{p-1,q+k-1} - \sigma_{p-1,q} + w_{q+k} + W_p\} + K\epsilon < K\epsilon.$$

Similarly, $|B|$, $|C|$ are each less than $K\epsilon$ and so

$$(2.24)\qquad |A+B+C| < K\epsilon.$$

2.3. *Bounded convergence.*

Since $|s_{p,q}| < K$, we have, from (2.12),

$$(2.31)\qquad \left|\sum_{m=1}^{p}\sum_{n=1}^{q} a_{m,n}v_{m,n}\right| < K(\sigma + w_q + W_p + K) < K,$$

which, together with (2.24), establishes the bounded convergence of $\Sigma a_{m,n}v_{m,n}$.

2.4. *Uniform bounded convergence.*

If the terms of the double series $\Sigma u_{m,n}$ are functions of z, the uniform bounded convergence of $\Sigma u_{m,n}$ is here used to imply that

(i) given $\epsilon > 0$, there exist numbers p_0, q_0, such that $p \geqslant p_0$, $q \geqslant q_0$ implies that, for every z,

$$|s - s_{p,q}| < \epsilon.$$

(ii) there is a constant K, such that $|s_{p,q}| < K$, for every p, q, z.

If, in Theorem III, the $v_{m,n}$ are functions of a complex variable z, then $\Sigma a_{m,n}v_{m,n}$ is of uniform bounded convergence with regard to z (*a*) if the conditions governing the $v_{m,n}$, namely i–iv of § 2.1, hold uniformly with regard to z; and, furthermore, (*b*) σ, w_q, W_p are each less than some K which is independent of z, p, q. Conditions (*a*) govern the uniform convergence, conditions (*b*) the uniform boundedness. If the $v_{m,n}$ are continuous, the latter conditions follow from the former.

If the $a_{m,n}$ are functions of z, we require the uniform bounded convergence of $\Sigma a_{m,n}$ in order to prove the same property of $\Sigma a_{m,n}v_{m,n}$.

Similar remarks apply to the Dirichlet test of the next sub-section.

2.5. THEOREM IV*. DIRICHLET'S TEST. *If* $|s_{p,q}| < K$, *and if, in addition to* i–iv *of Theorem III,* $v_{m,n} \to 0$ *as* $m, n \to \infty$, *the series* $\Sigma a_{m,n}v_{m,n}$ *is of bounded convergence.*

---

* Cf. T. Kojima, *Tôhoku Math. Journal*, 17 (1920), 213–219. His Dirichlet test is perhaps better, but his Abel test is very special and is not applicable here. It does show, by the special and artificial conditions which it requires, that for theorems of Types I and II something in addition to mere convergence in Pringsheim's sense is not only expedient, but necessary.

The proof is identical with that of Theorem III, save that in (2 . 23), and in the corresponding inequalities for B and C, the term of type

$$|v_{p,\,q+k}|.|s_{p,\,q+k}-s_{p,\,q}|$$

is now less than $K\epsilon$ because $v_{m,\,n}$ tends to zero, and not because $s_{p,\,q}$ tends to a limit.

3. *Proof of Theorem I.*

3 . 1. Given that the series $\Sigma a_{m,\,n}/(m\omega_1+n\omega_2)$ is boundedly convergent*, then, by § 2, the uniform convergence of $\Sigma a_{m,\,n}/(z+m\omega_1+n\omega_2)$ with respect to z will follow if, when we write

$$v_{m,\,n}=\frac{m\omega_1+n\omega_2}{z+m\omega_1+n\omega_2}, \tag{3 . 11}$$

the conditions i–iv of § 2 . 1 are satisfied uniformly with respect to z.

3 . 2. Let circles of small radius δ be described about each of the points $z=-m\omega_1-n\omega_2$ as centre, and let D be the domain which consists of all points, not in one of these circles, lying in $|z|<M$.

Then, for any m, n and for any z of D,

$$\left|\frac{m\omega_1+n\omega_2}{z+m\omega_1+n\omega_2}\right|<1+\left|\frac{z}{z+m\omega_1+n\omega_2}\right|<1+\frac{M}{\delta}=K;$$

and this proves that (i) is satisfied, uniformly with regard to z.

Also, this may be written as

$$\frac{1}{|z+m\omega_1+n\omega_2|}<\frac{K}{|m\omega_1+n\omega_2|}, \tag{3 . 21}$$

a form which we use later.

3 . 3. Using T to denote $m\omega_1+n\omega_2$, we have, by direct evaluation,

$$\Delta_{m,\,n}=-\frac{\omega_1\omega_2 z\{2z+2T+\omega_1+\omega_2\}}{(z+T)(z+T+\omega_1)(z+T+\omega_2)(z+T+\omega_1+\omega_2)},$$

so that, by (3 . 21),

$$|\Delta_{m,\,n}|<K\left\{\left|\frac{1}{(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)}\right|+\left|\frac{1}{T(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)}\right|\right\}$$

$$<K\left(1+\frac{1}{\eta}\right)\left|\frac{1}{(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)}\right|,$$

* Uniformly if the $a_{m,\,n}$ are functions of z.

where, since ω_2/ω_1 is not real, η is a definite positive number independent of m and n.

But we may write

$$|(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)| = \phi(1+\psi),$$

where $\phi = |m\omega_1+n\omega_2|^3$ and $\psi \to 0$ as $m, n \to \infty$, either separately or together.

Hence we have

$$|\Delta_{m,n}| < \frac{K}{|m\omega_1+n\omega_2|^3}.$$

and accordingly, by a well known theorem*, $\Sigma|\Delta_{m,n}|$ is uniformly convergent, so that (ii) is satisfied uniformly.

3.4. Again, if $\omega_2/\omega_1 = \Omega = \alpha+i\beta$, $\beta \neq 0$,

$$|{}_q\Delta_m| = \left|\frac{-\omega_1 z}{(z+m\omega_1+q\omega_2)(z+(m+1)\omega_1+q\omega_2)}\right| < \frac{K}{|(m+q\Omega)(m+1+q\Omega)|};$$

and, since K is independent of m and n,

$$\sum_{m=1}^{\infty}|{}_q\Delta_m| < K\sum_{m=1}^{\infty}\frac{1}{|(m+q\alpha+qi\beta)(m+1+q\alpha+qi\beta)|}.$$

Hence, if α is positive or zero,

$$w_q < K\sum_{1}^{\infty}\frac{1}{(m+q\alpha)^2+q^2\beta^2} < K\int_0^{\infty}\frac{dx}{(x+q\alpha)^2+q^2\beta^2}$$

$$= \frac{K}{q|\beta|}\left(\frac{\pi}{2}-\arctan\frac{\alpha}{|\beta|}\right) \to 0 \text{ as } q\to\infty.$$

Also, if $\alpha = -\gamma$, where γ is positive, and r denotes $[q\gamma]$,

$$\sum_{m=1}^{r-1}|{}_q\Delta_m| < K\sum_{m=1}^{r-1}\frac{1}{(q\gamma-m-1)^2+q^2\beta^2} = O\left(\frac{1}{q}\right),$$

$$|{}_q\Delta_r| = O(1/q^2),$$

$$\sum_{m=r+1}^{\infty}|{}_q\Delta_m| < K\sum_{m=r+1}^{\infty}\frac{1}{(m-q\gamma)^2+q^2\beta^2} = O\left(\frac{1}{q}\right),$$

so that, as in the case $\alpha \geqslant 0$, we have, uniformly with regard to z,

$$w_q \to 0 \text{ as } q\to\infty.$$

* Cf. Whittaker and Watson, *Modern analysis* (1920), 51, or Jordan, *Cours d'Analyse*, 1 (1909), 308.

Hence (iii) and, by a similar argument, (iv) are satisfied uniformly. Accordingly the convergence conditions (*cf.* § 3 . 1) are satisfied.

3 . 5. Again, from § 3 . 3, we see that (3 . 11) gives

$$\sigma = \sum_{m,n} |\Delta_{m,n}| < K \sum_{m,n} \frac{1}{|m\omega_1+n\omega_2|^3}, \tag{3.51}$$

and, from § 3 . 4,

$$w_q < K, \quad W_p < K, \tag{3.52}$$

the K being, of course, independent of z. Hence, from (2 . 31), the uniform boundedness of $\Sigma a_{m,n}/(z+m\omega_1+n\omega_2)$ follows from the bounded convergence of $\Sigma a_{m,n}/(m\omega_1+n\omega_2)$.

3 . 6. We have thus far proved the sufficiency of the condition in Theorem I. In order to prove the necessity of the condition we take

$$v_{m,n} = \frac{z+m\omega_1+n\omega_2}{m\omega_1+n\omega_2},$$

which gives

$${}_q\Delta_m = \frac{z\omega_1}{(m\omega_1+q\omega_2)\{(m+1)\omega_1+q\omega_2\}},$$

$$\Delta_{m,n} = \frac{2z\omega_1\omega_2}{(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)} + \frac{z\omega_1\omega_2(\omega_1+\omega_2)}{T(T+\omega_1)(T+\omega_2)(T+\omega_1+\omega_2)}.$$

Here $|v_{m,n}|<1+M/\eta$, where $|z|<M$ and $|m\omega_1+n\omega_2|>\eta$. As in §§ 3 . 3, 3 . 4,

$$|\Delta_{m,n}| < \frac{K}{|m\omega_1+n\omega_2|^3}, \quad |{}_q\Delta_m| < \frac{K}{|(m+q\Omega)(m+1+q\Omega)|},$$

and the necessity of our condition follows from the previous analysis.

3 . 7. *Corollary to Theorem I.*

If the series $\Sigma a_{m,n}$ oscillate finitely (*i.e.* $|s_{p,q}|<K$), it follows from the analysis already given that the Dirichlet test of Theorem IV will yield the uniform bounded convergence of $\Sigma a_m/(z+m\omega_1+n\omega_2)$.

4. *Proof of Theorem II.*

4 . 1. *Separate rows and columns.*

In the full enunciation of Theorem II the uniform regular convergence of a series $\Sigma u_{m,n}$, whose terms are functions of z, occurs. It may be pointed out that the theorem asserts that, for any given n, say,

$$\sum_{m=1}^{\infty} u_{m,n} \tag{4.11}$$

converges uniformly with regard to z. We make no attempt to consider the question whether the series (4.11) converges uniformly with regard to n.

Now, for a given n, the convergence of the series

$$\sum_{m=1}^{\infty} \frac{a_{m,n}}{m\omega_1+n\omega_2}$$

is a necessary and sufficient condition for the convergence, uniformly with regard to z, of the series

$$\sum_{m=1}^{\infty} \frac{a_{m,n}}{(z+m\omega_1+n\omega_2)}.$$

This is a direct consequence of J. M. Whittaker's theorem (quoted in § 1). Thus, in so far as it concerns the separate convergence of the rows and columns of the double series, the theorem states a result already known.

4.2. *The Pringsheim sum.*

Theorem II is based upon Hardy's theorem* (*loc. cit.*, Theorem 10):

"If (i) for every p and q, $\sum_{n=1}^{\infty} |{}_p\Delta_n|$, $\sum_{m=1}^{\infty} |{}_q\Delta_m|$ are convergent,

(ii) $\sum_{m,n} |\Delta_{m,n}|$ is convergent,

(iii) $\sum_{m,n} u_{m,n}$ is regularly convergent,

then $\sum_{m,n} u_{m,n} v_{m,n}$ is regularly convergent".

A direct extension of the proof of this theorem to results involving uniform convergence is not at once obvious: it would necessarily be lengthy. We accordingly do not construct any general uniformity theorem for regular convergence to correspond to Theorems III and IV, but use our previous analysis to establish Theorem II.

Now (*cf.* Hardy, *loc. cit.*, Lemma γ) if $\Sigma u_{m,n}$ is regularly convergent to the sum s, then, given $\epsilon > 0$, we can find w so that

$$|s_{m,n}-s| < \epsilon \tag{4.21}$$

if *either* m *or* n is greater than w.

If the $u_{m,n}$ are continuous functions of z, and the regular convergence of $\Sigma u_{m,n}$ is uniform with regard to z, the number w is independent of z, and s is continuous.

* We have changed the notation to that of Theorem III of the present paper.

Hence, if the $u_{m,n}$ are continuous functions of z in a closed domain D (in particular, if the $u_{m,n}$ are constants), (4 . 21) gives

$$(4.22) \qquad m \text{ or } n > w, \quad z \text{ in } D, \quad |s_{m,n}| < K,$$

while, since each $s_{m,n}$ is continuous,

$$(4.23) \qquad m \text{ and } n \leqslant w, \; z \text{ in } D, \quad |s_{m,n}| < K.$$

Accordingly, we have the

LEMMA. *If the $u_{m,n}$ are continuous functions of z in a closed domain D (in particular if they are constants) the uniform regular convergence of $\Sigma u_{m,n}$ in D implies the uniform bounded convergence of $\Sigma u_{m,n}$ in D.*

Hence, by Theorem I, if the $a_{m,n}$ are either continuous functions of z or are constants, the uniform *regular* convergence of either (1 . 1) or (1 . 2) in a closed domain D implies the uniform *bounded* convergence of the other in D.

This proves Theorem II in so far as the Pringsheim sums of the double series are concerned and, with § 4 . 1, establishes the complete theorem.

ON FERMAT'S LAST THEOREM

By S. W. P. STEEN.

[Received 31 March, 1928.—Read 26 April, 1928.]

I. *Introduction.*

The object of this paper is to give formulae for $\Sigma\nu^2\mu^2$ summed over the solutions of $a^n=\nu^n+\mu^n$, where a is a fixed integer and ν and μ are positive integers. In the course of the paper it soon becomes apparent that there is considerable simplification if $n=2$, and that there is a difference between n even and n odd. The final formulae are given for n odd and greater than one.

The method employed consists in forming the function

$$f_n(\xi)=\sum_{\nu=0}^{\infty}\frac{2\xi\nu^2}{\xi^2-\nu^{2n}}, \tag{1.0}$$

and finding a second expression for it by the Euler sum formula. The function

$$g_n(\xi)=f_n(\xi)f_n(a^n-\xi)\sin 2\pi\xi \tag{1.1}$$

is then formed. This function has simple poles at $\xi=\pm\nu^n$ if, and only if, $a^n=\pm\nu^n\pm\mu^n$, and the residue is $-2\pi\nu^2\mu^2$. Corresponding to the second expression for $f_n(\xi)$ there is a second expression for $g_n(\xi)$; this is integrated term by term round a contour enclosing the real axis from 0 to a^n: hence, by using Cauchy's theorem, a formula (2.70) is obtained for $\Sigma\nu^2\mu^2$, summed over the solutions of $a^n=\nu^n+\mu^n$ in positive integers ν and μ.

A second formula (2.94) is obtained by applying Cauchy's theorem to each term in the first formula in which the terms are integrals. In finding the first formula $g_n(\xi)$ was integrated round a contour enclosing the real axis from 0 to a^n. Had we integrated round a contour enclosing the real axis from a^n to infinity, or else enclosing the negative real axis, we should have obtained $\Sigma\nu^2\mu^2$, summed over the solutions of the con-

jugate problem $a^n = \nu^n - \mu^n$. The second formula for $\Sigma\nu^2\mu^2$ summed over the solutions of $a^n = \nu^n + \mu^n$, in positive integers ν and μ, appears at first sight to refer to $2\Sigma\nu^2\mu^2$, summed over the solutions of the conjugate problem. This is not so, since in the case of the conjugate problem we cannot interchange the orders of integration over an infinite range and summation of the series for $g_n(\xi)$. In fact, an interchange of this type is equivalent to proving Fermat's last theorem. For we can find $\Sigma\nu^{2h}\mu^{2k}$, where $h \geqslant 1$, $k \geqslant 1$, by using the function

$$\sum_{\nu=0}^{\infty} \frac{2\xi^{1+2h}}{\xi^2 - \nu^{2n}}. \tag{1.2}$$

If the interchange of the operations of integration and summation were possible we should obtain the result that, for any positive integral values of h and k, $\Sigma\nu^{2h}\mu^{2k}$ has the same value when summed over the solutions of $a^n = \nu^n + \mu^n$, in positive integers, as twice the corresponding sum taken over the solutions of $a^n = \nu^n - \mu^n$, in positive integers. This is clearly absurd unless one of ν and μ is zero.

The number of solutions of $a^n = \nu^n + \mu^n$ may be obtained from the function

$$\sum_{\nu=0}^{\infty} \frac{2\xi}{\xi^2 - \nu^{2n}}. \tag{1.3}$$

The advantage of using $\Sigma\nu^2\mu^2$ is that it does not contain any part corresponding to the trivial solutions when either ν or μ is zero.

Formulae of the nature of those dealt with in this paper by the theory of functions of a complex variable usually admit of treatment by real variable methods. The real variable counterpart of this paper would proceed by the use of the Hellinger-Stieltjes integral

$$\Sigma\phi(\nu, \mu) = \int_0^{a^n} \phi\{x, (a^n - x)^{1/n}\} \frac{d\psi(x)\, d\psi(a^n - x)}{d(x)}, \tag{1.4}$$

where $\psi(x)$ is the number of ν^n not exceeding x, and $(x) = [x] - x$, and the summation is over the solutions of $a^n = \nu^n + \mu^n$ in positive integers ν and μ. The equation (1.4) is a natural extension to Voronoï's† formula

$$\sum_{\lambda_n \leq x} f(\lambda_n) = \int_0^x f(x)\, d\phi(x), \tag{1.5}$$

where $\phi(x)$ is the number of λ_n not exceeding x, and the integral is a Stieltjes integral.

† Voronoï, *Ann. Sci. de l'Ecole Norm. Sup.*, 21 (1904), 207–267.

If we write m for a^n in the formulae given, where m is any positive integer, they will refer to the solutions of $m = +\nu^n + \mu^n$.

Lastly, a formula is given for $\Sigma\nu^2\mu^2$ summed over the solutions of $\lambda^n = \nu^n + \mu^n$, where λ, ν, and μ are positive integers, and $\lambda \leqslant X$. This formula may also be obtained by using the function

$$G_n(\xi) = F_n(\xi) f_n(\xi) \sin 2\pi\xi, \tag{1.6}$$

where

$$\begin{aligned} F_n(\xi) &= \sum_{\nu=0,\ \mu=0}^{\infty} \frac{2\xi\nu^2\mu^2}{\xi^2-(\nu^n+\mu^n)^2} \\ &= \frac{1}{n} \sum_{\nu=0,\ \mu=0}^{\infty} \epsilon_\nu\epsilon_\mu \int_0^\infty \frac{2\xi r^5\,dr}{\xi^2-r^{2n}} \\ &\quad \times \int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1} \cos\{2\nu\pi r t^{1/n}\} \cos\{2\mu\pi r(1-t)^{1/n}\}\,dt, \end{aligned} \tag{1.7}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ $(n \geqslant 1)$.

The function $G_n(\xi)$ has simple poles at $\xi = \pm\lambda^n$ if, and only if, $\pm\lambda^n = \pm(\nu^n+\mu^n)$.

The method employed in this paper is clearly applicable to finding formulae for the number of solutions, etc., of a wide range of Diophantine equations. Also problems of the Goldbach type may be dealt with by the use of the function

$$\begin{aligned} P(\xi) &= \sum_{p,\ m} \frac{2\xi\log p}{\xi^2-p^{2m}} \\ &= \log\frac{1-\xi}{1+\xi} - \frac{2}{(\xi^2-1)\xi}\log(1-\xi^2) + \frac{\xi}{\xi^2-1}\log\left(\frac{e}{2\pi}\right) + 2\pi i \sum_\rho{}^{*} \xi^{\rho-1} \\ &\quad + \int_1^{1+i\infty} \frac{4\xi t}{(\xi^2-t^2)^2} \sum_{\gamma>0} \frac{t^\rho}{\rho}\,dt + \int_1^{1-i\infty} \frac{4\xi t}{(\xi^2-t^2)^2} \sum_{\gamma<0} \frac{t^\rho}{\rho}\,dt, \end{aligned} \tag{1.8}$$

where the * denotes that we sum over $\gamma \gtrless 0$ according as $\mathfrak{I}\xi \gtrless 0$, and $\rho = \beta + i\gamma$ is a non-trivial zero of $\zeta(s)$. This method is different from that of Hardy and Littlewood†.

Fermat's last theorem, which states that a k-th power is not expressible as the sum of two k-th powers, raises the question as to how many k-th powers are required in order that there should be at least one number which can be expressed in two ways as the sum of k-th powers each way containing a *different* number of terms? In other words, what

† *Acta Math.*, 44 (1923), 1–70.

is the least value β can have in order that

$$x_1^k+x_2^k+\ldots+x_\alpha^k = y_1^k+y_2^k+\ldots+y_\beta^k \quad (\alpha<\beta) \tag{1.9}$$

should become possible with non-zero solutions? It is clear that $\beta=\beta(k)\leqslant G_1(k)+1$, where $G_1(k)$ is defined as being the least integer such that almost all numbers are representable as the sum of $G_1(k)$ k-th powers†. For if $\alpha=G_1(k)$, $\beta=G_1(k)+1$, then (1.9) is possible with non-zero solutions. Take $y_1=1$, then the number of numbers less than N and representable in the form $y_1^k+y_2^k+\ldots+y_\beta^k$, with non-zero y's, is $N+o(N)$; and the number of numbers less than N and representable in the form $x_1^k+x_2^k+\ldots+x_\alpha^k$, with non-zero x's, is $N+o(N)$; hence, if N be sufficiently large, there must be at least one number representable in both forms.

It is known that‡

$$\beta(3)=\beta(4)=3,\ \beta(5)\leqslant 6.$$

II. *Solutions of* $a^n=\nu^n+\mu^n$.

Let

$$f_n(\xi)=\sum_{\nu=0}^{\infty}\frac{2\xi\nu^2}{\xi^2-\nu^{2n}} \quad (n\geqslant 2). \tag{1.0}$$

Using Euler's sum formula, when ξ is complex, we obtain the equation

$$f_n(\xi)=\sum_{\nu=0}^{\infty}\epsilon_\nu\int_0^\infty\frac{2\xi x^2\cos(2\nu\pi x)}{\xi^2-x^{2n}}\,dx, \tag{2.0}$$

where

$$\epsilon_0=1,\quad \epsilon_\nu=2 \quad (\nu\geqslant 1). \tag{2.01}$$

Using Cauchy's theorem (2.0) becomes

$$\begin{aligned} f_n(\xi) &= \tfrac{1}{2}\sum_{\nu=0}^{\infty}\epsilon_\nu\int_{-\infty}^{\infty}\frac{2\xi x^2\exp(2\nu\pi i x)}{\xi^2-x^{2n}}\,dx \\ &= -\frac{\pi i}{n\xi}\sum_{\nu=0}^{\infty}\epsilon_\nu\sum_q(\xi_q^2)^{3/2n}\exp\{2\nu\pi i(\xi_q^2)^{1/2n}\}, \end{aligned} \tag{2.1}$$

where the summation for q is over the $2n$-th roots of ξ^2 which lie in the upper half-plane. Since $(\xi^2)^{1/2n}=\pm\xi^{1/n}$, we obtain the equation

$$f_n(\xi)=-\frac{\pi i}{n\xi}\sum_{\nu=0}^{\infty}\epsilon_\nu\left(\sum_h\xi_h^{3/n}\exp(2\nu\pi i\xi_h^{1/n})-\sum_k\xi_k^{3/n}\exp(-2\nu\pi i\xi_k^{1/n})\right), \tag{2.11}$$

† Hardy and Littlewood, *Math. Zeitschrift*, 23 (1925), 1–38.

‡ Dickson, *History of theory of numbers*, 2 (1920), 552, 648, 682.

where the summation for h is over the n-th roots of ξ in the upper half-plane, and the summation for k is over the n-th roots of ξ in the lower half-plane. The series (2.11) may be summed, giving the equation

$$(2.12) \qquad f_n(\xi) = -\frac{\pi}{n\xi}\left(\sum_h \xi_h^{3/n}\cot\pi\xi_h^{1/n} - \sum_k \xi_k^{3/n}\cot\pi\xi_k^{1/n}\right)$$

$$= -\frac{\pi}{n\xi}\sum_l \xi_l^{3/n}\cot\pi\xi_l^{1/n},$$

where the summation for l is over all the n-th roots of ξ.

Now consider the function

$$(2.20) \qquad g_n(\xi) = f_n(\xi)\, f_n(a_n-\xi)\sin 2\pi\xi,$$

where a is an integer. This function has simple poles when $\xi = \pm\nu^n$ and $a^n-\xi = \pm\mu^n$ simultaneously; thus it has simple poles at $\xi = \pm\nu^n$ if, and only if, $a^n = \pm\nu^n \pm\mu^n$. The residue is $-2\pi\nu^2\mu^2$. Clearly $g_n(\xi)$ has no pole at the origin or at $\xi = a^n$. Using Cauchy's theorem we obtain the equation

$$(2.21) \qquad -4\pi^2 i\Sigma\nu^2\mu^2 = \int_C g_n(\xi)\,d\xi,$$

where the summation is over all solutions of $a^n = \nu^n+\mu^n$ in positive integers ν and μ, and C is a contour enclosing the real axis from the origin to the point a^n. We shall take C to cross the real axis at the origin and at the point a^n. Let C_1 be the part of C which lies in the upper half-plane, and C_2 the part which lies in the lower half-plane. Then

$$(2.22) \qquad \int_C g_n(\xi)\,d\xi = -\int_{C_1} g_n(\xi)\,d\xi + \int_{C_2} g_n(\xi)\,d\xi,$$

where both contours are described from the origin to the point a^n. Taking the integration along C_1, we have

$$(2.30) \qquad \int_{C_1} g_n(\xi)\,d\xi = -\frac{\pi i}{n}\int_{C_1} \xi^{-1} f_n(a^n-\xi)\sin 2\pi\xi$$

$$\times \sum_{\nu=0}^{\infty} \epsilon_\nu \left[\sum_h e^{6h\pi i/n}\,\xi_0^{3/n}\exp(2\nu\pi i\xi_0^{1/n}e^{2h\pi i/n})\right.$$

$$\left. -\sum_k e^{6k\pi i/n}\,\xi_0^{3/n}\exp(-2\nu\pi i\xi_0^{1/n}e^{2k\pi i/n})\right] d\xi,$$

where $\xi_0^{1/n}$ is the principal value of $\xi^{1/n}$; and where

$$(2.31) \qquad 0 \leqslant h < \tfrac{1}{2}n, \quad \tfrac{1}{2}n \leqslant k < n.$$

The integral on the right-hand side of (2.30) is the sum of a finite

number of integrals, of which the following integral is typical:

$$(2.32)\quad -\frac{\pi i}{n}\int_{C_1}\xi^{-1}f_n(a^n-\xi)\sin 2\pi\xi\,.\,e^{6h\pi i/n}\,\xi_0^{3/n}\sum_{\nu=0}^{\infty}\epsilon_\nu\exp(2\nu\pi i\xi_0^{1/n}e^{2h\pi i/n})\,d\xi.$$

In this integral we shall interchange the orders of integration and summation; it is equal to

$$(2.33)\quad -\frac{\pi i}{n}e^{6h\pi i/n}\sum_{\nu=0}^{N}\epsilon_\nu\int_{C_1}\xi^{-1}f_n(a^n-\xi)\sin 2\pi\xi\,.\,\xi_0^{3/n}\exp(2\nu\pi i\xi_0^{1/n}e^{2h\pi i/n})\,d\xi$$

$$-\frac{\pi i}{n}e^{6h\pi i/n}\int_{C_1}\xi^{-1}f_n(a^n-\xi)\sin 2\pi\xi\,.\,\xi_0^{3/n}\frac{\exp\{2(N+1)\pi i\xi_0^{1/n}e^{2h\pi i/n}\}}{1-\exp(2\pi i\xi_0^{1/n}e^{2h\pi i/n})}\,d\xi.$$

In the last integral change the variable from ξ to t^n, and take the contour C_1' in the t-plane to be from 0 to $i\delta$, $i\delta$ to $a+i\delta$, $a+i\delta$ to a. Except for a constant factor the integral is then equal to

$$(2.34)\qquad \int_{C_1'}\frac{t^2f_n(a^n-t^n)\sin 2\pi t^n\exp\{2(N+1)\pi ite^{2\pi ih/n}\}}{1-\exp(2\pi ite^{2\pi ih/n})}\,dt.$$

When δ is fixed the absolute value of the integral from $i\delta$ to $a+i\delta$ can be made less than ϵ by taking N sufficiently large. The absolute value of the integral from 0 to $i\delta$ is less than

$$K\int_0^{\delta}t\exp\left\{-2(N+1)\pi t\cos\frac{2\pi h}{n}\right\}dt<\frac{K}{N},$$

and similarly for the integral from $a+i\delta$ to a. Hence the order of integration and summation may be changed in (2.32), which is thus equal to the sum

$$(2.35)\quad -\frac{\pi i}{n}e^{6h\pi i/n}\sum_{\nu=0}^{\infty}\epsilon_\nu\int_{C_1}\xi^{-1}f_n(a^n-\xi)\sin 2\pi\xi\,.\,\xi_0^{3/n}\exp(2\nu\pi i\xi_0^{1/n}e^{\quad 2l\pi i})\,d\xi.$$

Taking a typical term from (2.35), and replacing $f_n(a^n-\xi)$ by its expansion (2.11) we see that we may, as before, interchange the order of integration and summation. We thus obtain the equation

$$(2.36)\quad \int_{C_1}g_n(\xi)\,d\xi=-\frac{\pi^2}{n^2}\sum_{\nu=0,\ \mu=0}^{\infty}\epsilon_\nu\epsilon_\mu\int_{C_1}\xi_0^{(3/n)-1}(a^n-\xi)_0^{(3/n)-1}\sin 2\pi\xi$$

$$\times\left[\sum_h e^{6h\pi i/n}\exp(2\nu\pi i\xi_0^{1/n}e^{2h\pi i/n})-\sum_k e^{6k\pi i/n}\exp(-2\nu\pi i\xi_0^{1/n}e^{2k\pi i/n})\right]$$

$$\times\left[\sum_{h'}e^{6h'\pi i/n}\exp\{2\mu\pi i(a^n-\xi)_0^{1/n}e^{2h'\pi i/n}\}\right.$$

$$\left.-\sum_{k'}e^{6k'\pi i/n}\exp\{-2\mu\pi i(a^n-\xi)^{1/n}e^{2k'\pi i/n}\}\right]d\xi,$$

where

$$0 < h' \leqslant \tfrac{1}{2}n, \quad \tfrac{1}{2}n < k' \leqslant n. \tag{2.361}$$

The integral along C_2 may be dealt with in a similar manner. In the repeated sum in (2.36) it is clearly immaterial whether we sum for ν first or for μ. If we change the variable from ξ to $a^n t$, a typical integral from (2.36) is

$$-\frac{\pi^2}{n^2}\,\epsilon_\nu\,\epsilon_\mu\,e^{6\pi i(h+k)/n}\,a^{6-n} \tag{2.40}$$

$$\times\int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\exp\left[2\pi i a\left\{\nu t^{1/n}e^{2h\pi i/n}+\mu(1-t)^{1/n}e^{2h'\pi i/n}\right\}\right]\sin 2\pi a^n t\,dt,$$

where, if we have k instead of h, the sign of ν is changed, if we write $\epsilon_{-\nu} = -\epsilon_\nu$; and, similarly, if we have k' instead of h'. If $n = 2$ this integral can be simplified by the transformation $t = \sin^2\theta$. There is a similar set of integrals arising from the integration along C_2. From (2.22) we see that integrals arising from C_1 are to be multiplied by -1. The following table gives the values of h, k, h', and k' which occur in the typical integral (2.40) together with its sign.

From C_1 we have integrals corresponding to :—

$$\begin{aligned}
&0 \leqslant h < \tfrac{1}{2}n, \quad 0 < h' \leqslant \tfrac{1}{2}n \quad \text{(negative sign)};\\
&0 \leqslant h < \tfrac{1}{2}n, \quad \tfrac{1}{2}n < k' \leqslant n \quad \text{(positive sign)};\\
&\tfrac{1}{2}n \leqslant k < n, \quad 0 < h' \leqslant \tfrac{1}{2}n \quad \text{(positive sign)};\\
&\tfrac{1}{2}n \leqslant k < n, \quad \tfrac{1}{2}n < k' \leqslant n \quad \text{(negative sign)}.
\end{aligned} \tag{2.41}$$

From C_2 we have integrals corresponding to :—

$$\begin{aligned}
&0 < h \leqslant \tfrac{1}{2}n, \quad 0 \leqslant h' < \tfrac{1}{2}n \quad \text{(positive sign)};\\
&0 < h \leqslant \tfrac{1}{2}n, \quad \tfrac{1}{2}n \leqslant k' < n \quad \text{(negative sign)};\\
&\tfrac{1}{2}n < k \leqslant n, \quad 0 \leqslant h' < \tfrac{1}{2}n \quad \text{(negative sign)};\\
&\tfrac{1}{2}n < k \leqslant n, \quad \tfrac{1}{2}n \leqslant k' < n \quad \text{(positive sign)}.
\end{aligned} \tag{2.42}$$

Many of these integrals cancel, and we are left with integrals corresponding to the following values of h, k, h', and k' :—

$$\begin{aligned}
&h = 0, \quad 0 < h' < \tfrac{1}{2}n \ \text{(negative)}; \quad \tfrac{1}{2}n < k' < n \ \text{(positive)};\\
&h = \tfrac{1}{2}n, \quad 0 < h' < \tfrac{1}{2}n \ \text{(positive)}; \quad \tfrac{1}{2}n < k' < n \ \text{(negative)};\\
&k = \tfrac{1}{2}n, \quad 0 < h' < \tfrac{1}{2}n \ \text{(positive)}; \quad \tfrac{1}{2}n < k' < n \ \text{(negative)};\\
&k = n, \quad 0 < h' < \tfrac{1}{2}n \ \text{(negative)}; \quad \tfrac{1}{2}n < k' < n \ \text{(positive)};
\end{aligned} \tag{2.43}$$

and

$$(2.44)\qquad h=0,\quad h'=\tfrac{1}{2}n \text{ (negative)};\qquad k'=n \text{ (positive)};$$
$$k=\tfrac{1}{2}n,\quad h'=\tfrac{1}{2}n \text{ (positive)};\qquad k'=n \text{ (negative)};$$

and two more tables obtained from (2.43) and (2.44) by interchanging h with h' and k with k' and negative with positive.

We now take n to be odd, so that terms corresponding to $h=\frac{1}{2}n$, $k=\frac{1}{2}n$, etc., do not arise. If, in the integral (2.40), we interchange ν with μ, h with h', and k with k', we change its sign. Thus the integrals corresponding to values of h, k, h' and k' from the tables derived from (2.43) and (2.44), when summed for ν and μ, yield a value equal to the value of the corresponding sum coming from integrals corresponding to values of h, k, h', and k' from the tables (2.43) and (2.44). Also we may take $-\frac{1}{2}n<k'<0$ instead of $\frac{1}{2}n<k'<n$. We are thus left with the following table :—

$$(2.45)\qquad h=0,\quad 0<h'<\tfrac{1}{2}n \text{ (negative)};\quad -\tfrac{1}{2}n<k'<0 \text{ (positive)};$$
$$k=0,\quad 0<h'<\tfrac{1}{2}n \text{ (negative)};\quad -\tfrac{1}{2}n<k'<0 \text{ (positive)};$$
$$h=0,\quad k'=0 \qquad \text{(positive)};$$

and each term in the tables provides two integrals. Now write

$$(2.50)\qquad G_n^{(1)}(a;\ \nu,\ \mu)=\sum_{h=1}^{\frac{1}{2}n} e^{6h\pi i/n}\int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\sin 2\pi a^n t \times\exp\left[2\pi i a\left\{\nu t^{1/n}+\mu(1-t)^{1/n}e^{2\pi i h/n}\right\}\right]dt.$$

$$(2.51)\qquad G_n^{(2)}(a;\ \nu,\ \mu)=\sum_{h=1}^{\frac{1}{2}n} e^{6h\pi i/n}\int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\sin 2\pi a^n t \times\exp\left[2\pi i a\left\{-\nu t^{1/n}+\mu(1-t)^{1/n}e^{2\pi i h/n}\right\}\right]dt.$$

$$(2.52)\qquad G_n^{(3)}(a;\ \nu,\ \mu)=-\int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1} \times\exp\left[2\pi i a\left\{\nu t^{1/n}-\mu(1-t)^{1/n}\right\}\right]\sin 2\pi a^n t\,dt.$$

The integrals corresponding to values of h, h', and k' in the first line of the table (2.45) are equal to

$$(2.60)\qquad \frac{2\pi^2}{n^2}\,\epsilon_\nu\epsilon_\mu\,a^{6-n}\left[G_n^{(1)}(a;\ \nu,\ \mu)-\overline{G_n^{(2)}(a;\ \nu,\ \mu)}\right],$$

where the bar denotes the conjugate complex number. Integrals arising from the second line of the table (2.45) are equal to

$$(2.61)\qquad \frac{2\pi^2}{n^2}\,\epsilon_\nu\,\epsilon_\mu\,a^{6-n}\left[G_n^{(2)}(a\,;\ \nu,\ \mu)-\overline{G_n^{(1)}(a\,;\ \nu,\ \mu)}\right].$$

And the integral from the third line of the table (2.45) is equal to

$$(2.62)\qquad \frac{2\pi^2}{n^2}\,\epsilon_\nu\,\epsilon_\mu\,a^{6-n}\,G_n^{(3)}(a\,;\ \nu,\ \mu).$$

The equation (2.21) now becomes

$$(2.70)\quad \Sigma\nu^2\mu^2=-\frac{a^{6-n}}{2n^2}\sum_{\nu=0,\ \mu=0}^{\infty}\epsilon_\nu\,\epsilon_\mu \times\mathfrak{J}\left\{G_n^{(1)}(a\,;\ \nu,\ \mu)+G_n^{(2)}(a\,;\ \nu,\ \mu)+G_n^{(3)}(a\,;\ \nu,\ \mu)\right\},$$

where the summation on the left is taken over the solutions of $a^n=\nu^n+\mu^n$, in positive integers ν and μ.

Integrals of the type occurring in (2.50), (2.51), and (2.52) may be evaluated in terms of known functions by expanding the exponential function as a power series. Thus

$$(2.80)\quad \int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\exp\left[2\pi ia\left\{\nu t^{1/n}+\mu(1-t)^{1/n}e^{2h\pi i/n}\right\}\right]\sin 2\pi a^n t\,.\,dt$$

$$=\sum_{m=0}^{\infty}\frac{(2\pi ia)^m}{m!}\sum_{p+q=m}\nu^p\,\mu^q\,e^{2qh\pi i/n}\,\frac{\Gamma(p+q+1)}{\Gamma(p+1)\,\Gamma(q+1)} \times\int_0^1 t^{\{(p+3)/n\}-1}(1-t)^{\{(q+3)/n\}-1}\sin 2\pi a^n t\,dt$$

$$=\sum_{m=0}^{\infty}\frac{(2\pi ia)^m}{m!}\sum_{p+q=m}\nu^p\,\mu^q\,e^{2qh\pi i/n}\,\frac{\Gamma(p+q+1)}{\Gamma(p+1)\,\Gamma(q+1)}\,\frac{\Gamma\{(p+3)/n\}\,\Gamma\{(q+3)/n\}}{\Gamma\{(p+q+6)/n\}} \times\left[{}_1F_1\left(\frac{p+3}{n}\,;\ \frac{q+3}{n}\,;\ 2\pi ia^n\right)-{}_1F_1\left(\frac{p+3}{n}\,;\ \frac{q+3}{n}\,;\ -2\pi ia^n\right)\right].$$

Integrals of the type occurring in (2.50), (2.51), and (2.52) are also expressible as integrals over an infinite range. Take

$$(2.90)\quad \int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\exp\left[2\pi ia\left\{\nu t^{1/n}e^{i\alpha}+\mu(1-t)^{1/n}e^{i\beta}\right\}+2\pi ia^n t\right]dt,$$

and integrate round a rectangle whose corners are 0, 1, $1+iT$ and iT. Cut the t-plane from 0 to $-\infty$ and from 1 to $+\infty$, and choose that branch

of $t^{1/n}$ which is positive for positive t, and that branch of $(1-t)^{1/n}$ which is positive for $t<1$. The integration along the portion of the contour from $1+iT$ to iT clearly tends to zero as T tends to infinity. Thus the integral (2.90) is equal to

$$(2.91)\quad e^{(3\pi i)/2n}\int_0^\infty t^{(3/n)-1}(1-it)^{(3/n)-1}\times\exp\left[2\pi ia\left\{\nu t^{1/n}e^{i\alpha+(i\pi/2n)}+\mu(1-it)^{1/n}e^{i\beta}\right\}-2\pi a^n t\right]dt$$
$$+e^{(-3\pi i)/2n}\int_0^\infty (1+it)^{(3/n)-1}t^{(3/n)-1}\times\exp\left[2\pi ia\left\{\nu(1+it)^{1/n}e^{i\alpha}+\mu t^{1/n}e^{i\beta-(i\pi/2n)}\right\}-2\pi a^n t\right]dt.$$

Similarly, we have

$$(2.92)\quad \int_0^1 t^{(3/n)-1}(1-t)^{(3/n)-1}\exp\left[2\pi ia\left\{\nu t^{1/n}e^{i\alpha}+\mu(1-t)^{1/n}e^{i\beta}\right\}-2\pi ia^n t\right]dt$$
$$=e^{-(3\pi i/2n)}\int_0^\infty t^{(3/n)-1}(1+it)^{(3/n)-1}\times\exp\left[2\pi ia\left\{\nu t^{1/n}e^{i\alpha-(\pi i/2n)}+\mu(1+it)^{1/n}e^{i\beta}\right\}-2\pi a^n t\right]dt$$
$$+e^{3\pi i/2n}\int_0^\infty (1-it)^{(3/n)-1}t^{(3/n)-1}\times\exp\left[2\pi ia\left\{\nu(1-it)^{1/n}e^{i\alpha}+\mu t^{1/n}e^{i\beta+(\pi i/2n)}\right\}-2\pi a^n t\right]dt.$$

In equations (2.91) and (2.92) the amplitudes of $1+it$ and $1-it$ are determined by

$$(2.93)\qquad -\pi<\operatorname{amp}(1+it)<\pi,\quad -\pi<\operatorname{amp}(1-it)<\pi.$$

Let the result of adding the first integral in (2.91) to the last integral in (2.92), taking $\alpha=0$, $\beta=2\pi h/n$, multiplying by $e^{3i\beta}$, and summing for $h=1, 2, \ldots, [\frac{1}{2}n]$ be called $G_n^{(4)}(a;\nu,\mu)$. Similarly define $G_n^{(5)}(a;\nu,\mu)$, using the last integral of (2.91) and the first integral of (2.92). Let $G_n^{(6)}(a;\nu,\mu)$ and $G_n^{(7)}(a;\nu,\mu)$ be similarly defined, when $\alpha=\pi$, $\beta=2\pi h/n$, and the summation is for $h=1, 2, \ldots, [\frac{1}{2}n]$. Lastly, let $G_n^{(8)}(a;\nu,\mu)$ and $G_n^{(9)}(a;\nu,\mu)$ be similarly defined, when $\alpha=0$, $\beta=\pi$. The equation (2.70) now becomes

$$(2.94)\quad \Sigma\nu^2\mu^2=-\frac{a^{6-n}}{2n^2}\sum_{\nu=0,\ \mu=0}\epsilon_\nu\epsilon_\mu\,\mathfrak{J}\left\{G_n^{(4)}(a;\nu,\mu)+\ldots+G_n^{(9)}(a;\nu,\mu)\right\},$$

where the summation on the left-hand side is taken over the solutions of $a^n=\nu^n+\mu^n$ in positive integers ν and μ.

III. *Solutions of* $\lambda^n = \nu^n + \mu^n$. $\lambda \leqslant X$.

In equation (2 . 70) write λ instead of a, and sum for λ from 1 to X, where X is an integer. We then have

$$(3\,.\,0)\quad \sum_{\nu=0,\ \mu=0}^{X}{}' \nu^2\mu^2 = -\frac{1}{2n^2}\sum_{\nu=0,\ \mu=0}^{\infty} \epsilon_\nu \epsilon_\mu \times \mathfrak{J} \sum_{\lambda=0}^{X}{}' \lambda^{6-n}\{G_n^{(1)}(\lambda\,;\ \nu,\ \mu)+G_n^{(2)}(\lambda\,;\ \nu,\ \mu)+G_n^{(3)}(\lambda\,;\ \nu,\ \mu)\},$$

where the summation on the left-hand side is over all positive ν and μ which satisfy $\nu^n+\mu^n = \lambda^n \leqslant X^n$; and the dashes denote that terms arising from $\lambda = 0$ or $\lambda = X$ are to be multiplied by $\frac{1}{2}$. Using the Euler sum formula for the inner sum on the right-hand side of (3 . 0), we obtain the equation

$$(3\,.\,1)\quad \sum_{\nu=0,\ \mu=0}^{X}{}' \nu^2\mu^2 = -\frac{1}{2n^2}\sum_{\nu=0,\ \mu=0,\ \lambda=0} \epsilon_\nu \epsilon_\mu \epsilon_\lambda \times \mathfrak{J} \int_0^X u^{6-n} \cos 2\lambda\pi u \{G_n^{(1)}(u\,;\ \nu,\ \mu)+G_n^{(2)}(u\,;\ \nu,\ \mu)+G_n^{(3)}(u\,;\ \nu,\ \mu)\}\, du,$$

where the summation for λ is taken first. There is a similar formula obtainable from (2 . 94).

DIRICHLET'S PRINCIPLE FOR A FLAT RING

By E. G. C. POOLE.

[Received 17 May, 1928.—Read 14 June, 1928.]

I.

A thin plane lamina bounded by two concentric circles may be briefly called a flat ring. Some interesting potential problems suggested by it have been discussed by Nicholson†, whose methods yield good numerical approximate results. The methods to be developed in this paper are of purely theoretical interest, since they ultimately depend on computations too formidable to be undertaken except for a problem of the first importance. But it appears to be of some interest to show that "Dirichlet's principle" can be established for this configuration, by specializing well known analysis. Given an arbitrary function (subject to the usual conditions of continuity of Fourier type), which takes assigned values on both sides of the flat ring, we can construct a harmonic function, regular and single-valued in external space, assuming those values on the ring. The solution appears as a doubly infinite series of "normal solutions" $\Sigma\Sigma P f_{mn}(\lambda) g_{mn}(\mu) h_{mn}(\nu)$, where λ, μ, ν are orthogonal curvilinear coordinates, and P is a function of position characteristic of the whole family.

Wangerin‡ and Darboux§ have reduced the cases in which Laplace's equation admits an infinite family of such normal product solutions to those in which the triple system consists of confocal cyclides or their degenerate forms. The latter were enumerated by Bôcher‖, by the theory

† J. W. Nicholson, *Proc. Royal Soc.* (A), 101 (1922), 195–210.

‡ A. Wangerin, *Journal für Math.*, 82 (1876), 145-157; *Encyklopädie* II, A 10; II, 1 (ii), 732–742.

§ G. Darboux, *Systèmes orthogonaux*, 2nd ed. (Paris, 1910), 212–263; 283-287.

‖ M. Bôcher, *Die Reihenentwicklungen der Potentialtheorie* (Leipzig, 1894), *vassim.*

of invariant factors of two quadratic forms. The typical system has five spheres or planes of symmetry, and its canonical form in pentaspherical coordinates is

$$\Sigma \frac{u_k^2}{\theta-\theta_k} = 0, \quad \text{where} \quad \Sigma u_k^2 \equiv 0 \quad [k = 1, 2, 3, 4, 5]; \quad (1)$$

and degeneration occurs when the θ_k become equal.

We require the special case in which one family of surfaces consists of planes through the axis of symmetry and the other two are cyclides of revolution. This occurs when two values of θ_k in (1) become equal without any change of the canonical form. One pair of spheres of symmetry is then indeterminate and can be inverted into two planes, while a third sphere of symmetry is simultaneously inverted into an equatorial plane. If (ϖ, z, ϕ) are cylindrical coordinates, we can use a set of four homogeneous tetracyclic coordinates in the meridan plane instead of pentaspherical coordinates. We put

$$u_1 : u_2 : u_3 : iu_4 = 2a\varpi : 2az : a^2-\varpi^2-z^2 : a^2+\varpi^2+z^2, \quad (2)$$

where

$$\Sigma u_k^2 = 0 \quad [k = 1, 2, 3, 4]. \quad (3)$$

In this case Wangerin showed that the triple system of orthogonal curvilinear coordinates is reducible to ϕ, u, v, where $u+iv = f(\varpi+iz)$ and where f denotes an elliptic integral of the first kind. We shall find that the u, v curves are confocal bicircular quartics. More general cases have been given by Haentzschel*.

We consider a pair of equations similar to (1), but with only four terms in each, instead of five, the four values of θ_k being unequal. It is easily seen that the form of these equations is unchanged by any homographic transformation of the parameters θ, θ_k. We may without loss of generality assign to the latter the values

$$\theta_1 = \infty, \quad \theta_2 = 0, \quad \theta_3 = 1, \quad \theta_4 = \frac{1}{k^2}. \quad (4)$$

We now impose the definite restriction that k is real, with $0 < k < 1$, and we adopt the notation of Jacobian elliptic functions. The family of cyclides of revolution becomes

$$\frac{4a^2z^2}{\theta} + \frac{(a^2-\varpi^2-z^2)^2}{\theta-1} - \frac{(a^2+\varpi^2+z^2)^2}{\theta-(1/k^2)} = 0. \quad (5)$$

* E. Haentzschel, *Reduction der Potentialgleichung . . .* (Berlin, 1893).

Every real point gives two real values of θ, lying in the intervals $(-\infty, 0)$ and $(0, 1)$, which we call $\mathrm{sn}^2 iv$ and $\mathrm{sn}^2 u$ respectively; for the present we suppose that $0 \leqslant u \leqslant K$, $0 \leqslant v \leqslant K'$.

The cyclide $\theta = \mathrm{sn}^2 u$ has two closed sheets, which are inverse to one another with respect to the sphere $u_3 = 0$, and each of which is a surface of revolution with an equatorial plane of symmetry.

The cyclide $\theta = \mathrm{sn}^2 iv$ is a ring-cyclide of revolution, which is unaltered by inversion with respect to the sphere $u_3 = 0$, and is symmetrical about the equator. When $v = 0$ it reduces to the double surface of the flat ring

$$z = 0, \qquad a(1-k) \leqslant k'\varpi \leqslant a(1+k).$$

The geometric mean of the inner and outer radii is a, which is the radius of the sphere $u_3 = 0$.

The equation (5) can be treated like the corresponding equation in another classical special case, that of confocal quadrics, and the tetracyclic coordinates are expressed in terms of u, v by the formulae

$$\frac{u_1^2}{k'^2} = \frac{u_2^2}{-k^2k'^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 iv} = \frac{u_3^2}{k^2\,\mathrm{cn}^2 u\,\mathrm{cn}^2 iv} = \frac{u_4^2}{-\mathrm{dn}^2 u\,\mathrm{dn}^2 iv}. \tag{6}$$

This gives a set of eight points symmetrically situated in the meridian plane, one of which is

$$\frac{u_1}{k'} = \frac{u_2}{-ikk'\,\mathrm{sn}\,u\,\mathrm{sn}\,iv} = \frac{u_3}{k\,\mathrm{cn}\,u\,\mathrm{cn}\,iv} = \frac{iu_4}{\mathrm{dn}\,u\,\mathrm{dn}\,iv}. \tag{7}$$

When u, v are restricted as above, this will be the point for which ϖ, z, $a^2-\varpi^2-z^2$ are positive. From (7) we have

$$\begin{aligned}\varpi + iz &= \frac{a(u_1+iu_2)}{u_3+iu_4} \\ &= \frac{ak'(1+k\,\mathrm{sn}\,u\,\mathrm{sn}\,iv)}{\mathrm{dn}\,u\,\mathrm{dn}\,iv+k\,\mathrm{cn}\,u\,\mathrm{cn}\,iv} \\ &= \frac{ak'}{\mathrm{dn}\,(u+iv)+k\,\mathrm{cn}\,(u+iv)}, \\ \varpi + iz &= a\{\mathrm{dn}\,(u+iv)-k\,\mathrm{cn}\,(u+iv)\}/k'. \end{aligned} \tag{8}$$

Differentiating logarithmically,

$$\begin{aligned} d(\varpi+iz) &= k(\varpi+iz)\,\mathrm{sn}\,(u+iv)\,d(u+iv), \\ d\varpi^2+dz^2 &= k^2(\varpi^2+z^2)\,|\,\mathrm{sn}\,(u+iv)\,|^2\,(du^2+dv^2) \\ &= k^2\varpi^2(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 iv)\,|\,\mathrm{sn}\,(u+iv)\,|^2\,(du^2+dv^2), \\ d\varpi^2+dz^2 &= k^2\varpi^2(\mathrm{sn}^2 u-\mathrm{sn}^2 iv)(du^2+dv^2). \end{aligned} \tag{9}$$

To obtain the required form of Laplace's equation, we write, after Poisson,

$$\varpi^{\frac{1}{2}}\nabla^2 V \equiv \left[\frac{\partial^2}{\partial\varpi^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{\varpi^2}\left(\frac{\partial^2}{\partial\phi^2}+\frac{1}{4}\right)\right](\varpi^{\frac{1}{2}}V),$$

and put

$$M \equiv k^2\varpi^2(\mathrm{sn}^2 u-\mathrm{sn}^2 iv).$$

Then

$$\varpi^{\frac{1}{2}}M\nabla^2 V \equiv \left[\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+k^2(\mathrm{sn}^2 u-\mathrm{sn}^2 iv)\left(\frac{\partial^2}{\partial\phi^2}+\frac{1}{4}\right)\right](\varpi^{\frac{1}{2}}V). \quad (10)$$

Substituting the normal product

$$V = \varpi^{-\frac{1}{2}}\cos m(\phi-\phi_0)\,F(u)\,G(iv), \quad (11)$$

we obtain the equations

$$\frac{F''(u)+(\frac{1}{4}-m^2)\,k^2\,\mathrm{sn}^2 u\,F(u)}{F(u)} = \frac{G''(iv)+(\frac{1}{4}-m^2)\,k^2\,\mathrm{sn}^2 iv\,G(iv)}{G(iv)} = B, \quad (12)$$

where B is a constant.

It follows that $F(u)$ and $G(iv)$ satisfy the same Lamé equation of order $m-\frac{1}{2}$, namely

$$F''(u) = \{(m^2-\tfrac{1}{4})\,k^2\,\mathrm{sn}^2 u+B\}\,F(u), \quad (13)$$

$$G''(iv) = \{(m^2-\tfrac{1}{4})\,k^2\,\mathrm{sn}^2 iv+B\}\,G(iv). \quad (14)$$

To solve the problem, we must determine the admissible values of the parameters m, B, and we must choose the right branches of the solution in (13) and (14).

The determination of m is immediate; for the required harmonic function is a uniform function of position and takes the same value at any point after describing a circle about the axis of z parallel to the equator. This restricts m to integral values.

The arbitrary function assigned on the surface of the ring $v = 0$ will be taken therefore as a Fourier series in ϕ, whose coefficients are arbitrary functions of u. In view of Poisson's substitution, which was used in (10) and (11), it is convenient to consider the typical term in the form

$$V'_m = \varpi^{-\frac{1}{2}}\cos m(\phi-\phi_0)\,A_m(u). \quad (15)$$

Since $\varpi^{-\frac{1}{2}}$ is continuous and does not vanish on the ring, this involves no loss of generality.

For every integer m, an infinite set of values of B and corresponding functions $F_n^{(m)}(u)$ must be found, forming a complete system of orthogonal

functions for the expansion of an arbitrary function $A_m(u)$, assigned in an interval of length $4K$, half of which corresponds to each side of the ring.

The Lamé equation of order $m-\frac{1}{2}$, called by Haentzschel the Lamé-Wangerin equation, was shown by Brioschi and Crawford to be soluble in finite terms for m special values of B†. But these are not the systems with which we are here concerned, although such solutions may lie concealed among the infinite system which we shall find. The difference in the method of expansion would probably cause them to remain unnoticed.

We may here mention also that elegant results connected with the potential of rings and cyclides were found by A. L. Dixon‡, using methods different from ours, but also connected with pentaspherical coordinates.

II

In principle the admissible values of B are determined by the uniformity of the harmonic function, when we consider a closed circuit threading the ring.

The relation (8) gives the conformal representation of a circular quadrant in the meridian plane of $\varpi+iz$ defined by

$$\varpi \geqslant 0, \quad z \geqslant 0, \quad \varpi^2+z^2 \leqslant a^2,$$

on a rectangle in the plane of $u+iv$ defined by

$$0 \leqslant u \leqslant K, \quad 0 \leqslant v \leqslant K'.$$

Keeping the value of v restricted, but allowing u to increase, we find that the exterior of the same quadrant corresponds to $K \leqslant u \leqslant 2K$; the range $2K \leqslant u \leqslant 3K$ gives the reflexion of the exterior quadrant in $z=0$, and $3K \leqslant u \leqslant 4K$ gives the reflexion of the interior quadrant. Thus, when u increases by $4K$, while v remains constant, the point in the meridian plane returns to its original position, after describing a circuit threading the ring, which is a loop of a bicircular quartic. This circuit must leave the required potential unaltered, so that the fundamental condition is

$$F(u+4K) \equiv F(u). \qquad (16)$$

† Whittaker and Watson, *Modern analysis* (3rd ed.), 575-578.

‡ A. L. Dixon, *Proc. London Math. Soc.* (1), 27 (1896), 226-249; 28 (1897), 439-442.

The equation (13) is unaltered if u is changed in sign or is increased by $2K$. If $F(u)$ is a solution, so is $F(-u)$; by considering the expressions $F(u)\pm F(-u)$, one of which may vanish identically, we see that there is no loss of generality if we suppose $F(u)$ to be either an odd or an even function of u. Now

$$F(2K+u) = F(-2K+u) = \pm F(2K-u),$$

so that $F(u)$ is of the same parity regarded as a function of $u-2K$ as it is when regarded as a function of u. The differential equation (13) being satisfied by $F(u)$ and $F(u+2K)$, and these being of the same parity in u, we must have

$$F(u) = cF(u+2K) = c^2F(u+4K) = c^2F(u),$$

$$c^2 = 1,$$

and (16) breaks up into either

$$F(u+2K) \equiv F(u), \tag{16 a}$$

or

$$F(u+2K) \equiv -F(u). \tag{16 b}$$

Thus $2K$ is either a period or a "semi-period" of $F(u)$.

We can next show that $F(u)$ contains only terms of the same parity in $u-K$ as well as in u, though the parity need not be identical in the two variables. For

$$F(2K-u) = cF(-u) = \pm cF(u),$$

which proves the statement.

Since there are no singularities of (13) for any real value of u, and since there is an even or an odd solution in $u-sK$ for any integer s, we see that the condition $F(0) = 0$, or $F(sK) = 0$, is sufficient to identify $F(u)$ with the *odd* branch of the solution at $u = 0$, or $u = sK$, and similarly $F'(0) = 0$, or $F'(sK) = 0$ identifies it with the *even* branch. The problem can now be restated as a Sturm-Liouville boundary-value problem for the differential equation (13) and the real interval $0 \leqslant u \leqslant K$.

If $2K$ is a full period (16a) we have either

$$\left.\begin{array}{l} \text{(I)} \quad F'(0) = F'(K) = 0 \text{ if } F(u) \text{ is } \textit{even} \text{ in both } u \text{ and } u-K \\ \text{or (II)} \quad F(0) = F(K) = 0 \text{ if } F(u) \text{ is } \textit{odd} \text{ in both } u \text{ and } u-K \end{array}\right\}. \tag{17 a}$$

In either case a segment of the real axis of length $2K$ contains an *even* number of zeros of $F(u)$.

If $2K$ is a semi-period (16b) we have either

$$\left.\begin{array}{ll} \text{(III)} & F'(0) = F(K) = 0 \text{ if } F(u) \text{ is } \textit{even} \text{ in } u \text{ and odd in } u-K \\ \text{or (IV)} & F(0) = F'(K) = 0 \text{ if } F(u) \text{ is } \textit{odd} \text{ in } u \text{ and even in } u-K \end{array}\right\}. \quad (17\,b)$$

In either case a segment of the real axis of length $2K$ contains an *odd* number of zeros of $F(u)$.

For real values of u we have

$$k^2 \operatorname{sn}^2 u = \frac{K-E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos \frac{n\pi u}{K}, \quad (18)$$

and, putting

$$\left.\begin{array}{l} \pi u = 2Kx, \qquad F(u) \equiv F\left(\dfrac{2Kx}{\pi}\right) \equiv f(x), \qquad m^2 - \tfrac{1}{4} = p, \\ \qquad\qquad B + p\,\dfrac{K-E}{K} = -\dfrac{\pi^2 b}{4K^2} \end{array}\right\}, \quad (19)$$

the equation (13) becomes

$$f''(x) + \left\{b + 8p \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos 2nx\right\} f(x) = 0. \quad (20)$$

This is a particular case of Hill's equation, and our treatment follows the familiar analysis of Hill's and Mathieu's equations†.

If we substitute formally for $f(x)$ in (20) a trigonometric series with undetermined coefficients, the terms involving sines and cosines and those involving even and odd multiples of x fall into separate groups. Hill's infinite determinant thus breaks up into four factors, each of which can be formed separately by substituting respectively for $f(x)$ one of the four series

$$\left.\begin{array}{ll} \text{(I)} & \Sigma a_n \cos 2nx \\ \text{(II)} & \Sigma b_n \sin 2nx \\ \text{(III)} & \Sigma a'_n \cos (2n+1)x \\ \text{(IV)} & \Sigma b'_n \sin (2n+1)x \end{array}\right\}. \quad (21)$$

† H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, 2 (Paris, 1893), 260–277; Whittaker and Watson, *loc. cit.*, Ch. 19, *passim*; E. L. Ince, *Ordinary differential equations*, 175–178, 381–388.

The analysis is easily completed for the limiting case where $q = 0$, and this throws valuable light on the general case.

If $f(x) \to f_0(x)$ and $b \to b_0$ as $q \to 0$, and if $\Delta_0(b_0) = 0$ is the limiting form of Hill's discriminant, we have respectively

Case	(I)	(II)	(III)	(IV)
$f_0(x) =$	$\cos 2sx,$	$\sin 2sx,$	$\cos(2s-1)x,$	$\sin(2s-1)x,$

$$\Delta_0(b_0) = b_0 \prod_{s=1}^{\infty}\left(1-\frac{b_0}{4s^2}\right), \quad \prod_{s=1}^{\infty}\left(1-\frac{b_0}{4s^2}\right), \quad \prod_{s=1}^{\infty}\left(1-\frac{b_0}{(2s-1)^2}\right), \quad \prod_{s=1}^{\infty}\left(1-\frac{b_0}{(2s-1)^2}\right). \tag{22}$$

Each discriminant has simple zeros. The limiting forms (III) and (IV) are identical; those corresponding to Cases (I) and (II) differ by the factor b_0, because the equation yielding this factor is multiplied in one case by $\cos 0x$ and in the other by $\sin 0x$, and the former is to be retained and the latter erased before forming the discriminant.

When $q \neq 0$, each of the four cases gives a discriminant $\Delta(b, q) = 0$, whose left-hand side is analytic in b and in q, for all finite values of b and for $|q| < 1$. As $q \to 0$, $\Delta(b, q) \to \Delta_0(b)$; and, because the zeros of $\Delta_0(b)$ are simple, so also are those of $\Delta(b, q)$, provided that $|q| < c_1 < 1$. Hence each zero is analytic in q and can be expanded as a series

$$b = b_0 + b_1 q + b_2 q^2 + \ldots, \quad \text{where} \quad |q| < c_2 \leqslant c_1 < 1. \tag{23}$$

In practice, q is real and positive, and rough limits for each value of b are given by the oscillation theorem. Comparing the equation (20) with its limiting form when $q \to 0$, and taking solutions satisfying the same boundary conditions and having the same number of oscillations in the interval $0 \leqslant x \leqslant \frac{1}{2}\pi$, we find

$$|b - b_0| \leqslant 8|p| \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \frac{4|p|K(K-E)}{\pi^2}. \tag{24}$$

Hill's method gives the solution as an infinite determinant, which may be expanded as a Fourier series (21) whose coefficients are proportional to minors of $\Delta(b, q)$. This form of the solution is valid for all finite values of b and for $|q| < 1$, the value of b being a zero of $\Delta(b, q)$. When q tends to zero, $f(x)$ reduces to a single term $f_0(x)$, and we may seek an expansion of the form

$$f(x) \equiv f(x\,;\ s, p, q) = f_0(x) + qf_1(x) + q^2 f_2(x) \ldots. \tag{25}$$

We shall presently discuss this method of expansion and we shall see that it is valid for some region $|q|<c_3$. But, in obtaining it, we have to divide by one of the coefficients of the original expansion (21), and we also substitute for b the series (23). When the result is finally rearranged in powers of q, we must not expect it to converge over the whole unit circle for which Hill's determinant was valid.

Further light is thrown on the forms of the expansions (23) and (25) if we substitute $(x+\frac{1}{2}\pi, -q, b^*)$ for (x, q, b) respectively in (20). The solution $f(x)$ is replaced by $f^*(x) \equiv f(x+\frac{1}{2}\pi;\ s,\ p,\ -q)$. From the parity of $f(x)$ in x and in $x-\frac{1}{2}\pi$ and from its limiting form for small values of q, we see that in Cases (I) and (II) the new b^*, $\pm f^*(x)$ correspond to the same Sturm-Liouville problem as the old, and, since the number of "oscillations" in an interval is the same, we have $b^* = b$, $f^*(x) = \pm f(x)$. Hence the expansion (23) for b contains only even powers of q, while the coefficients of the Fourier series (21) are alternately odd and even in q.

But in Cases (III) and (IV), the two Sturm-Liouville problems are interchanged and the conclusion does not hold. If, however, b and $f(x)$ fulfil one set of conditions, $b^* = b(-q)$ and $f^*(x)$ fulfil the other. To illustrate the method of calculating the expansions, consider an even solution for which $b_0 = s^2$, and $f_0(x) = \cos sx$. We find on substituting formally in (20) and collecting powers of q,

$$\left.\begin{aligned}
f_0''+s^2 f_0 &= 0\\
f_1''+s^2 f_1+(b_1+8p\cos 2x)f_0 &= 0\\
f_2''+s^2 f_2+(b_1+8p\cos 2x)f_1+(b_2+16p\cos 4x)f_0 &= 0\\
f_3'''+s^2 f_3+(b_1+8p\cos 2x)f_2+(b_2+16p\cos 4x)f_1 &\\
+(b_3+24p\cos 6x+8p\cos 2x)f_0 &= 0\\
\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots &
\end{aligned}\right\}. \quad (26)$$

We know b_0 and $f_0(x)$. Suppose that

$$b_1,\ b_2,\ \ldots,\ b_{r-1}, \qquad f_1(x),\ f_2(x),\ \ldots,\ f_{r-1}(x),$$

are known; then b_r and $f_r(x)$ are uniquely given as follows:—

(i) $f_r(x)$ must be an even periodic function of x; since it is given by $f_r''(x)+s^2 f_r(x) = \phi(x)$, where $\phi(x)$ is a finite trigonometric series extending from terms in $\cos(s-2r)x$ to $\cos(s+2r)x$, it is necessary and sufficient that $\phi(x)$ should have no term in $\cos sx$. This can be secured by choosing b_r in one way only.

(ii) $f_r(x)$ must not contain $\cos sx$, because all such terms are collected together in $f_0(x)$. This makes the constant of integration zero, and $f_r(x)$ is uniquely determined by an elementary differential equation.

If s is even, the first few terms are

$$\left.\begin{aligned}
&b_0 = s^2, \quad b_2 = \frac{8p^2}{s^2-1}, \ \ldots, \quad b_1 = b_3 = \ldots = b_{2n+1} = 0\\
&f_0 = \cos sx\\
&f_1 = \frac{p}{s+1}\cos(s+2)x - \frac{p}{s-1}\cos(s-2)x\\
&f_2 = \frac{p}{s+2}\left(1+\frac{p}{2s+2}\right)\cos(s+4)x - \frac{p}{s-2}\left(1-\frac{p}{2s-2}\right)\cos(s-4)x\\
&f_3 = \frac{p}{s+3}\left(1+\frac{2p}{3s+3}+\frac{p}{3s+6}+\frac{p^2}{6(s+1)(s+2)}\right)\cos(s+6)x\\
&\qquad + \frac{p}{s+1}\left(1-\frac{2p}{s-1}+\frac{p}{s+2}+\frac{p^2}{2(s+1)(s+2)}\right.\\
&\qquad\qquad \left.+\frac{2p^2}{(s+1)^2(s-1)}\right)\cos(s+2)x\\
&\qquad - \frac{p}{s-3}\left(1-\frac{2p}{3s-3}-\frac{p}{3s-6}+\frac{p^2}{6(s-1)(s-2)}\right)\cos(s-6)x\\
&\qquad - \frac{p}{s-1}\left(1+\frac{2p}{s+1}-\frac{p}{s-2}+\frac{p^2}{2(s-1)(s-2)}\right.\\
&\qquad\qquad \left.-\frac{2p^2}{(s-1)^2(s+1)}\right)\cos(s-2)x
\end{aligned}\right\}. \quad (27)$$

We shall establish the convergence of our process by an argument based on *fonctions majorantes*, leaving some details of the proof to the reader.

If $\phi(x) \equiv \Sigma a_n \cos nx$ is a trigonometric series with a finite number of terms, let $A[\phi(x)] = \Sigma|a_n|$, $A[f_r(x)] = A_r$. It is easy to show that $A[\phi(x)\cos mx] \leqslant A[\phi(x)]$.

Now, from the definition of b_r as the coefficient of $q_r \cos sx$ in a certain trigonometric series, we can easily prove that

$$\sum_{n=1}^{\infty}|b_n|q^n \ll 8p\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)\left(1+\sum_{n=1}^{\infty}A_nq^n\right) \ll \frac{M_1}{1-d_1q}\left(1+\sum_{n=1}^{\infty}A_nq^n\right). \quad (28)$$

Again, from the definition of $f_r(x)$ and the fact that the particular solution of $y''+s^2y = \cos mx$ is $y = \cos mx/(s^2-m^2)$, and

$\big||s|-|m|\big| \geqslant 2$, we have

$$4\sum_{n=1}^{\infty} A_n q^n \ll \Big(\sum_{n=1}^{\infty} |b_n| q^n + \frac{M_1 q}{1-d_1 q}\Big)\Big(1+\sum_{n=1}^{\infty} A_n q^n\Big),$$

$$4\sum_{n=1}^{\infty} A_n q^n \ll \frac{M_1 q}{1-d_1 q}\Big(2+\sum_{n=1}^{\infty} A_n q^n\Big)\Big(1+\sum_{n=1}^{\infty} A_n q^n\Big). \tag{29}$$

By expanding in powers of q the solution of the equation

$$4(1-d_1 q)\, y = M_1 q(2+3y+y^2), \tag{30}$$

which vanishes with q, we have the two *fonctions majorantes* which establish the convergence

$$\sum_{n=1}^{\infty} A_n q^n \ll \frac{M_2 q}{1-d_2 q}, \quad \text{and hence} \quad \sum_{n=1}^{\infty} |b_n| q^n \ll \frac{M_3 q}{1-d_3 q}. \tag{31}$$

In connexion with (27), we note that each $f_r(x)$ is an even function of s, so that if half the coefficients are calculated, the rest are given by symmetry. We also find that, if s is odd, the first odd power to appear in (23) is $b_s q^s$, and the expressions (27) are valid for suffixes up to $s-1$.

III.

We must now leave the problem of expanding the individual function $f(x)$, and consider the properties of the family of these functions corresponding to the different values of b, and to a single fixed value of m.

The arbitrary function $A_m(u)$ in (15) may be divided into four parts, which are respectively *even* or *odd* in the variables u and $u-K$. We require the values of each part in the range $0 \leqslant u \leqslant K$, or say $0 \leqslant x \leqslant \frac{1}{2}\pi$. Taking one of the four parts, we associate it with one of the four families of functions satisfying the differential equation (13) and one of the pairs of boundary conditions (17). The next step of the theory is important, but need not detain us, as it is fully covered by classical theorems†. From the Sturm-Liouville theory we know that each family forms a complete system of orthogonal functions for the range $0 \leqslant x \leqslant \frac{1}{2}\pi$; for large values of s, the functions behave asymptotically like a single sine or cosine of a multiple of x. Any function which can be

† Riemann-Weber, *Differentialgleichungen der Physik*, 7th ed. (Braunschweig, 1925), 1, Ch. 7; E. Goursat, *Cours d'analyse*, 3, Ch. 33; A. Kneser, *Die Integralgleichungen . . .*, 2nd ed. (Braunschweig, 1922); Courant-Hilbert, *Methoden der Mathematischen Physik* (Berlin, 1924).

expanded as a Fourier series can also be expanded as a series of these normal functions.

Assuming that $A_m(u)$ can be written in such a form, we consider a typical term, and our problem is to construct a harmonic function $V_{m,s}(u, v, \phi)$, which is regular in free space and zero at infinity, and which takes the values

$$V_{m,s}(u, 0, \phi) = \varpi^{-\frac{1}{2}} \cos m(\phi - \phi_0) F_s^{(m)}(u) \tag{32}$$

on the ring $v = 0$, where $F_s^{(m)}(u) \equiv f(x; s, p, q)$ in the notation of the last section.

The solution will be given by a normal product

$$V_{m,s}(u, v, \phi) \equiv \varpi^{-\frac{1}{2}} \cos m(\phi - \phi_0) F_s^{(m)}(u) G_s^{(m)}(iv), \tag{11 a}$$

provided that the last factor satisfies the following conditions :—

(i) it is a solution of the Lamé equation (14), where all the parameters have been fixed by the former discussion ;

(ii) $\varpi^{-\frac{1}{2}} G(iv)$ must be finite for all real values of u and $0 \leqslant v \leqslant K'$, and must vanish at infinite distances from the ring ;

(iii) on the surface of the ring $v = 0$, we must have $G(0) = 1$. The most unfavourable value of $\varpi^{-\frac{1}{2}}$ being given by $u = 0$, and ϖ being of the order $v - K'$ on this locus, the first and second conditions will be satisfied if $G(iv)$ is a branch of the solution which is $O\{\sqrt{(v - K')}\}$ as $v \to K'$. Now $iv = iK'$ is a regular singularity with the exponents $\frac{1}{2} \pm m$; so that $G(iv)$ is to be identified with the branch of exponent $\frac{1}{2} + m$ if $m \geqslant 0$, or with the branch free from logarithms if $m = 0$.

The third condition could not be satisfied if this branch vanished for $v = 0$; but this would make the normal product zero at all points of the ring and at infinity, and therefore identically zero everywhere, which is absurd.

Any obvious method of computing $G(iv)$ is laborious for a single specimen and prohibitive for the doubly-infinite system. In general terms, $G(iv)$ is represented by different expansions near $v = 0$ and $v = K'$; and the analytical continuation is theoretically effected by comparing the values of $G(iv_0)$ and $G'(iv_0)$ at a convenient point, where both expansions hold. Since $G(0) = 1$, we have, near $v = 0$,

$$G_s^{(m)}(iv) = 1 + \sum_{r=1}^{\infty} a_{2r} \operatorname{sn}^{2r} iv + A \sum_{r=1}^{\infty} a_{2r-1} \operatorname{sn}^{2r-1} iv, \tag{33}$$

where

$$(r+1)(r+2)\, a_{r+2} - [r^2(1+k^2) + B]\, a_r + k^2 [r^2 - 3r + \tfrac{9}{4} - m^2]\, a_{r-2} = 0. \tag{34}$$

On the other hand, since the solution belongs to the exponent $\frac{1}{2}+m$ at $v = K'$, we have an expansion of the form

$$G_s^{(m)}(iv) = (\mathrm{dn}\, iv)^{-\frac{1}{2}-m} \sum_{r=0}^{\infty} b_{2r}\, \mathrm{dn}^{-2r}\, iv, \tag{35}$$

where

$$4(r+1)(r+2)\, b_{2r+2} - [4r^2+4mr+2r+m+\tfrac{1}{2}+k'^2(m+2r+\tfrac{1}{2})^2 - B]\, b_{2r} + k'^2[(m+2r-1)^2-\tfrac{1}{4}]\, b_{2r-2} = 0. \tag{36}$$

These formulae give some idea of the mathematical labours which must confront the inhabitants of Saturn's ring!

If we suppose every arithmetical difficulty surmounted, and if a given function is expressed as a double series which converges uniformly over the surface of the ring, then, by an adaptation of Harnack's theorem, we infer that the corresponding double series of harmonic functions also converges in free space. We thus have a method, or programme of calculation, for the construction of the complementary function in a typical potential problem, the singularities in free space being known in advance.

Note added 6 *March*, 1929. By Brioschi's method, putting

$$iv = 2\xi - iK', \quad G(iv) = \{\mathrm{sn}\,\xi\, \mathrm{cn}\,\xi\, \mathrm{dn}\,\xi\}^{\frac{1}{2}-m}.\, w,$$

we obtain an expansion of w in descending powers of $\mathrm{dn}\,\zeta$ valid in the whole interval. Details for the case $m = 1$ will appear in another paper (cf. Halphen, *Fonctions Elliptiques*, II, 471).

THE ADDITION THEOREMS FOR GENERAL LEGENDRE FUNCTIONS

By E. W. HOBSON.

[Received 11 June, 1928.—Read 14 June, 1928.]

1. The present communication contains an investigation of the addition theorems for the general Legendre functions

$$P_n\{\mu\mu'-\sqrt{(\mu^2-1)}\sqrt{(\mu'^2-1)}\cos\phi\}, \quad Q_n\{\mu\mu'-\sqrt{(\mu^2-1)}\sqrt{(\mu'^2-1)}\cos\phi\},$$

of the first and second kinds, where n is not restricted to be integral or real, ϕ is real, and μ, μ' are points in the plane of a complex variable, with a cross-cut extending along the real axis from the point 1 to $-\infty$. The definitions, notation, and methods of my memoir*, "On a type of spherical hormonics of unrestricted degree, order, and argument", are employed here; reference to this memoir will be denoted by H., with the numbered formula, section, or page added. The advantage of an investigation of this general character is that the results may be applied to all the special cases, in which n has either real or complex values, and in which the arguments are either real and greater than unity, or real and less than unity, such as those in which the boundaries of the spaces considered in potential problems are either spheres, or spheroids, or cones, or anchor rings. The most general investigation for the addition theorem for the function P_n with which I am acquainted is that given in Whittaker and Watson's work *Modern analysis*; the result there established imposes for conditions of validity of the theorem that

$$R(\mu), \quad R(\mu'), \quad \text{and} \quad R\{\mu\mu'-\sqrt{(\mu^2-1)}\sqrt{(\mu'^2-1)}\cos\phi\},$$

for all real values of ϕ, should be positive. I shall here show that the last of these conditions is unnecessary, and that the theorem is also valid in certain cases when one of the two numbers $R(\mu)$, $R(\mu')$ is negative. The method of investigation is different from that of Whittaker and Watson. A very general investigation of the addition theorem for the

* *Phil. Trans.*, 186 (1896), 443.

function Q_n has been given* by Heine; I have, however, endeavoured to make this investigation more exact and more general. The error made by Heine in his definition of the function P_n, which I have already pointed out (H., § 37), has an effect which influences Heine's work at various points, and introduces some inexactitudes into his results.

2. In the ensuing investigations two theorems will be required, the first of which is the following :—

If $R(\mu) > 0$, *and* n *is unrestricted, the series*

$$P_n(\mu) + 2 \sum_{m=1}^{\infty} \frac{\Pi(n)}{\Pi(n+m)} P_n^m(\mu) \cos m(\phi \pm iu),$$

where u *is real and* $|u| < \frac{1}{2} \log \left| \frac{\mu+1}{\mu-1} \right|$, *converges to*

$$\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n;$$

the convergence being uniform for all values of ϕ *and for all values of* u, *such that*

$$|u| \leqslant u_0 < \tfrac{1}{2} \log \left| \frac{\mu+1}{\mu-1} \right|.$$

It is clear that the restriction that u must be real may be removed, because an imaginary part of u can be absorbed in ϕ; the condition which u must satisfy then becomes

$$|R(u)| < \tfrac{1}{2} \log \left| \frac{\mu+1}{\mu-1} \right|.$$

In the following proof we restrict u to be real.

Let $w = (\mu^2-1)^{\frac{1}{2}} e^{i\phi - u}$, where u is positive; we then find that

$$\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi + iu)\}^n = 2^{-n}(\mu+1)^n \left(1 + \frac{\mu-1}{w}\right)^n \left(1 + \frac{w}{\mu+1}\right)^n;$$

and

$$\left| \frac{\mu-1}{w} \right| < \left| \frac{\mu-1}{\mu+1} \right|^{\frac{1}{2}} e^{u_0} < 1,$$

also

$$\left| \frac{w}{\mu+1} \right| < \left| \frac{\mu-1}{\mu+1} \right|^{\frac{1}{2}} < 1.$$

The case in which we take $w = (\mu^2-1)^{\frac{1}{2}} e^{i\phi + u}$ can be treated similarly in relation to $\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi - iu)\}^n$; we need consider the first case only.

---

* *Kugelfunctionen*, 1 (1878), 332.

Let the binomial series for $\left(1+\frac{\mu-1}{w}\right)^n$ be $a_0+a_1+a_2+\ldots$; and let that for $\left(1+\frac{w}{\mu+1}\right)^n$ be $b_0+b_1+b_2+\ldots$: these series are absolutely convergent for each value of ϕ, and for each value of u in the interval $(0, u_0)$.

Let $a_0+a_1+a_2+\ldots$ and $\beta_0+\beta_1+\beta_2+\ldots$ be the binomial expansions of

$$\left(1-\left|\frac{\mu-1}{\mu+1}\right|^{\frac{1}{2}} e^{u_0}\right)^{-|n|} \quad \text{and} \quad \left(1-\left|\frac{\mu-1}{\mu+1}\right|^{\frac{1}{2}}\right)^{-|n|};$$

all the numbers a_r, β_r are then real and positive. Moreover, $|a_r| < a_r$, $|b_r| < \beta_r$, for all values of r.

The Cauchy product

$$a_0\beta_0+(a_0\beta_1+a_1\beta_0)+\ldots+(a_0\beta_r+a_1\beta_{r-1}+\ldots+a_r\beta_0)+\ldots$$

of the two absolutely convergent series $a_0+a_1+a_2+\ldots$, $\beta_0+\beta_1+\beta_2+\ldots$ is convergent, its sum being

$$\left(1-\left|\frac{\mu-1}{\mu+1}\right|^{\frac{1}{2}} e^{u_0}\right)^{-|n|}\left(1-\left|\frac{\mu-1}{\mu+1}\right|^{\frac{1}{2}}\right)^{-|n|}.$$

Thus the double series $\sum_{r=0}\sum_{s=0} a_r\beta_s$ is absolutely convergent, and it may be rearranged in a series of any type without alteration of its sum; thus the series

$$(a_0\beta_0+a_1\beta_1+a_2\beta_2+\ldots)+(a_0\beta_1+a_1\beta_2+\ldots)+(a_1\beta_0+a_2\beta_1+\ldots)$$
$$+\ldots+(a_0\beta_r+a_1\beta_{r+1}+\ldots)+(a_r\beta_0+a_{r+1}\beta_1+\ldots)+\ldots$$

is convergent.

Since $|a_rb_s| < a_r\beta_s$, the double series $\sum_{r=0}\sum_{s=0} a_rb_s$ is absolutely convergent, and it may be rearranged in a series of any type, which is convergent, and of which the sum is independent of the type. It follows that the series

$$(a_0b_0+a_1b_1+a_2b_2+\ldots)+(a_0b_1+a_1b_2+\ldots)+(a_1b_0+a_2b_1+\ldots)$$
$$+\ldots+(a_0b_r+a_1b_{r+1}+\ldots)+(a_rb_0+a_{r+1}b_1+\ldots)+\ldots.$$

converges to the same sum as the Cauchy product of the two series $a_0+a_1+\ldots$, $b_0+b_1+\ldots$, which is

$$\left(1+\frac{\mu-1}{w}\right)^n\left(1+\frac{w}{\mu+1}\right)^n,$$

since the two series are absolutely convergent.

By applying Weierstrass's test it is seen that the series of which the general term is $(a_0 b_r + a_1 b_{r+1} + \ldots) + (a_r b_0 + a_{r+1} b_1 + \ldots)$ converges uniformly for all values of ϕ, and for all values of u in the interval $(0, u_0)$, where

$$0 < u_0 < \tfrac{1}{2} \log \left| \frac{\mu+1}{\mu-1} \right|.$$

For this general term is, in absolute value, less than

$$(a_0 \beta_r + a_1 \beta_{r+1} + \ldots) + (a_r \beta_0 + a_{r+1} \beta_1 + \ldots),$$

which is the general term of a numerical series that is absolutely convergent.

The series

$$(a_0 b_0 + a_1 b_1 + \ldots) + (a_0 b_1 + a_1 b_2 + \ldots) + (a_1 b_0 + a_2 b_1 + \ldots) + \ldots$$

is a series of terms which consist of powers of w, positive and negative, or in cosines and sines of multiples of $\phi + iu$. Its sum is

$$\left(1 + \frac{\mu-1}{w}\right)^n \left(1 + \frac{w}{\mu+1}\right)^n.$$

It has thus been shown that $\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n$ can be expressed as the sum of a series

$$\sum_{m=0}^{\infty} (W_m \cos m\phi + V_m \sin m\phi),$$

which converges uniformly for all values of ϕ and for all values of u in $(0, u_0)$.

Remembering that $R(\mu) > 0$, we have

$$W_m = \frac{1}{\pi} \int_0^{2\pi} \{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n \cos m\phi \, d\phi$$

$$= \frac{2\Pi(n)}{\Pi(n+m)} P_n^m(\mu) \cos m(\mp iu) \quad \text{by [H. (81)]},$$

except that $W_0 = P_n(\mu)$. Also

$$V_m = \frac{2\Pi(n)}{\Pi(n+m)} P_n^m(\mu) \sin m(\mp iu), \qquad V_0 = P_n(\mu).$$

It follows that

$$\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n = P_n(\mu) + 2 \sum_{m=1}^{\infty} \frac{\Pi(n)}{\Pi(n+m)} P_n^m(\mu) \cos m(\phi \pm iu),$$

where $R(\mu) > 0$, n is unrestricted, and the convergence is uniform for all values of ϕ, and for

$$|u| \leqslant u_0 < \tfrac{1}{2} \log \left| \frac{\mu+1}{\mu-1} \right|,$$

where u is real.

In case n is a positive real number, the series is finite, having only $n+1$ terms.

Changing n into $-n-1$, and remembering that

$$P_n^m(\mu) = P_{-n-1}^m(\mu), \qquad \frac{\Pi(-n-1)}{\Pi(m-n-1)} = (-1)^m \frac{\Pi(n-m)}{\Pi(n)},$$

we see that

$$P_n(\mu) + 2 \sum_{m=1}^{\infty} (-1)^m \frac{\Pi(n-m)}{\Pi(n)} P_n^m(\mu) \cos m(\phi \pm iu)$$

converges, uniformly as before, to

$$\frac{1}{\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^{n+1}}.$$

It is easy to deduce the theorem which corresponds to (1) in the case in which $R(\mu) < 0$. Let $\bar{\mu} = -\mu$, then we have

$$\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n = e^{\pm n\pi i} \{\bar{\mu} + (\bar{\mu}^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n,$$

where the upper or the lower sign of the exponential is to be taken, according as $I(\mu) > 0$ or $I(\mu) < 0$.

Since $\{\bar{\mu} + (\bar{\mu}^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n$ is represented by the series

$$P_n(\bar{\mu}) + 2 \sum_{m=0}' \frac{\Pi(n)}{\Pi(n+m)} P_n^m(\bar{\mu}) \cos m(\phi \pm iu)$$

and

$$P_n^m(\bar{\mu}) = e^{\mp n\pi i} P_n^m(\mu) - \frac{2 \sin n\pi}{\pi} Q_n^m(\mu) \quad [\text{H. (20)}],$$

where the upper or lower sign in the exponential is to be taken according as $I(\mu) > 0$ or $I(\mu) < 0$, we see that the series which represents $\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\phi \pm iu)\}^n$, when $R(\mu) < 0$, is

$$\left[P_n(\mu) - \frac{2 \sin n\pi}{\pi} e^{\pm n\pi i} Q_n(\mu) \right]$$

$$+ 2 \sum_{m=1}' \frac{\Pi(n)}{\Pi(n+m)} \left[P_n^m(\mu) - \frac{2 \sin n\pi}{\pi} e^{\mp n\pi i} Q_n^m(\mu) \right] \cos m(\phi \pm iu), \quad (2)$$

where the upper or the lower sign in $e^{\pm n\pi i}$ is to be taken according as $I(\mu) > 0$ or $I(\mu) < 0$. The convergence is uniform as before.

3. It is known that $Q_n(\mu)$ is expressible in the form

$$Q_n(\mu) = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}},$$

provided that $R(n+1) > 0$ [H. (86)]. This holds good for all values of μ that are not real and between 1 and $-\infty$.

If λ be a real number such that, in the plane of u, there is no singular point of the integrand in the infinite rectangle bounded by the real axis of u and a straight line parallel to it at a distance λ from it, the integral taken, as in the above expression, along the real axis of u is equal in value to the integral taken along the parallel side of the rectangle. Thus we can write

$$Q_n(\mu) = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\lambda)\}^{n+1}},$$

provided that, if z_0 be a real or complex number for which

$$\mu+(\mu^2-1)^{\frac{1}{2}}\cosh z_0 = 0,$$

the point z_0 does not lie in the rectangle bounded by the real axis of u and a parallel to it at distance λ. It is also clear that λ may be taken to be complex, because the real part of $i\lambda$ causes only the addition of a constant to u, which does not affect the integration between infinite limits. We proceed to consider the case in which the condition given is not satisfied. In order to find the points z_0 for which $\mu+(\mu^2-1)^{\frac{1}{2}}\cosh z_0 = 0$, let $-\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}} = \rho(\cos\chi+i\sin\chi)$, where ρ is positive and χ is in the interval $(-\pi, \pi)$. Let $z_0 = u_0+i\lambda_0$, then

$$\cosh u_0 \cos\lambda_0 = \rho\cos\chi, \qquad \sinh u_0 \sin\lambda_0 = \rho\sin\chi;$$

these equations determine u_0 and λ_0. If χ is in the interval $(0, \pi)$, we may take λ_0 to be in the same interval, and u_0 to be positive; then z_0 may have the value $u_0+i\lambda_0$. If χ is in the interval $(-\pi, 0)$, we may take u_0 to be negative and λ_0 to be in the interval $(0, \pi)$. In case $\lambda < \lambda_0$ the value of the integral is, as above, $Q_n(\mu)$. If $\lambda > \lambda_0 > 0$, we have

$$\int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\lambda)\}^{n+1}} = \int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\pi)\}^{n+1}},$$

since there is no zero of $\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\lambda)$ at any point in the rectangle bounded by the straight lines at distances λ and π from the real axis of u. In order to evaluate the integral on the right-hand side, we see

that, when u_0 is positive,

$$\int_{-\infty}^{0} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}} - \int_{-\infty}^{0} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\pi)\}^{n+1}}$$
$$= -i\int_0^{\pi} \frac{d\lambda}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos\lambda\}^{n+1}};$$

for the sum of the integrals along the three straight lines for which $-\infty < u \leqslant 0$, $\lambda = 0$; $u = 0$, $0 \leqslant \lambda \leqslant \pi$; $0 \geqslant u > -\infty$, $\lambda = \pi$ is zero, since there is no singularity of the integrand in the rectangle. When $R(\mu) > 0$, the value of the expression on the right-hand side is $-i\pi P_n(\mu)$, and when $R(\mu) < 0$ it is

$$-i\pi\left\{P_n(\mu) - \frac{2}{\pi}e^{\pm n\pi i}\sin n\pi\, Q_n(\mu)\right\},$$

the upper or the lower sign being taken in $e^{\pm n\pi i}$, according as $I(\mu) \gtrless 0$.

Similarly, in case u_0 is negative, we have

$$\int_0^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}} - \int_0^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\pi)\}^{n+1}}$$
$$= i\int_0^{\pi} \frac{d\lambda}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos\lambda\}^{n+1}};$$

the value of the expression on the right-hand side is

$$i\pi P_n(\mu) \text{ when } R(\mu) > 0 \quad \text{and} \quad i\pi\left\{P_n(\mu) - \frac{2}{\pi}e^{\pm n\pi i}Q_n(\mu)\right\} \text{ when } R(\mu) < 0,$$

where the sign in $e^{\pm n\pi i}$ is fixed as before.

Since the integrals taken from 0 to ∞, and from $-\infty$ to 0, have the same value, we see that, when $\lambda > \lambda_0 > 0$,

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}} - \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^{n+1}}$$

has the value $-i\pi P_n(\mu)$ or $i\pi P_n(\mu)$ according as u_0 is positive or negative, when $R(\mu) > 0$; and when $R(\mu) < 0$ it has the value

$$+i\pi\left\{e^{\mp 2n\pi i}P_n(\mu) - \frac{2}{\pi}e^{+n\pi i}\sin n\pi\, Q_n(\mu)\right\}$$

if u_0 is positive, and the value with the sign changed if u_0 is negative; the sign in the exponential being the upper or the lower according as $I(\mu) \gtrless 0$.

We now see that, *when* $\lambda > \lambda_0$,

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh(u+i\lambda)\}^{n+1}}$$

has the value $Q_n(\mu) \pm i\pi P_n(\mu)$, *when* $R(\mu) > 0$, *where the upper or the lower sign is taken according as* u_0 *is positive or negative. When* $R(\mu) < 0$, *the value is*

$$Q_n(\mu) \mp i\pi \left\{ e^{\mp 2n\pi i} P_n(\mu) - \frac{2}{\pi} e^{\mp n\pi i} \sin n\pi Q_n(\mu) \right\},$$

where the sign in $e^{\mp n\pi i}$ *depends upon that of* $I(\mu)$. *The number* u_0 *is positive or negative according as the argument of* $\frac{-\mu}{(\mu^2-1)^{\frac{1}{2}}}$ *is in the interval* $(0, \pi)$ *or in* $(-\pi, 0)$.

The above theorem gives no result for the case $R(\mu) = 0$. In that case, if $\mu = i\nu$, we have

$$\frac{-\mu}{(\mu^2-1)^{\frac{1}{2}}} = \rho(\cos\chi + i\sin\chi),$$

where $\chi = \pi$, $\rho = \frac{\nu}{(\nu^2+1)^{\frac{1}{2}}}$; hence $u_0 = 0$, $\cos\lambda_0 = -\frac{\nu}{(\nu^2+1)^{\frac{1}{2}}}$; thus λ_0 is in the interval $(\frac{1}{2}\pi, \pi)$.

In this case we have, for the value of $Q_n(\mu)$

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu + (\mu^2-1)^{\frac{1}{2}} \cosh(u+i\lambda)\}^{n+1}}, \quad \text{in case } \lambda < \lambda_0,$$

and

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{du}{\{\mu - (\mu^2-1)^{\frac{1}{2}} \cosh u\}^{n+1}}, \quad \text{in case } \lambda > \lambda_0.$$

4. Let μ, μ' be two points in the plane of μ which are outside the cross-cut along the real axis from 1 to $-\infty$; and let it be assumed that $R(\mu) > 0$, $R(\mu') > 0$.

It has been shown in § 2 that the Fourier series for the two functions

$$\{\mu + (\mu^2-1)^{\frac{1}{2}} \cos(\psi-\phi)\}^n, \quad \{\mu' + (\mu'^2-1)^{\frac{1}{2}} \cos\psi\}^{-n-1}$$

are

$$P_n(\mu) + 2\sum_{m=1}^{\infty} \frac{\Pi(n)}{\Pi(n+m)} P_n^m(\mu) \cos m(\psi-\phi),$$

$$P_n(\mu') + 2\sum_{m=1}^{\infty} (-1)^m \frac{\Pi(n-m)}{\Pi(n)} P_n^m(\mu') \cos m\phi;$$

and it was shown that these series converge uniformly for all values of the real numbers ψ, ϕ. The number n is unrestricted, but the first series is finite in case n is a positive integer.

From Parseval's theorem, it follows that the series

$$2P_n(\mu)\,P_n(\mu')+4\sum_{m=1}^{\infty}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\mu)\,P_n^m(\mu')\cos m\phi$$

converges to $$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)\}^n}{\{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi\}^{n+1}}\,d\psi.$$

It can readily be shown that the convergence of the series is uniform for all real values of ϕ. For we have

$$\left|\int_{-\pi}^{\pi}\left\{2\sum_{m=m_1}^{m_1+r}\frac{\Pi(n)}{\Pi(n+m)}P_n^m(\mu)\cos m(\psi-\phi)\right\}\right.$$
$$\left.\times\left\{2\sum_{m=m_1}^{m_1+r}(-1)^m\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\mu')\cos m\psi\right\}d\psi\right|$$
$$\leqslant\int_{-\pi}^{\pi}\left|2\sum_{m=m_1}^{m_1+r}\frac{\Pi(n)}{\Pi(n+m)}P_n^m(\mu)\cos m(\psi-\phi)\right|$$
$$\times\left|2\sum_{m=m_1}^{m_1+r}(-1)^m\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\mu')\cos m\psi\right|d\psi;$$

if, now, m_1 be chosen sufficiently large, the expression on the right-hand side is less than $2\pi\epsilon^2$, for all values of ϕ, and all values 1, 2, 3, ... of r, since the two series with which we started are uniformly convergent, where ϵ is arbitrarily chosen. The expression on the left-hand side is equal to

$$\left|4\pi\sum_{m=m_1}^{m_1+r}(-1)^m\frac{\Pi(n-m}{\Pi(n+m}P_n^m(\mu)\,P_n^m(\mu')\cos m\phi\right|;$$

and thus the condition for uniform convergence is satisfied.

5. If the two points $\mu+(\mu^2-1)^{\frac{1}{2}}$, $\mu-(\mu^2-1)^{\frac{1}{2}}$ be denoted by P, Q, and the two points $\mu'+(\mu'^2-1)^{\frac{1}{2}}$, $\mu'-(\mu'^2-1)^{\frac{1}{2}}$ be denoted by P', Q', the points $\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)$, $\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi$ are points R, R' lying respectively on the segments PQ, $P'Q'$. It follows that

$$\left|\frac{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)}{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi}\right|=\left|\frac{OR}{OR'}\right|;$$

and this has its value, for all values of ψ and ϕ, in the interval bounded by the greatest and least of the four numbers

$$\left|\frac{OP}{OP'}\right|,\ \left|\frac{OP}{OQ'}\right|,\ \left|\frac{OQ}{OP'}\right|,\ \left|\frac{OQ}{OQ'}\right|.$$

Thus the points $$\frac{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)}{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi},$$

when all values of ψ and ϕ are taken into account, form a bounded set of points, at a finite minimum distance from the origin. Let ϕ have any fixed value; then all the points OR/OR' may be included in the interior of a contour S, to which the origin is exterior, and which cuts the real axis only in points on the right of the origin.

In order to transform the definite integral

$$\int_{-\pi}^{\pi} \frac{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)\}^n}{\{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi\}^{n+1}}\,d\psi,$$

where ϕ has any fixed value, we see, by Cauchy's theorem, that

$$\left\{\frac{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)}{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi}\right\}^n = \frac{1}{2\pi i}\int_{(S)} \frac{h^n}{h-\dfrac{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)}{\mu'+(\mu'^2-1)\cos\psi}}\,dh;$$

and we thus have

$$\int_{-\pi}^{\pi} \frac{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)\}^n}{\{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi\}^{n+1}}\,d\psi = \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\psi \int_{(S)} \frac{h^n}{A+B\cos\psi+C\sin\psi}\,dh,$$

where

$$A = h\mu'-\mu, \quad B = h(\mu'^2-1)^{\frac{1}{2}}-(\mu^2-1)^{\frac{1}{2}}\cos\phi, \quad C = -(\mu^2-1)^{\frac{1}{2}}\sin\phi.$$

Since the modulus of the integrand on the right-hand side is bounded, the order of the repeated integration may be reversed without changing the result; we thus have

$$\int_{-\pi}^{\pi} \frac{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)\}^n}{\{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi\}^{n+1}}\,d\psi = \frac{1}{2\pi i}\int_{(S)} h^n\,dh \int_{-\pi}^{\pi} \frac{d\psi}{A+B\cos\psi+C\sin\psi}.$$

We find at once that $A^2-B^2-C^2 = 1-2h\xi_\phi+h^2$, where ξ_ϕ denotes $\mu\mu'-(\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}\cos\phi$.

It was shown by Jacobi* that the integral $\displaystyle\int_{-\pi}^{\pi} \frac{d\psi}{A+B\cos\psi+C\sin\psi}$ has the value $\dfrac{2\pi}{(A^2-B^2-C^2)^{\frac{1}{2}}}$, where the sign of the radical is to be so taken that $|A-(A^2-B^2-C^2)^{\frac{1}{2}}|$ is less than the greater of the two numbers $|B-iC|$, $|B+iC|$, subject to the following conditions:

(*a*) If $A = x+ix'$, $B = y+iy'$, $C = z+iz'$, and $yz'-y'z$, $zx'-z'x$, $xy'-x'y$ do not all vanish, then the condition is that

$$(yz'-y'z)^2-(zx'-z'x)^2-(xy'-x'y)^2 > 0.$$

* *Journal für Math.*, 32 (1846), 8-13. An account of this investigation is given by Heine, *Kugelfunctionen*, 1 (1878), 27-31.

In case this inequality is reversed, the value of the integral is zero. In case the expression vanishes, the integral has no value.

(*b*) If $yz'-y'z$, $zx'-z'x$, $xy'-x'y$ are all zero, the condition is that $x^2-y^2-z^2>0$, the sign of the radical being determined as before.

In order to apply this theorem to the case above, let $\mu=\cosh(\xi+i\eta)$, $\mu'=\cosh(\xi'+i\eta')$; when $0\leqslant\xi<\infty$, $-\frac{1}{2}\pi<\eta<\frac{1}{2}\pi$, $0\leqslant\xi'<\infty$, $-\frac{1}{2}\pi<\eta'<\frac{1}{2}\pi$.

Thus μ, μ' lie on ellipses with foci at the points 1, -1.

The semi-major axes of the ellipses are $\cosh\xi$, $\cosh\xi'$, and the points μ, μ' are at the points on these ellipses of which the eccentric angles are η, η' respectively. Let $h=h_1+ik_1$, we have then

$$x=h_1\cosh\xi'\cos\eta'-k_1\sinh\xi'\sin\eta'-\cosh\xi\cos\eta,$$

$$x'=h_1\sinh\xi'\sin\eta'+k_1\cosh\xi'\cos\eta'-\sinh\xi\sin\eta,$$

$$y=h_1\sinh\xi'\cos\eta'-k_1\cosh\xi'\sin\eta'-\sinh\xi\cos\eta\cos\phi,$$

$$y'=h_1\cosh\xi'\sin\eta'+k_1\sinh\xi'\cos\eta'-\cosh\xi\sin\eta\cos\phi,$$

$$z=-\sinh\xi\cos\eta\sin\phi,\quad z'=-\cosh\xi\sin\eta\sin\phi.$$

We then find that

$$\begin{aligned}yz'-y'z=\sin\phi\,[&h_1(\sinh\xi\cosh\xi'\cos\eta\sin\eta'-\cosh\xi\sinh\xi'\sin\eta\cos\eta')\\&+k_1(\sinh\xi\sinh\xi'\cos\eta\cos\eta'+\cosh\xi\cosh\xi'\sin\eta\sin\eta')],\end{aligned}$$

$$\begin{aligned}xz'-x'z=\sin\phi\,[&h_1(-\cosh\xi\cosh\xi'\sin\eta\cos\eta'+\sinh\xi\sinh\xi'\cos\eta\sin\eta')\\&+k_1(\cosh\xi\sinh\xi'\sin\eta\sin\eta'+\sinh\xi\cosh\xi'\cos\eta\cos\eta')]\\&+\sin\eta\cos\eta\sin\phi,\end{aligned}$$

$$\begin{aligned}xy'-x'y=(h_1^2+k_1^2)\sin\eta'\cos\eta'+h_1[&-\cosh\xi\cosh\xi'\sin\eta\sin\eta'\cos\phi\\&-\cosh\xi\cosh\xi'\cos\eta\sin\eta'\\&+\sinh\xi\sinh\xi'\cos\eta\sin\eta'\cos\phi\\&+\sinh\xi\sinh\xi'\sin\eta\cos\eta']\\+k_1[&\cosh\xi\sinh\xi'\sin\eta\sin\eta'\cos\phi-\cosh\xi\sinh\xi'\cos\eta\cos\eta'\\&+\cosh\xi'\sinh\xi\cos\eta\cos\eta'-\sinh\xi\sinh\xi'\sin\eta\sin\eta']\\&+\sin\eta\cos\eta\cos\phi.\end{aligned}$$

When $\sin\phi \neq 0$, there is, in general, one point (h_1, k_1) at which the first two expressions vanish; if, and only if, this point is such that the third expression vanishes at it, there is a single point at which all three expressions vanish. If ϕ is 0, π, or $-\pi$, the first two expressions vanish, and the third will vanish only on the circumference of a certain circle. This case may be omitted because the integral must be a continuous function of ϕ at each of these points.

Except when $\sin\eta' = 0$, it is seen that, when $|h|$ is sufficiently large, on account of the dominant term $(h_1^2+k_1^2)\sin\eta'\cos\eta'$ in the last expression, the condition $(yz'-z'y)^2-(xy'-x'y)^2-(xz'-x'z)^2 < 0$ is satisfied for all sufficiently large values of $|h|$. When $\eta' = 0$, the values, at the origin, of the three expressions are 0, $\sin\phi\sin\eta\cos\eta$, $\sin\eta\cos\eta\cos\phi$; hence, at the origin, $(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'z)^2 = -\sin^2\eta\cos^2\eta < 0$, unless $\eta = 0$.

It has thus been shown that, when η, η' are not both zero, there exists a point exterior to S at which the condition

$$(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'z)^2 < 0$$

is satisfied. When η and η' both vanish, the points μ and μ' are both on the real axis and are greater than 1; $yz'-y'z$, $xy'-x'y$, $xz'-x'z$ all vanish at the origin, and $x = -\cosh\xi$, $y = -\sinh\xi\cos\phi$, $z = -\sinh\xi\sin\phi$, and thus $x^2-y^2-z^2 = 1$, and therefore the condition is satisfied at the origin for the value of the integral to be $2\pi/(A^2-B^2-C^2)^{\frac{1}{2}}$.

At a point at which $A+B\cos\psi+C\sin\psi = 0$, for some value of ψ we have $x+y\cos\psi+z\sin\psi = 0$, $x'+y'\cos\psi+z'\sin\psi = 0$, and thus

$$\frac{1}{yz'-y'z} = \frac{\cos\psi}{zx'-z'x} = \frac{\sin\psi}{xy'-x'y},$$

hence $(yz'-y'z)^2-(zx'-x'z)^2-(xy'-x'y)^2 = 0$, and by hypothesis all such points are interior to S. Conversely, if this expression vanishes at a point, there is a value of ψ for which $A+B\cos\psi+C\sin\psi = 0$. If $yz'-y'z$, $zx'-z'x$, $xy'-xy$ are all zero, or $x/x' = y/y' = z/z'$, or $A+B\cos\psi+C\sin\psi = 0$, we have $x+y\cos\psi+z\sin\psi = 0$, for such values of ψ, and thus $x^2 \leqslant y^2+z^2$; such a point is interior to S.

Starting from a point X, outside S, at which

$$(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'z)^2 < 0,$$

and moving along a curve which avoids any point at which $yz'-y'z$, $zx'-z'x$, $xy'-x'y$ all vanish, up to a point q, on the contour S (at

which the same restriction holds), it follows that the condition

$$(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'z)^2<0$$

must be satisfied at q, for otherwise there would be on the arc X_η some point at which the expression vanishes, and all such points are interior to S. It is thus seen that, at all points of S, the condition

$$(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'y)^2<0$$

is satisfied, except possibly at a single point, at which $yz'-y'z$, $zx'-z'x$, $xy'-x'y$ all vanish.

At the points $\xi\pm(\xi^2-1)^{\frac{1}{2}}$, we have $A^2-B^2-C^2=0$, or $1-2h\xi+h^2=0$. It follows that, at either of these points, $x^2-y^2-z^2=x'^2-y'^2-z'^2$, and $xx'-yy'-zz'=0$; it then follows that $(yz'-y'z)^2-(xy'-x'y)^2-(xz'-x'z)^2$, which is equivalent to $(x^2-y^2-z^2)(x'^2-y'^2-z'^2)-(xx'-yy'-zz')^2$, or to $(x^2-y^2-z^2)^2$, is positive unless $x^2-y^2-z^2=0$. It is thus seen that, when $x^2-y^2-z^2\neq 0$, the points $\xi\pm(\xi^2-1)^{\frac{1}{2}}$ are interior to S. When $x^2-y^2-z^2=0$, we have also $x'^2-y'^2-z'^2=0$, $xx'=yy'+zz'$; hence $(y^2+z^2)(y'^2+z'^2)=(yy'+zz')^2$, or $yz'-y'z=0$; and thus

$$\frac{y}{y'}=\frac{z}{z'}=\frac{y^2+z^2}{yy'+zz'}=\frac{x}{x'};$$

and hence $yz'-y'z=zx'-z'x=xy'-x'y=0$. A point at which these conditions hold, and at which $x^2-y^2-z^2=0$, is interior to S. It has thus been shown that, in every case, the two points $\xi\pm(\xi^2-1)^{\frac{1}{2}}$ are interior to S.

It has now been proved that, on the contour S, the value of the integral

$$\int_{-\pi}^{\pi}\frac{d\psi}{A+B\cos\psi+C\sin\psi} \quad \text{is} \quad \frac{2\pi}{(1-2h\xi+h^2)^{\frac{1}{2}}}.$$

To determine the phase of the radical, we must have

$$|\,h\mu'-\mu-(1-2h\xi+h^2)^{\frac{1}{2}}\,|$$

less than the greater of the numbers

$$|\,h(\mu'^2-1)^{\frac{1}{2}}-(\mu^2-1)^{\frac{1}{2}}\cos\phi\pm i(\mu^2-1)^{\frac{1}{2}}\sin\phi\,|.$$

Let h be taken to be real and large, then the first expression has the asymptotic value $h\,|\,\mu'\mp 1\,|$ and the second $h\,|\,\mu'^2-1\,|^{\frac{1}{2}}$; it is clear that we must take the upper sign in the first of these. Therefore the phase of $(1-2h\xi+h^2)^{\frac{1}{2}}$ is such that it tends to zero as h is increased indefinitely through positive real values.

We have now shown that

$$\int_{-\pi}^{\pi} \frac{\{\mu+(\mu^2-1)^{\frac{1}{2}}\cos(\psi-\phi)\}^n}{\{\mu'+(\mu'^2-1)^{\frac{1}{2}}\cos\psi\}^{n+1}}\,d\psi = \frac{1}{2\pi i}\int_{(S)} \frac{h^n}{(1-2h\xi+h^2)^{\frac{1}{2}}}\,dh.$$

By H. (51) the expression on the right-hand side has the value $P_n(\xi)$; for the contour S encloses both the points $\xi \pm (\xi^2-1)^{\frac{1}{2}}$, and is such that the origin is exterior to it and on its left-hand side. Moreover, the phase of the radical is in accordance with that in the formula for $P_n(\xi)$.

The following theorem has now been established:

If $R(\mu) > 0$, $R(\mu') > 0$, *the points* μ, μ' *being outside the cross-cut, the series*

$$P_n(\mu)\,P_n(\mu') + 2\sum_{m=1}^{\infty}(-1)^m \frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\mu)\,P_n^m(\mu')\cos m\phi$$

converges, uniformly for all real values of ϕ, *to the function*

$$P_n\{\mu\mu' - \sqrt{(\mu^2-1)}\sqrt{(\mu'^2-1)}\cos\phi\}.$$

The index n *is unrestricted, but when it is a positive integer the series is finite, with* $n+1$ *terms.* (4)

Cases in which the conditions $R(\mu) > 0$, $R(\mu') > 0$ can be relaxed will be considered later.

6. Since the sum of the series in (4) is continuous, it is clear that the line joining the points

$$\mu\mu' - (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}, \quad \mu\mu' + (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$$

either does not cross the real axis or else crosses the real axis in the part of it between -1 and $+\infty$; in such a crossing the function P_n has no breach of continuity. That this is actually the case we shall now verify.

Let $$\mu = \cosh(\xi+i\eta), \quad \mu' = \cosh(\xi'+i\eta'),$$

where $$-\tfrac{1}{2}\pi < \eta < \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi < \eta' < \tfrac{1}{2}\pi,$$

so that $$R(\mu) > 0, \quad R(\mu') > 0.$$

We denote by p the point $\mu\mu' + (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$,

or $$\cosh(\xi+\xi')\cos(\eta+\eta') + i\sinh(\xi+\xi')\sin(\eta+\eta'),$$

and we denote by q the point $\mu\mu' - (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$,

or $$\cosh(\xi-\xi')\cos(\eta-\eta')+i\sinh(\xi-\xi')\sin(\eta-\eta').$$

These points p, q have then the coordinates $\cosh(\xi+\xi')\cos(\eta+\eta')$, $\sinh(\xi+\xi')\sin(\eta+\eta')$, and $\cosh(\xi-\xi')\cos(\eta-\eta')$, $\sinh(\xi-\xi')\sin(\eta-\eta')$, respectively.

The point r in which pq cuts the real axis has the coordinate

$$Or=\tfrac{1}{2}\frac{\sinh 2\xi\sin 2\eta'+\sinh 2\xi'\sin 2\eta}{\sinh(\xi+\xi')\sin(\eta+\eta')-\sinh(\xi-\xi')\sin(\eta-\eta')}.$$

If $\xi\neq\xi'$, we may assume that $\xi>\xi'$; if $\eta+\eta'$, $\eta-\eta'$ have the same sign, p and q are on the same side of the real axis, and r is then not in the segment pq. If $\eta+\eta'$, $\eta-\eta'$ have opposite signs, p and q are on opposite sides of the real axis, and then r is in the segment pq; and we have only this case to consider. If η and η' are both positive and $\eta<\eta'$, the numerator and the denominator in Or are both positive, and therefore $Or>0$. If η, η' are both negative and $\eta>\eta'$, we have also $Or>0$. If η is positive and η' is negative, $\eta-\eta'$ is positive and $\eta+\eta'$ is negative, on account of the assumption that $\eta+\eta'$, $\eta-\eta'$ have opposite signs; the denominator is then negative, and the numerator is less than

$$\tfrac{1}{2}\sinh 2\xi(\sin 2\eta+\sin 2\eta') \quad\text{or}\quad \sinh 2\xi\sin(\eta+\eta')\cos(\eta-\eta'),$$

which is negative; thus the numerator is negative, and $Or>0$. If η is negative and η' positive, $\eta-\eta'$ is negative and $\eta+\eta'$ is positive; the denominator is positive and the numerator is greater than

$$\sinh 2\xi'\sin(\eta+\eta')\sin(\eta-\eta'),$$

and is consequently positive; thus $Or>0$.

If $\eta=\eta'$, q is on the real axis at a distance $\cosh(\xi-\xi')$ from the origin; thus $Or>1$. If $\eta=-\eta'$, p is on the real axis at a distance $\cosh(\xi+\xi')$ from 0.

In case $\xi=\xi'\neq 0$, q is on the real axis at a distance $\cos(\eta-\eta')$ which is greater than -1 from the origin.

7. We proceed to consider the addition theorem for $Q_n(\xi)$. It has been shown in § 2 that, when $R(\mu')>0$, $R(n+1)>0$,

$$\frac{1}{\{\mu'-(\mu'^2-1)^{\frac{1}{2}}\cos(\phi\pm iu)\}^{n+1}}$$
$$=P_n(\mu')+2\sum_{m=1}^{\infty}\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\mu')\cos m(\phi\pm iu),$$

provided that $$|R(u)| < \tfrac{1}{2}\log\left|\frac{\mu'+1}{\mu'-1}\right|;$$

and it was shown that the series converges uniformly for all values of ϕ and of u, where $|R(u_0)|$ is in the interval $(0, u_0)$, and

$$|R(u_0)| < \tfrac{1}{2}\log\left|\frac{\mu'+1}{\mu'-1}\right|.$$

The value of u in iu is not necessarily real, because its imaginary part can be absorbed in ϕ. Let us multiply both sides of the above equation by $\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^n$, and integrate with respect to u over the interval

$$\left(0,\ \tfrac{1}{2}\log\frac{\mu+1}{\mu-1}\right).$$

We assume that this path of integration lies wholly between lines parallel, in the u-plane, to the imaginary axis, and at distances

$$\log\left|\frac{\mu'+1}{\mu'-1}\right|^{\frac{1}{2}}, \quad -\log\left|\frac{\mu'+1}{\mu'-1}\right|^{\frac{1}{2}}$$

from the origin. This condition will be satisfied, either if $R(\mu) > 0$ and $\left|\frac{\mu+1}{\mu-1}\right| < \left|\frac{\mu'+1}{\mu'-1}\right|$, or if $R(\mu) < 0$ and $\left|\frac{\mu-1}{\mu+1}\right| < \left|\frac{\mu'+1}{\mu'-1}\right|$. We then have

$$\int_0^{\frac{1}{2}\log[(\mu+1)/(\mu-1)]} \frac{\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^n}{\{\mu'-(\mu'^2-1)^{\frac{1}{2}}\cos(\phi\pm iu)\}^n}\,du$$

$$= P_n(\mu')\int_0^{\frac{1}{2}\log[(\mu+1)/(\mu-1)]} \{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^n\,du$$

$$+2\sum_{m=1}^{\infty}\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\mu')\int_0^{\frac{1}{2}\log[(\mu+1)/(\mu-1)]}\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^n$$

$$\times\cos m(\phi\pm iu)\,du.$$

In order to show that the term by term integration is valid when $R(n+1) > 0$, and that the convergence of the series is uniform with respect to ϕ, we observe that

$$\left|\int_0^{\frac{1}{2}\log[(\mu+1)/(\mu-1)]}\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}^n\right.$$

$$\left.\times\left[\sum_{m=m_1}^{m_1+s}2\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\mu')\cos m(\phi\pm iu)\right]du,\right.$$

where m_1 is so chosen that

$$\left| \sum_{m=m_1}^{m_1+s} 2 \frac{\Pi(n-m)}{\Pi(n)} P_n^m(\mu') \cos m(\phi \pm iu) \right| < \epsilon,$$

for all values 1, 2, 3, ... of s, and for all values of u and ϕ, such that $|R(u)|$ is in an interval $(0, u_0)$, where $|u_0| < \log \left| \frac{\mu'+1}{\mu'-1} \right|^{\frac{1}{2}}$ is equivalent to (see H., § 42)

$$\left| \int_{-1}^{1} \frac{1}{2^{n+1}} \frac{(1-t^2)^n}{(\mu-t)^{n+1}} \left[\sum_{m=m_1}^{m_1+s} 2 \frac{\Pi(n-m)}{\Pi(n)} P_n^m(\mu') \cos m(\phi \pm iu) \right] dt,$$

where t denotes $\mu-(\mu^2-1)^{\frac{1}{2}} e^u$; and this is less than

$$\epsilon \int_{-1}^{1} \frac{1}{2^{n+1}} \frac{(1-t^2)^{R(u)}}{|\mu-t|^{R(u)+1}} dt,$$

which is less than a fixed multiple of ϵ, since $R(n+1) > 0$. The uniform convergence of the series has thus been established. That the series can be integrated term by term and converges to the integral follows from the fact that s can become indefinitely great.

It has been assumed that the point μ is not in the cross-cut, and thus has not the value $\cos\theta + 0 \cdot i$. In order to extend the proof to this case we have only to suppose the cross-cut joining the points $(-1, 1)$ to be displaced into a curve joining these points and wholly below the real axis, the integration being taken along this curve, so that $|\mu-t|^{-R(n)-1}$ remains finite in the integral.

Employing the formula [H. (92)] for $Q_n^m(\mu)$, and adding the two series obtained by taking both signs in $\pm iu$, we see that the series

$$P_n(\mu') Q_n(\mu) + 2 \sum_{m=1}^{\infty} (-1)^m \frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\mu') Q_n^m(\mu) \cos m\phi$$

converges to

$$\frac{1}{2} \int_0^{\frac{1}{2}\log[(\mu+1)/(\mu-1)]} \{\mu-(\mu^2-1)^{\frac{1}{2}} \cosh u\}^n \times \left[\frac{1}{\{\mu'-(\mu'^2-1)^{\frac{1}{2}} \cos(\phi+iu)\}^{n+1}} + \frac{1}{\{\mu'-(\mu'^2-1)^{\frac{1}{2}} \cos(\phi-iu)\}^{n+1}} \right] du,$$

provided that $R(n+1) > 0$, $R(\mu') > 0$, $\left| \log \left| \frac{\mu+1}{\mu-1} \right| \right| < \log \left| \frac{\mu'+1}{\mu'-1} \right|$.

The last condition is satisfied either (1), if $R(\mu) > 0$, and

$$\log\left|\frac{\mu+1}{\mu-1}\right| < \log\left|\frac{\mu'+1}{\mu'-1}\right|,$$

in which case μ is outside the circle on which the ratio of the distances of a point from the point -1 and from the point $+1$ has the constant value $\left|\frac{\mu'+1}{\mu'-1}\right|$; or (2), if $R(\mu) < 0$, and μ is outside the circle which is the optical image of the former circle in the imaginary μ-axis. This last condition may be expressed in the form that the point $-\mu$ for which $R(-\mu) > 0$ is outside the circle on which the ratio for μ' has the constant value $\left|\frac{\mu'+1}{\mu'-1}\right|$.

In order to transform the sum of the series, we apply to the two integrands the Jacobian transformation

$$\{\mu-(\mu^2-1)^{\frac{1}{2}}\cosh u\}\{\mu+(\mu^2-1)^{\frac{1}{2}}\cosh v\} = 1.$$

The limits for v are then ∞ and 0. The integrals become

$$\int_0^\infty \frac{dv}{(A+B\cosh v \pm C\sinh v)^{n+1}},$$

where

$$A = \xi_\phi = \mu\mu'-(\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}\cos\phi, \quad B = \mu'(\mu^2-1)^{\frac{1}{2}}-\mu(\mu'^2-1)^{\frac{1}{2}}\cos\phi,$$

$$C = i(\mu'^2-1)^{\frac{1}{2}}\sin\phi.$$

We thus have $A^2-B^2+C^2 = 1$;

let $B = (\xi_\phi^2-1)^{\frac{1}{2}}\cosh ip, \quad C = (\xi_\phi^2-1)^{\frac{1}{2}}\sinh ip.$

The expression for the sum then becomes

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{dv}{\{\xi_\phi+(\xi_\phi^2-1)^{\frac{1}{2}}\cosh(v+ip)\}^{n+1}},$$

where p is determined by the above conditions. If $p = p_0+iq$, the sum of the series becomes

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{dv}{\{\xi_\phi+(\xi_\phi^2-1)^{\frac{1}{2}}\cosh(v+ip_0)\}^{n+1}},$$

since the value of the integral is unaltered by changing $v-q$ into v.

In accordance with the results obtained in § 3, in case $R(\xi_\phi) > 0$, the integral has one of the three values $Q_n(\xi_\phi)$, $Q_n(\xi_\phi) \pm i\pi P_n(\xi_\phi)$, according to the value of p_0 in relation to the zero of the function $\xi_\phi+(\xi_\phi^2-1)^{\frac{1}{2}}\cosh v$.

In case $R(\zeta_\phi) < 0$ the three values are

$$Q_n(\zeta_\phi), \quad Q_n(\zeta_\phi) \pm i\pi \left\{ e^{\mp 2n\pi i} P_n(\zeta_\phi) - \frac{2}{\pi} e^{\mp n\pi i} \sin n\pi Q_n(\zeta_\phi) \right\},$$

where the upper or lower sign is taken in the exponentials according as $I(\zeta_\phi)$ is positive or negative.

Whenever the segment joining the two points ζ_π, ζ_0 or

$$\mu\mu' \pm (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$$

does not intersect the cross-cut from $-\infty$ to 1 along the real axis, the sum of the series will have a form which is the same for all values of ϕ. This follows from the fact that the series converges uniformly with respect to ϕ, and accordingly has a continuous sum. If the segment joining the points ζ_ϕ and ζ_0 intersects the cross-cut, the form of the sum must change as ϕ passes through the value for which ζ is on the cross-cut, since the functions which represent the sum of the series for two values of ϕ, such that the corresponding points ζ_ϕ are on opposite sides of the cross-cut, must be continuous with one another in crossing the cross-cut. It is accordingly necessary, for given values of μ and μ', to consider the values of ζ_π and ζ_0 only. We shall provisionally assume that $R(\mu) > 0$. It can be shown that the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is always in the interval $(\frac{1}{2}\pi, \pi)$ or in $(-\pi, -\frac{1}{2}\pi)$. Let α, β, γ be the angles, each within the interval $(-\pi, \pi)$, that the lines joining the point ζ_ϕ with the three points $1, -1, 0$ on the real axis make with the positive direction of the real axis; then the phase of ζ_ϕ is γ, and that of $(\zeta_\phi^2-1)^{\frac{1}{2}}$ is $\frac{1}{2}(\alpha+\beta)$. When ζ_ϕ has its imaginary part positive, and $R(\zeta) > 0$, we have $\alpha-\gamma > \gamma-\beta$; and thus the phase $\gamma-\frac{1}{2}(\alpha+\beta)$ of $\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $-\frac{1}{2}\pi$ and 0; it follows that the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $\frac{1}{2}\pi$ and π. When $R(\zeta_\phi) < 0$, $I(\zeta_\phi) > 0$, the phase of $\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between 0 and $\frac{1}{2}\pi$; hence the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $-\pi$ and $-\frac{1}{2}\pi$. When $R(\zeta_\phi) < 0$, $I(\zeta_\phi) < 0$, the phase of $\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between 0 and $\frac{1}{2}\pi$; and thus the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $-\pi$ and $-\frac{1}{2}\pi$. When $R(\zeta_\phi) < 0$, $I(\zeta_\phi) < 0$, the phase of $\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $-\frac{1}{2}\pi$ and 0; and thus the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is between $\frac{1}{2}\pi$ and π. It has thus been seen that the phase of $-\frac{\zeta_\phi}{(\zeta_\phi^2-1)^{\frac{1}{2}}}$ is always numerically greater than $\frac{1}{2}\pi$.

Let us consider first the case when $\phi = \pi$; we have then

$$\xi_\pi = A = \mu\mu' + (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$$

$$= \cosh(\xi+\xi')\cos(\eta+\eta'(+i\sinh(\xi+\xi')\sin(\eta+\eta'),$$

$$B = \mu'(\mu^2-1)^{\frac{1}{2}} + \mu(\mu'^2-1)^{\frac{1}{2}}$$

$$= \sinh(\xi+\xi')\cos(\eta+\eta') + i\cosh(\xi+\xi')\sin(\eta+\eta'),$$

$$C = 0.$$

It follows that $\sinh ip = 0$, and thus that p is zero or $\pm\pi$. The imaginary parts of ξ_π and $\xi_\pi^2-1)^{\frac{1}{2}}$ have the same sign; hence p is zero or $\pm\pi$ according as the imaginary parts of B and ξ_π have or have not the same sign, since $B = (\xi_\pi^2-1)^{\frac{1}{2}}\cosh ip$. But, since ξ and ξ' are both positive, the imaginary parts of ξ_π and B both have the sign of $\sin(\eta+\eta')$. It follows that $p = 0$. In accordance with § 3, the sum of the series is $Q_n(\xi_\pi)$.

In case $\eta+\eta' = 0$, or, in particular, if η and η' are both zero, which is the case when μ and μ' are both real and greater than 1, A and B are real and positive, and so also is $(\xi_\pi^2-1)^{\frac{1}{2}}$; therefore $\cosh ip = 1$, and thus $p = 0$.

If $\eta+\eta'$ is π or $-\pi$, ξ_π is negative, real, and numerically greater than 1; then $Q_n(\xi_\pi)$ has no definite value.

Next, let $\phi = 0$; then

$$\xi_0 = A = \mu\mu' - (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$$

$$= \cosh(\xi-\xi^2)\cos(\eta-\eta') + i\sinh(\xi-\xi')\sin(\eta-\eta'),$$

$$B = \mu'(\mu^2-1)^{\frac{1}{2}} - \mu(\mu'^2-1)^{\frac{1}{2}}$$

$$= \sinh(\xi-\xi')\cos(\eta-\eta') + i\cosh(\xi-\xi')\sin(\eta-\eta'),$$

$$C = 0.$$

We thus have $\sinh ip = 0$, and therefore p is 0 or $\pm\pi$.

When the imaginary parts of B and ξ_0 have the same sign, which is the case when $\xi > \xi'$, we have $p = 0$. But when $\xi < \xi'$, the imaginary parts of B and ξ have opposite signs, and then $p = \pm\pi$.

It thus follows that, when $\xi > \xi'$, the sum of the series is $Q_n(\xi_0)$; and that, when $\xi < \xi'$, the sum of the series is

$$Q_n(\xi_0) + i\pi P_n(\xi_0) \quad \text{or} \quad Q_n(\xi_0) - i\pi P_n(\xi_0),$$

according as the argument of χ is positive or negative, provided that

$R(\xi_0) > 0$; but when $R(\xi_0) < 0$, the sum of the series is

$$Q_n(\xi_0) \pm i\pi \left[P_n(\xi_0)\, e^{\mp 2n\pi i} - \frac{2}{\pi}\, e^{\pm n\pi i} \sin n\pi\, Q_n(\xi_0) \right],$$

where the sign of the exponential is positive or negative according as $I(\xi_0)$ is positive or negative; it will, however, be shown that this last case does not arise in the present connexion.

Assuming in the first instance that $R(\mu) > 0$, and thus that

$$\left| \frac{\mu+1}{\mu-1} \right| < \left| \frac{\mu'+1}{\mu'-1} \right|,$$

this condition can easily be seen to be equivalent to

$$\cosh \xi \cos \eta' > \cosh \xi' \cos \eta.$$

If $\xi \leqslant \xi'$, we see that $|\eta| > |\eta'|$. When $\xi < \xi'$, the two points ξ_0, ξ_ϕ are on opposite sides of the real axis, if $\eta+\eta'$ and $\eta-\eta'$ have the same sign and the segment joining them cuts the real axis at a point r such that $Or < 0$ (see § 6). We have

$$1 - Or = 1 - \frac{\sinh \xi \cosh \xi \sin \eta' \cos \eta' + \sinh \xi' \cosh \xi' \sin \eta \cos \eta}{\sinh \xi \cosh \xi' \cos \eta \sin \eta' + \cosh \xi \sinh \xi' \sin \eta \cos \eta'}$$

$$= \frac{(\cosh \xi \cos \eta' - \cosh \xi' \cos \eta)(\sinh \xi' \sin \eta - \sinh \xi \sin \eta')}{\sinh \xi \cosh \xi' \cos \eta \sin \eta' + \cosh \xi \sinh \xi' \sin \eta \cos \eta'};$$

the first factor in the numerator is positive, as has been shown above.

If η and η' are both positive, the numerator is positive; also

$$\sinh \xi' \sin \eta - \sinh \xi \sin \eta' > \sinh \xi (\sin \eta - \sin \eta') > 0,$$

and therefore we have $0 < Or < 1$, or the segment crosses the real axis between the points 0 and 1. We see, by changing the signs of η and η', that the same result holds good when η and η' are both negative. If η is positive and η' is negative, we have $\eta > -\eta'$, and the numerator is positive, and the denominator is

$$\cosh \xi \cos |\eta'| \sinh \xi' \sin \eta - \cosh \xi' \cos \eta \sinh \xi \sin |\eta'|,$$

$$> \cos \xi' \cos \eta (\sinh \xi' \sin \eta - \sinh \xi \sin |\eta'|)$$

$$> \cosh \xi' \cos \eta \sin |\eta'| (\sinh \xi' - \sinh \xi) > 0;$$

hence also, in this case, $0 < Or < 1$. If η is negative and η' is positive, $\eta-\eta'$ and $\eta+\eta'$ have the same signs, and the segment does not cut the real axes.

When $\xi > \xi'$, the two points ζ_0 and ζ_π are on opposite sides of the real axis, if $\sin(\eta+\eta')$, $\sin(\eta-\eta')$ have opposite signs. If η and η' are positive

and $\eta < \eta'$, we see that $\sinh \xi' \sin \eta - \sinh \xi \sin \eta'$ is negative, and thus $Or > 1$; therefore the line joining ζ_π and ζ_0 does not intersect the cross-cut. The same holds good if η and η' are both negative and $\eta > \eta'$.

It has now been shown that the segment joining ζ_π and ζ_0 in no case intersects the cross-cut on the negative side of the origin.

The following results have now been obtained:

If $R(n+1) > 0$, $R(\mu) > 0$, $R(\mu') > 0$, *and*

$$\left|\frac{\mu+1}{\mu-1}\right| < \left|\frac{\mu'+1}{\mu'-1}\right|,$$

neither of the points μ, μ' *lying in the cross-cut* $(-1, 1)$, *the series*

$$Q_n(\mu)\, P_n(\mu') + 2 \sum_{m=1}^{\infty} (-1)^m \frac{\Pi(n-m)}{\Pi(n+m)} Q_n^m(\mu)\, P_n^m(\mu) \cos mp$$

converges uniformly for all real values of ϕ. *Moreover,*

(*a*) *If* μ *and* μ' *are both real and greater than* 1, *in which case the condition*

$$\left|\frac{\mu+1}{\mu-1}\right| > \left|\frac{\mu'+1}{\mu'-1}\right|$$

reduces to $\mu > \mu'$, *the sum to which the series converges is* $Q_n(\xi_\phi)$, *where*

$$\xi_\phi \equiv \mu\mu' - (\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}} \cos \phi.$$

(*b*) *More generally, whenever the segment joining the two points* ξ_π, ξ_0 *does not contain a point of the cross-cut, the sum to which the series converges is* $Q_n(\xi_\phi)$. *This is the case when* $\xi > \xi'$ *and* $\eta+\eta'$, $\eta-\eta'$ *have the same sign; also when* $\xi > \xi'$ *and* η, η' *have the same sign and* $|\eta| < |\eta'|$; *also when* $\xi > \xi'$ *and* $\eta+\eta'$, $\eta-\eta'$ *have opposite signs.*

(*c*) *When the segment joining the two points* ξ_π, ξ_0 *contains a point of the cross-cut* $(-1, 1)$, *the sum of the series is* $Q_n(\xi_\phi)$ *or* $Q_n(\xi_\phi) \mp i\pi P_n(\xi_\phi)$, *according as the point* ξ_ϕ *is on the same side, or the opposite side, of the real axis, as* ξ_π; *the upper or lower sign in* $\mp i\pi$ *is to be taken, according as* $I(\xi_\pi)$ *is positive or negative.*

It will now be shown that the condition $R(n+1) > 0$ is unnecessary for the truth of the above theorem. We have

$$P_n^m(\mu) = P_{-n-1}^m(\mu), \quad P_n^m(\mu) = \frac{1}{\pi} \tan n\pi \left[Q_n^m(\mu) - Q_{-n-1}^m(\mu)\right] \quad \text{[see H. (18)]}.$$

The theorem becomes, on substituting for $P_n^m(\mu)$, $Q_n^m(\mu)$ their values in

terms of $P^m_{-n-1}(\mu)$, $Q^m_{-n-1}(\mu)$,

$$Q^m_{-n-1}(\xi_\phi)+\pi\cot n\pi\, P^m_{-n-1}(\xi_\phi)$$
$$= P_{-n-1}(\mu')\left[Q_{-n-1}(\mu)+\pi\cot n\pi\, P_{-n-1}(\mu)\right]$$
$$+2\sum_{m=1}^{\infty}(-1)^n\frac{\Pi(n-m)}{\Pi(n+m)}P^m_{-n-1}(\mu')\left[Q^m_{-n-1}(\mu)+\pi\cot n\pi\, P^m_{-n-1}(\mu)\right]\cos m\phi,$$

subject to the condition stated.

Employing the theorem of § 5, we see that

$$P_{-n-1}(\mu')\,Q_{-n-1}(\mu)+2\sum_{m=1}^{\infty}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P^m_{-n-1}(\mu')\,Q^m_{-n-1}(\mu)\cos m\phi$$

converges to $Q^m_{-n-1}(\xi_\phi)$, subject to the conditions of the above theorem.

Observing that $\dfrac{\Pi(n-m)}{\Pi(n+m)}=\dfrac{\Pi(-n-m-1)}{\Pi(-n+m-1)}$, we see that the theorem holds good when n is changed into $-n-1$; and thus the condition $R(n+1)>0$ is unnecessary.

The corresponding theorem for the case in which $R(\mu)<0$, $R(\mu')>0$ may be deduced by employing the relation $Q^m_n(\mu)=e^{\pm n\pi i}Q^m_n(-\mu)$ [see H. (21)], where the upper or lower sign in the exponential is to be taken, according as $I(-\mu)\gtrless 0$.

Since $R(-\mu)>0$, we obtain the following theorem:

When $R(\mu')>0$, $R(\mu)<0$, $\left|\dfrac{1-\mu}{1+\mu}\right|<\left|\dfrac{\mu'+1}{\mu'-1}\right|$, *the series*

$$Q_n(\mu)\,P_n(\mu')+2\sum_{m=0}^{\infty}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}Q^m_n(\mu)\,P^m_n(\mu')\cos m\phi$$

converges uniformly to $Q_n(\xi_\phi)$, *whenever the corresponding series, with* $-\mu$ *written for* μ, *satisfies the conditions in* (*b*) *of the foregoing theorem. When the conditions in* (*c*) *are satisfied for* $-\mu$, μ', *the sum can be obtained from the result there given, it being remembered that*

$$P_n(-\xi_\phi)=e^{\pm n\pi i}P_n(\xi_\phi)-\frac{2}{\pi}\sin n\pi\, Q_n(\xi_\phi).$$

The number n *is unrestricted.*

8. An extension of the theorem of § 5 can now be obtained. Let $R(\mu)<0$ and $R(\mu')>0$, and let $-\mu$ be denoted by $\bar{\mu}$; then the series

$$P_n(\mu)\,P_n(\mu')+2\Sigma(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n(\mu)\,P_n(\mu')\cos m\phi$$

is equivalent to

$$P_n(\mu')\left[e^{\mp n\pi i}P_n(\bar{\mu})-\frac{2\sin n\pi}{\pi}Q_n(\bar{\mu})\right]$$

$$+2\sum_{m=0}^{\infty}(-)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\mu')\left[e^{\mp n\pi i}P_n^m(\bar{\mu})-\frac{2\sin n\pi}{\pi}Q_n^m(\bar{\mu})\right]\cos m\phi,$$

the upper or lower sign being taken in the exponential, according as $I(\bar{\mu})\gtrless 0$; this follows from H. (20). The series converges to

$$P_n(\mu\mu'-\sqrt{\mu^2-1}\sqrt{\mu'^2-1}\cos\phi),$$

or to

$$e^{\mp n\pi i}P_n(\bar{\mu}\mu'-\sqrt{\bar{\mu}^2-1}\sqrt{\mu'^2-1}\cos\phi)$$

$$-\frac{2\sin n\pi}{\pi}Q_n\{\bar{\mu}\mu'-\sqrt{\bar{\mu}^2-1}\sqrt{\mu'^2-1}\cos\phi\}.$$

Now if $\left|\frac{\bar{\mu}+1}{\bar{\mu}-1}\right|<\left|\frac{\mu'+1}{\mu'-1}\right|$, and the segment joining the two points $\bar{\mu}\mu'\pm(\bar{\mu}^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$ does not intersect the cross-cut, the series

$$P_n(\mu')\,Q_n(\bar{\mu})+2\sum_{m=0}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\mu')\,Q_n^m(\bar{\mu})\cos m\phi$$

converges uniformly to $Q_n\{\bar{\mu}\mu-\sqrt{\bar{\mu}^2-1}\sqrt{\mu'^2-1}\cos\phi\}$; it follows then that

$$P_n(\mu')\,P_n(\bar{\mu})+2\sum_{m=1}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\mu')\,P_n^m(\bar{\mu})\cos m\phi$$

converges uniformly to $P_n(\bar{\mu}\mu'-\sqrt{\bar{\mu}^2-1}\sqrt{\mu^2-1}\cos\phi)$.

The following extension of the theorem of § 5 has now been established:

If $R(\mu)<0$, $R(\mu')>0$, *and* $\left|\frac{1-\mu}{1+\mu}\right|<\left|\frac{\mu'+1}{\mu'-1}\right|$, *the series*

$$P_n(\mu)\,P_n(\mu')+2\sum_{m=1}(-1)^m\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\mu)\,P_n^m(\mu')\cos m\phi$$

converges uniformly to $P_n(\mu\mu'-\sqrt{\mu^2-1}\sqrt{\mu'^2-1}\cos\phi)$, *provided that the segment joining the two points* $\mu\mu'\pm(\mu^2-1)^{\frac{1}{2}}(\mu'^2-1)^{\frac{1}{2}}$ *does not contain a point of the cross-cut* $(-1, 1)$. *The convergence to the same sum also*

occurs when the segment contains a point of the cross-cut, for those values of ϕ for which ξ_ϕ is on the same side of the real axis as ξ_π.

9. There remains for consideration the case in which μ and μ' are in the real axis, between the points 1 and -1. We may assume that the cross-cut between these points is displaced provisionally so that it is on a curve joining these points and below the real axis. The functions $P_n^m(\mu)$ $Q_n^m(\mu)$ will then be continued across the real axis between 1 and -1 so that they have no discontinuities in crossing this part of the real axis; the values of the functions on this part of the real axis then become $P_n^m(\mu+0\,.\,i)$ and $Q_n^m(\mu+0\,.\,i)$. The investigations in §§ 2, 3, 4 can then be adapted to the case in which $\mu=\cos\eta$, $\mu'=\cos\eta'$. The values of $P_n(\xi_\phi)$, $Q_n(\xi_\phi)$ then become $P_n(\cos\gamma+0\,.\,i)$, $Q_n(\cos\gamma+0\,.\,i)$, where

$$\cos\gamma=\cos\eta\cos\eta'+\sin\eta\sin\eta'\cos\phi.$$

Remembering the relations

$$P_n^m(\cos\theta+0\,.\,i)=e^{-\frac{1}{2}m\pi i}P_n^m(\cos\theta),$$

$$Q_n^m(\cos\theta+0\,.\,i)=e^{\frac{3}{2}m\pi i}\{Q_n^m(\cos\theta)-\tfrac{1}{2}\pi iP_n^m(\cos\theta)\}\quad\text{(see H., 270–471)};$$

we deduce from § 3, that

$$P_n(\cos\eta)\,P_n(\cos\eta')+2\sum_{m=1}^{\infty}\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\cos\eta)\,P_n^m(\cos\eta')\cos m\phi$$

converges to $P_n(\cos\gamma)$ uniformly with respect to ϕ, provided that $0<\eta<\frac{1}{2}\pi$, $0<\eta'<\frac{1}{2}\pi$.

In § 7 the condition $\left|\frac{\mu+1}{\mu-1}\right|<\left|\frac{\mu'+1}{\mu'-1}\right|$ becomes, when $0<\eta<\frac{1}{2}\pi$, $0<\eta'<\frac{1}{2}\pi$, $\cos\eta<\cos\eta'$, or $\eta>\eta'$; hence we have, provided that this condition $\eta>\eta'$ is satisfied

$$P_n(\cos\eta')[Q_n(\cos\eta)-\tfrac{1}{2}\pi iP_n(\cos\eta)]$$

$$+2\sum_{m=1}^{n}\frac{\Pi(n-m)}{\Pi(n+m)}P_n^m(\cos\eta')[Q_n^m(\cos\eta)-\tfrac{1}{2}\pi iP_n^m(\cos\eta)]\cos m\phi,$$

converges uniformly to $Q_n(\cos\gamma)-\frac{1}{2}\pi iP_n(\cos\gamma)$. For, in this case, we have $\xi_\pi=A=\cos(\eta+\eta')$, $B=i\sin(\eta+\eta')$, $C=0$, and the phase of $-\frac{\xi_\pi}{(\xi_\pi^2-1)^{\frac{1}{2}}}$ is $\frac{1}{2}\pi$; hence $\cosh i\pi=\frac{i\sin(\eta+\eta')}{i\sin(\eta+\eta')}=1$, and thus $p=0$. Again $\xi_0=\cos(\eta-\eta')$, $B=i\sin(\eta-\eta')$, $C=0$, and, as before, we have $p=0$.

Employing the preceding result for $P_n(\cos\gamma)$, we see that:

If $0<\eta<\frac{1}{2}\pi$, $0<\eta'<\frac{1}{2}\pi$, *and* $\eta>\eta'$, *the series*

$$P_n(\cos\eta')\,Q_n(\cos\eta)+2\sum_{m=0}^{n}\frac{\Pi(n-m)}{\Pi(n+m)}\,P_n^m(\cos\eta')\,Q_n^m(\cos\eta)\cos m\phi$$

converges, uniformly with respect to ϕ, *to* $Q_n(\cos\gamma)$.

In order to extend these theorems to the case in which η is in the interval $(\frac{1}{2}\pi,\ \pi)$, let $\bar\eta=\pi-\eta$, and thus $0<\bar\eta<\frac{1}{2}\pi$. We have from H. (33),

$$P_n^m(\cos\eta)=(-1)^m\,P_n^m(\cos\bar\eta)\cos n\pi-(-1)^m\,\frac{2}{\pi}\,\sin n\pi\,Q_n^m(\cos\eta).$$

Since the two series

$$P_n(\cos\bar\eta)\,P_n(\cos\eta')+2\Sigma\,\frac{\Pi(n-m)}{\Pi(n+m)}\,P_n^m(\cos\bar\eta)\,P_n^m(\cos\eta')\cos m\phi,$$

$$Q_n(\cos\bar\eta)\,P_n(\cos\eta')+2\Sigma\,\frac{\Pi(n-m)}{\Pi(n+m)}\,Q_n^m(\cos\bar\eta)\,P_n^m(\cos\eta')\cos m\phi$$

converge uniformly to

$$P_n(\cos\eta\cos\bar\eta+\sin\eta\sin\bar\eta\cos\phi),\quad Q_n(\cos\eta\cos\bar\eta+\sin\eta\sin\eta'\cos\phi),$$

respectively, provided that $\bar\eta>\eta'$ or $\eta+\eta'<\pi$, we have, on multiplication of the series by $\cos n\pi$ and $\frac{2}{\pi}\sin n\pi$ and subtraction,

$$P_n(\cos\eta)\,P_n(\cos\eta')+2\sum_{m=0}^{\infty}\frac{\Pi(n-m)}{\Pi(n+m)}\,(-1)^m\,P_n^m(\cos\eta)\,P_n^m(\cos\eta')\cos m\phi$$

converges uniformly to

$$P_n(-\cos\eta\cos\eta'+\sin\eta\sin\eta'\cos\phi)\cos n\pi$$
$$-\tfrac{1}{2}\pi\sin n\pi\,Q_n(-\cos\eta\cos\eta'+\sin\eta\sin\eta'\cos\phi),$$

or to $\qquad P_n(\cos\eta\cos\eta'-\sin\eta\sin\eta'\cos\phi).$

On changing ϕ into $\pi-\phi$, we have the theorem that:

If $0<\eta'<\frac{1}{2}\pi$, $\frac{1}{2}\pi<\eta<\pi$, *and* $\eta+\eta'<\pi$, *the series*

$$P_n(\cos\eta)\,P_n(\cos\eta')+2\sum_{m=0}^{n}\frac{\Pi(n-m)}{\Pi(n+m)}\,P_n^m(\cos\eta)\,P_n^m(\cos\eta')\cos m\phi$$

converges to $P_n(\cos\eta\cos\eta'+\sin\eta\sin\eta'\cos\phi)$, *uniformly with respect to* ϕ.

If we employ the relation H. (34),

$$Q_n^m(\cos\eta) = (-1)^{m+1}\cos n\pi\, Q_n^m(\cos\overline{\eta}) - \tfrac{1}{2}\pi(-1)^m \sin n\pi\, P_n^m(\cos\overline{\eta}),$$

we obtain, in a similar manner, the theorem:

If $0 < \eta' < \frac{1}{2}\pi$, $\frac{1}{2}\pi < \eta < \pi$, *and* $\eta + \eta' < \pi$, *the series*

$$Q_n(\cos\eta)\, P_n(\cos\eta') + 2\sum_{m=0}^{\infty} \frac{\Pi(n-m)}{\Pi(n+m)}\, Q_n^m(\cos\eta)\, P_n^m(\cos\eta')$$

converges, uniformly with respect to ϕ, *to the sum*

$$Q_n(\cos\eta\cos\eta' + \sin\eta\sin\eta'\cos\phi).$$

It has thus been shown that the addition theorems, in the case of points on the cross-cut between -1 and 1 are valid, for unrestricted values of n, when η and η' are such that $0 < \eta + \eta' < \pi$, where η and η' are both positive and $\eta < \frac{1}{2}\pi$.

BEMERKUNGEN ZUR BEHANDLUNG DES MAJOR ARC BEI DER ANWENDUNG DER HARDY-LITTLEWOOD'SCHEN METHODE AUF DAS WARING'SCHE PROBLEM

Von ERICH BESSEL-HAGEN.

[Received 10 June, 1928.—Read 14 June, 1928.]

INHALTSÜBERSICHT.

1. Einleitung.

2. Bezeichnungen und Vorbereitungen.

TEIL I.

3. Die Bernoulli'schen Functionen und die Euler'sche Summenformel.

4. Aufstellung der asymptotischen Entwickelung mittels der Euler'schen Summenformel.

5. Abschätzung von $|f(x)-\phi_\rho(x)|$.

6. Schärfere Abschätzung von $|f(x)-\phi_\rho(x)|$.

7. Aufstellung der asymptotischen Entwickelung mittels der Mellin'schen Formel und Abschätzungen des Restes.

TEIL II.

8. Über die Brauchbarkeit der Abschätzung $|f(x)-\phi_\rho(x)| < C_1 q^{-(1,K)+\epsilon} n^a$.

1. *Einleitung**.

Zur Abschätzung der Ausdrücke $|f(x)-\phi_\rho(x)|$ auf den major arcs sind seit der Entdeckung der Hardy-Littlewood'schen Behandlungsweise

* Sämtliche in der Einleitung vorkommenden Bezeichnungen sind in § 2 erklärt.

des Waring'schen Problems verschiedene Methoden entwickelt worden*. Im ersten Teile der vorliegenden Note unterziehe ich einige von ihnen einer kritischen Vergleichung: dabei betrachte ich die Methodenanalyse durchaus als Selbstzweck, da eine Verschärfung der Abschätzung auf dem major arc—die ich nicht gebe—für das Waring'sche Problem nutzlos bleibt, ehe die von den minor arcs herrührenden Schwierigkeiten aus dem Wege geräumt sind. Im zweiten Teile werde ich zeigen, dass bei passender Wahl des Exponenten a in der Ordnung $N=[n^{1-a}]$ der zu Grunde gelegten Fareyzerschneidung die ganz schwache Abschätzung

$$|f(x)-\phi_\rho(x)|<C_1 q^{-(1/K)+\epsilon} n^a \quad \text{auf } \mathfrak{M}_\rho$$

bereits ausreicht, um den "ersten Hardy-Littlewood'schen Satz" (La. Vorl. Satz 279)

$$(1.1) \qquad \left| r_{k,s}(n)-\frac{\Gamma(1+a)^s}{\Gamma(sa)}\, n^{sa-1}\mathfrak{S}\right| < b_1 n^{sa-1-c_1}$$

für alle $s \geqq (k-2)K+5$ zu beweisen.

In der Literatur treten vor allem zwei Typen von Abschätzungen auf; in P.N. I und in La. Mel. wird für die Fareyzerschneidung der Ordnung $[n^{1-a}]$ bewiesen

$$(1.2) \qquad |f(x)-\phi_\rho(x)|<C_2 n^{a\kappa+\epsilon} \quad \text{auf } \mathfrak{M}_\rho,$$

in P.N. VI (§ 4) dagegen wird für jede Fareyzerschneidung einer Ordnung $[n^{1-a}]$ mit $0<a<a$

$$(1.3) \qquad |f(x)-\phi_\rho(x)|<D_1 q^{\kappa+\epsilon} \quad \text{auf } \mathfrak{M}_\rho$$

hergeleitet; diese Herleitung ist in veränderter Form in La. War II und in La. Vorl. (VI. Teil, Kap. 3, § 2) reproducirt. Der Beweis von (1.3) bei Hardy-Littlewood und Landau beruht auf der Anwendung der Euler'schen Summenformel. Mein Ziel in § 3–§ 5 ist es, die Beweisführung ohne Änderung des Grundgedankens dadurch übersichtlicher zu gestalten, dass ich die Euler'sche Summenformel gleich

* G. H. Hardy und J. E. Littlewood, "Some problems of 'Partitio Numerorum', I and VI", *Göttinger Nachrichten*, (1920), 33–54, bezw. *Math. Zeitschrift*, 23 (1925), 1–37, im Folgenden citirt als P.N. I, bezw. P.N. VI. E. Landau, "Zur Hardy-Littlewoodschen Lösung des Waringschen Problems", *Göttinger Nachrichten* (1921), 88–92, im Folgenden citirt als La. Mel. E. Landau, "Zum Waringschen Problem", *Math. Zeitschrift*, 12 (1922), 219–247, im Folgenden citirt als La. War. E. Landau, "Zum Waringschen Problem", *Proc. London Math. Soc.* (2), 25 (1925), 484–486, im Folgenden citirt als La. War, II. E. Landau, *Vorlesungen über Zahlentheorie* (Leipzig, Hirzel, 1927), insbes. Teil VI in Bd. 1, im Folgenden citirt als La. Vorl.

zu Beginn in einer dem vorliegenden Fall unmittelbar angepassten Form benutze. Es stellt sich dabei von selbst eine Art asymptotischer Entwickelung für $f(x)$ ein, indem für jedes ganze $r \geqq 0$

$$f(x) = \phi_\rho(x) + \sum_{\nu=0}^{r} \beta_1(k, \rho, \nu)\, Y^\nu + R_r \tag{1.4}$$

gesetzt werden kann, wo die $\beta_1(k, \rho, \nu)$ wohlbestimmte von k, ρ, ν abhängige Werte bedeuten [vgl. unten (4.9) und (4.6)] und wo das Restglied R_r durch ein bestimmtes Integral dargestellt ist. Die selbe asymptotische Entwickelung stellt sich auch ein, wie ich in § 7 zeige, wenn man nach dem Muster von La. Mel. $f(x)$ mittels der Mellin'schen Integralformel in ein Integral längs einer verticalen Geraden verwandelt und dann den Integrationsweg nach links verschiebt und die Residuen anschreibt; auch hier erscheint R_r in Form eines bestimmten Integrales dargestellt. Die Abschätzung des Bestandteiles $\sum_{\nu=0}^{r} \beta_1(k, \rho, \nu)\, Y^\nu$ verursacht keine Schwierigkeiten und liefert ein Ergebnis, das zur Erreichung von (1.2) wie (1.3) ausreicht.

Um nun zu (1.3) zu gelangen, genügt es, die Integrale, durch welche R_r bei den geschilderten Ansätzen dargestellt ist, dadurch abzuschätzen, dass man die Integranden durch ihre Beträge ersetzt und r hinreichend gross wählt, d.h. die asymptotische Entwickelung (1.4) genügend weit fortsetzt. Will man dagegen (1.2) gewinnen, so ist es unerlässlich, die Oscillationen der Integranden der genannten Integrale zu berücksichtigen; dafür genügt es aber, $r = 0$ zu wählen, d.h. die Entwickelung (1.4) bereits bei ihrem ersten Gliede abzubrechen. Da die Schwierigkeit der feineren Integralabschätzungen gegenüber derjenigen der Bestimmung der Form der Glieder $\beta_1(k, \rho, \nu)$ für $\nu \geqq 1$ bei weitem überwiegt, muss die Abschätzung (1.2) als wesentlich tiefer liegend bezeichnet werden als (1.3).

Zur schärferen Abschätzung der Integrale, durch welche R_r dargestellt ist, hat *Landau* in La. Mel. den bequemsten Weg gewiesen: Landau-van d. Corput'schen Hilfssatz verbunden mit dem zweiten Mittelwertsatz. In § 5 werde ich zeigen, wie dieser Weg auch für die mittels der Euler'schen Summenformel gewonnene Gestalt des Restgliedes R_r gangbar ist; dabei ist es allerdings vorteilhaft, die Euler'sche Summenformel durch die mit ihr ungefähr aequivalente Fourierentwickelung zu ersetzen, also wieder zu dem Ausgangspunkt in P.N. I zurückzukehren.

Ich bemerke noch, dass ich aus methodischen Gründen x bezw. die mit x gekoppelten Grössen X, y, Y so lange wie möglich als freie

Variabele behandele und jeweils erst ganz zuletzt die Specialisirung auf den major arc eintreten lasse.

2. *Bezeichnungen und Vorbereitungen.*

In den Bezeichnungen schliesse ich mich bis auf geringfügige Abweichungen an La. Vorl. an.

k, n, q, s bedeuten natürliche Zahlen, von denen $k \geqq 3$ sein soll. Weiter sei

$$a = \frac{1}{k}, \qquad K = 2^{k-1}, \qquad \kappa = 1 - \frac{1}{K},$$

ρ eine primitive q-te Einheitswurzel,

$$S_\rho = \sum_{h=0}^{q-1} \rho^{h^k}, \qquad S_{\rho,m} = \sum_{h=0}^{q-1} \rho^{h^k} e^{2\pi i m h/q},$$

$$S^{(c)}_{\rho,m} = \sum_{h=0}^{q-1} \rho^{h^k} \cos\frac{2\pi m h}{q}, \qquad S^{(s)}_{\rho,m} = \sum_{h=0}^{q-1} \rho^{h^k} \sin\frac{2\pi m h}{q},$$

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\rho(q)} \left(\frac{S_\rho}{q}\right)^s \rho^{-n},$$

$$f(x) = \sum_{m=0}^{\infty} x^{m^k}, \quad \{f(x)\}^s = \sum_{n=0}^{\infty} r_{k,s}(n)\, x^n \quad \text{für} \quad |x| < 1,$$

$$x = \rho X, \qquad X = e^{-y}, \qquad Y = q^k y,$$

$$Y = \eta + i\zeta = |Y| e^{i\psi} \quad (\eta, \zeta \text{ reell}, \ \eta > 0, \ -\tfrac{1}{2}\pi < \psi < \tfrac{1}{2}\pi),$$

$$\phi_\rho(x) = \frac{\Gamma(1+a)\, S_\rho}{q y^a},$$

$$\psi_\rho(x) = \frac{\Gamma(1+a)\, S_\rho}{q(1-X)^a},$$

$$g(Y, t) = \sum_{m=0}^{\infty} e^{-Y(m+t)^k}.$$

Auf dem Keise $|x| = e^{-1/n}$ sei in üblicher Weise eine Einteilung in Fareybögen $\mathfrak{B}_\rho$ mittels der Fareyreihe der Ordnung $[n^{1-\alpha}]$ hergestellt, wo $0 < \alpha \leqq a$; es sei $\delta = a - \alpha$. Die Bögen mit $q \leqq n^\alpha$ heissen major arcs $\mathfrak{M}_\rho$ (oder kurz $\mathfrak{M}$) diejenigen mit $n^\alpha < q \leqq [n^{1-\alpha}]$ minor arcs $\mathfrak{m}_\rho$ (oder kurz $\mathfrak{m}$). Auf dem Kreise $|x| = e^{-1/n}$ werde ferner gesetzt

$$X = e^{-(1/n)+\vartheta i}, \quad \text{also} \quad y = \frac{1}{n} - \vartheta i, \quad Y = \frac{q^k}{n}(1 - n\vartheta i), \quad \eta = \frac{q^k}{n}.$$

ϵ bedeutet eine von Fall zu Fall beliebig—ausser wo das Gegenteil gesagt ist—zu wählende positive Zahl.

$$c_1,\ c_2,\ \ldots,\ C_1,\ \ldots,\ b_1,\ \ldots,\ B_1,\ \ldots,\ D_1,\ \ldots$$

bezeichnen positive Constanten, die höchstens von den sogleich anzugebenden Argumenten abhängen:

$$c(k),\quad C(k,\epsilon),\quad b(k,s),\quad B(k,s,\epsilon),\quad D(k,\epsilon,\delta).$$

$\gamma_1(\ldots)$, ... bezeichnen positive Constanten und $\beta_1(\ldots)$, ... wohlbestimmte complexe Zahlen, die höchstens von den jeweils angegebenen Argumenten abhängen.

Ich merke gleich folgende bekannte Abschätzungen an:

$$(2.11)\qquad \begin{cases} |S_\rho| < C_3 q^{\kappa+\epsilon}, & |S_{\rho,m}| < C_3 q^{\kappa+\epsilon}, \quad |S^{(c)}_{\rho,m}| < C_3 q^{\kappa+\epsilon}, \\ & |S^{(s)}_{\rho,m}| < C_3 q^{\kappa+\epsilon}, \end{cases}$$

$$(2.12)\qquad \left| \sum_{h=m_1}^{m_2} \left(\frac{h}{q}\right)^\mu \rho^{hk} \right| < C_3 q^{\kappa+\epsilon}$$

für ganzes $\mu \geqq 0$, ganze m_1 und m_2 mit $0 \leqq m_1 \leqq m_2 \leqq q-1$,

$$(2.2)\qquad \frac{1}{\frac{1}{2}\pi - |\psi|} < \frac{1}{\sin\left(\frac{1}{2}\pi - |\psi|\right)} = \frac{1}{\cos\psi} = \frac{|Y|}{\eta}.$$

Auf einem $\mathfrak{B}_\rho$ ist

$$|Y| < \frac{q^k}{n} \cdot 8 \operatorname{Max}\left(1,\ \frac{n^a}{q}\right), \qquad \frac{|Y|}{\eta} < 8 \operatorname{Max}\left(1,\ \frac{n^a}{q}\right),$$

insbesondere gilt auf einem $\mathfrak{M}$

$$(2.3)\qquad |Y| < 8q^{k-1} n^{a-1} \leqq 8n^{-\delta}, \quad \eta = \frac{q^k}{n}, \quad \frac{|Y|}{\eta} < 8\frac{n^a}{q}.$$

3. *Die Bernoulli'schen Functionen und die Euler'sche Summenformel.*

Die Bernoulli'schen Zahlen benutze ich in der Numerirung, die den Festsetzungen $B_0 = 1$, $(B+1)^\lambda = B^\lambda$ für $\lambda \geqq 2$ entspricht; die Bernoulli'schen Functionen sind dann für beliebiges reelles x erklärt durch

$$(3.1)\qquad b_\lambda(x) = \frac{(B+x-\lceil x\rceil)^\lambda}{\lambda!} \quad (\lambda \geqq 1 \text{ ganz}).$$

Es genügt zum Verständnis dieser Note, von ihnen folgende Eigenschaften zu kennen, durch die sie vollständig charakterisirt sind (vgl. La. Vorl. Satz 329):

(1) $b_1(x) = x - [x] - \frac{1}{2}$.

(2) $b_\lambda(x)$ ist periodisch mit der Periode 1 und es ist

$$\int_0^1 b_\lambda(u)\,du = 0.$$

(3) Bei passender Wahl der Constanten $\beta_2(\lambda)$ ist

$$b_{\lambda+1}(x) = \int_0^x b_\lambda(u)\,du + \beta_2(\lambda).$$

—Nebenbei bemerkt, es ist $\beta_2(\lambda) = B_{\lambda+1}/(\lambda+1)!$ zu wählen.—Aus diesen Eigenschaften ergibt sich sofort

$$(3.2) \qquad |b_\lambda(x)| < \gamma_1(\lambda) \quad \text{für} \quad -\infty < x < \infty,$$

ferner, dass für $0 \leqq x < 1$, $b_\lambda(x)$ ein Polynom in x vom Grade λ ist. Für später notire ich noch die [auf Grund von (1) bis (3) leicht zu beweisenden] Fourierentwickelungen der $b_\lambda(x)$:

$$(3.31) \qquad b_1(x) = \begin{cases} -\sum\limits_{m=-\infty}^{\infty}{}' \dfrac{e^{2\pi i m x}}{2\pi i m} & \text{für} \quad x \neq 0, \\ -\frac{1}{2} - \lim\limits_{M\to\infty} \sum\limits_{m=-M}^{M}{}' \dfrac{e^{2\pi i m x}}{2\pi i m} = -\frac{1}{2} & \text{für} \quad x = 0, \end{cases}$$

$$(3.32) \qquad b_\lambda(x) = -\sum_{m=-\infty}^{\infty}{}' \frac{e^{2\pi i m x}}{(2\pi i m)^\lambda} \quad (\lambda > 1),$$

Σ' bedeutet hier und später, dass $m = 0$ auszulassen ist.

Die Euler'sche Summenformel brauche ich in der folgenden Fassung:

SATZ 1. *a und b seien ganz, $a < b$, $l \geqq 0$ sei ganz, es sei $0 \leqq t < 1$, $\phi(u)$ sei eine complexe Function des reellen Argumentes u, die für $a \leqq u \leqq b$ $(l+1)$-mal statig differentiirbar ist. Dann ist*

$$(3.4) \qquad \sum_{m=a}^{b-1} \phi(m+t) = \int_a^b \phi(u)\,du + \sum_{\lambda=0}^{l} \{\phi^{(\lambda)}(b)\, b_{\lambda+1}(t-b) - \phi^{(\lambda)}(a)\, b_{\lambda+1}(t-a)\}$$
$$- \int_a^b \phi^{(l+1)}(u)\, b_{l+1}(t-u)\,du.$$

Beweis. Für $l = 0$ heisst die Formel

$$(3.5)\qquad \sum_{m=a}^{b-1} \phi(m+t) = \int_a^b \phi(u)\,du + \phi(b)\,b_1(t-b) - \phi(a)\,b_1(t-a) - \int_a^b \phi'(u)\,b_1(t-u)\,du.$$

Wegen der "Additivität" beider Seiten hinsichtlich des Intervalles und auf Grund der trivialen Transformation $\phi(u) = \psi(u-a) = \psi(v)$, $u-a = v$, genügt es, den Specialfall $a = 0$, $b = 1$, d.h die Formel

$$\phi(t) = \int_0^1 \phi(u)\,du + \phi(1)(t-\tfrac{1}{2}) - \phi(0)(t-\tfrac{1}{2}) - \int_0^1 \phi'(u)(t-u-[t-u]-\tfrac{1}{2})\,du$$

zu beweisen. Diese bestätigt sich sofort durch Zerlegung des letzten Integrales in $\int_0^t + \int_t^1$ und partielle Integration der beiden Bestandteile.

Für $l > 0$ ergibt sich (3.4) aus (3.5) durch l-malige partielle Integration.

4. *Aufstellung der asymptotischen Entwickelung mittels der Euler'schen Summenformel.*

Ich gehe von der aus den Definitionen unmittelbar folgenden Formel

$$(4.1)\qquad f(x) = \sum_{m=0}^{\infty} x^{m^k} = \sum_{h=0}^{q-1} \rho^{h^k} \sum_{m=0}^{\infty} X^{(mq+h)^k} = \sum_{h=0}^{q-1} \rho^{h^k} \sum_{m=0}^{\infty} e^{-Y[m+(h/q)]^k} = \sum_{h=0}^{q-1} \rho^{h^k} g\left(Y, \frac{h}{q}\right)$$

aus und wende auf $g(Y, t) = \sum_{m=0}^{\infty} e^{-Y(m+t)^k}$ die Euler'sche Summenformel an. Um dies auszuführen, sind drei Hilfssätze voranzuschicken.

Satz 2. *Für beliebige complexe Y, u, und ganzes $\lambda \geqq 0$ gilt*

$$(4.2)\qquad \frac{d^\lambda e^{-Yu^k}}{du^\lambda} = P_\lambda(u, Y)\, e^{-Yu^k},$$

wo $P_\lambda(u, Y)$ ein Polynom in u und Y ist von der speciellen Gestalt

$$(4.3)\qquad P_\lambda(u, Y) \sum_{\alpha\lambda \leq \mu \leq \lambda} \beta_3(k, \lambda, \mu)\, Y^\mu\, u^{k\mu-\lambda}.$$

Beweis. Für $\lambda = 0$ ist (4.2) mit $P_0(u, Y) = 1$, also einem P_λ der Gestalt (4.3) richtig. Es seien schon die Gleichungen (4.2), (4.3)

für irgend ein ganzes $\lambda \geqq 0$ bewiesen. Dann wird

$$\frac{d^{\lambda+1} e^{-Yu^k}}{du^{\lambda+1}} = \left\{-kYu^{k-1} P_\lambda(u, Y) + \frac{\partial}{\partial u} P_\lambda(u, Y)\right\} e^{-Yu^k}$$

$$= P_{\lambda+1}(u, Y)\, e^{Yu^k},$$

wo
$$P_{\lambda+1}(u, Y) = \sum_{\alpha\lambda \leqq \mu \leqq \lambda} -k\beta_3(k, \lambda, \mu)\, Y^{\mu+1} u^{k(\mu+1)-(\lambda+1)} + \sum_{\alpha\lambda < \mu \leqq \lambda} (k\mu-\lambda)\, \beta_3(k, \lambda, \mu)\, Y^\mu u^{k\mu-(\lambda+1)}$$

ist, also die Gestalt

$$\sum_{\alpha(\lambda-1) \leqq \mu \leqq \lambda+1} \beta_3(k, \lambda+1, \mu)\, Y^\mu u^{k\mu-(\lambda+1)}$$

hat.

SATZ 3.

$$P_\lambda(0, Y) = \begin{cases} 0 & \text{wenn} \quad \lambda \not\equiv 0 \pmod{k}, \\ \dfrac{(-1)^\nu (k\nu)!}{\nu!}\, Y^\nu & \text{wenn} \quad \lambda = k\nu. \end{cases} \tag{4.4}$$

Beweis. $P_\lambda(0, Y) = \left(\dfrac{d^\lambda e^{-Yu^k}}{du^\lambda}\right)_{u=0} = \lambda!$ mal dem Coefficienten von u^λ in der Potenzreihe von e^{-Yu^k}.

SATZ 4. *Es sei $l \geqq 0$ ganz, $r = [l/k]$, $0 \leqq t < 1$; dann wird*

$$\sum_{\lambda=0}^{l} P_\lambda(0, Y)\, b_{\lambda+1}(t) = \sum_{\nu=0}^{r} \frac{(-1)^\nu (k\nu)!}{\nu!}\, Y^\nu b_{k\nu+1}(t) \tag{4.51}$$

$$= -\sum_{\nu=0}^{r} \sum_{\mu=0}^{k\nu+1} \beta_4(k, \nu, \mu)\, Y^\nu t^\mu. \tag{4.52}$$

Beweis. (4.51) folgt aus (4.4), (4.52) aus (4.51), da $b_{k\nu+1}(t)$ für $0 \leqq t < 1$ ein Polynom in t vom Grade $k\nu+1$ ist.—Nebenbei bemerkt folgt aus (3.1)

$$\beta_4(k, \nu, \mu) = \frac{(-1)^{\nu+1}}{(k\nu+1)\nu!} \binom{k\nu+1}{\mu} B_{k\nu+1-\mu}. \tag{4.6}$$

SATZ 5. *Es sei $\Re(Y) > 0$, $0 \leqq t < 1$, $r \geqq 0$ ganz, l ganz und $rk \leqq l < (r+1)k$. Dann gilt*

$$g(Y, t) = \Gamma(a+1)\, Y^{-a} + \sum_{\nu=0}^{r} \sum_{\mu=0}^{k\nu+1} \beta_4(k, \nu, \mu)\, Y^\nu t^\mu - \int_0^\infty e^{-Yu^k} P_{l+1}(u, Y)\, b_{l+1}(t-u)\, du. \tag{4.7}$$

Beweis. Ich wende Satz 1 an auf $\phi(u) = e^{-Yu^k}$, $a = 0$, $b > 0$ mit nachfolgendem Grenzübergang $b \to \infty$. Dann ergibt sich wegen (4 . 2)

$$g(Y, t) = \sum_{m=0}^{\infty} e^{-Y(m+t)^k}$$

$$(4.8) \qquad = \int_0^{\infty} e^{-Yu^k}\, du - \sum_{\lambda=0}^{l} P_\lambda(0, Y)\, b_{\lambda+1}(t) - \int_0^{\infty} P_{l+1}(u, Y)\, e^{-Yu^k}\, b_{l+1}(t-u)\, du.$$

Hieraus folgt die Behauptung, da

$$\int_0^{\infty} e^{-Yu^k}\, du = \int_0^{\infty} e^{-v} (aY^{-a} v^{a-1})\, dv = \Gamma(a+1)\, Y^{-a}$$

ist und (4 . 52) gilt.

SATZ 6. *Es sei* $|x| < 1$, $r \geqq 0$ *ganz und* $R_r = R_r(k, \rho, Y)$ *erklärt durch*

$$(4.9) \qquad f(x) = \phi_\rho(x) + \sum_{\nu=0}^{r} \sum_{\mu=0}^{k\nu+1} \beta_4(k, \nu, \mu)\, Y^\nu \sum_{h=0}^{q-1} \left(\frac{h}{q}\right)^\mu \rho^{hk} + R_r.$$

Dann ist für jedes ganze l *mit* $rk \leqq l < (r+1)k$

$$(4.10) \qquad R_r = -\sum_{h=0}^{q-1} \rho^{hk} \int_0^{\infty} e^{-Yu^k}\, P_{l+1}(u, Y)\, b_{l+1}\left(\frac{h}{q} - u\right) du.$$

Beweis. Folgt unmittelbar aus (4 . 1), (4 . 7) und der Definition von $\phi_\rho(x)$.

5. *Abschätzung von* $|f(x) - \phi_\rho(x)|$.

Aus (2 . 1) ergibt sich für jedes $\epsilon > 0$

$$(5.1) \qquad \left| \sum_{\nu=0}^{r} \sum_{\mu=0}^{k\nu+1} \beta_4(k, \nu, \mu)\, Y^\nu \sum_{h=0}^{q-1} \left(\frac{h}{q}\right)^\mu \rho^{hk} \right| < \gamma_2(k, r, \epsilon)\, q^{\kappa+\epsilon} \operatorname{Max}(1, |Y|^r),$$

insbesondere also auf einem $\mathfrak{M}_\rho$ wegen (2 . 3)

$$(5.2) \qquad \left| \sum_{\nu=0}^{r} \sum_{\mu=0}^{k\nu+1} \beta_4(k, \nu, \mu)\, Y^\nu \sum_{h=0}^{q-1} \left(\frac{h}{q}\right)^\mu \rho^{hk} \right| < \gamma_3(k, r, \epsilon)\, q^{\kappa+\epsilon}.$$

Die rechte Seite hat die Form $Cq^{\kappa+\epsilon}$, wenn r als absolute Constante fixirt wird (z.B. $r = 0$), und sie hat die Form $Dq^{\kappa+\epsilon}$, wenn r von der Ordnung der Fareyzerschneidung, d.h. von δ, abhängig gemacht wird.

SATZ 7. *Unter der Voraussetzung von Satz 6 ist*

$$|R_r| < \gamma_4(k, l)\, q\eta^{al}\left(\frac{|Y|}{\eta}\right)^{l+1}. \tag{5.3}$$

Beweis. Da nach (3.2) für alle reellen u

$$\left|\sum_{h=0}^{q-1} \rho^{h^k} b_{l+1}\left(\frac{h}{q}-u\right)\right| < \gamma_1(l+1)\, q^{*}$$

und nach (4.3) für $u \geqq 0$

$$|e^{-Yu^k} P_{l+1}(u, Y)| \leqq \sum_{a(l+1)\leq\mu\leq l+1} |\beta_3(k, l+1, \mu)|\, |Y|^{\mu} u^{k\mu-l-1} e^{-\eta u^k}$$

ist, wird

$$\begin{aligned}
|R_r| &< q \sum_{a(l+1)\leq\mu\leq l+1} \gamma_8(k, l, \mu)|Y|^{\mu}\int_0^{\infty} e^{-\eta u^k} u^{k\mu-l-1}\, du \\
&= q \sum_{a(l+1)\leq\mu\leq l+1} \gamma_8(k, l, \mu)|Y|^{\mu} a\eta^{-\mu+al}\int_0^{\infty} e^{-v} v^{\mu-al-1}\, dv \\
&= q\eta^{al} \sum_{a(l+1)\leq\mu\leq l+1} \gamma_9(k, l, \mu)\left(\frac{|Y|}{\eta}\right)^{\mu} \\
&< \gamma_4(k, l)\, q\eta^{al}\left(\frac{|Y|}{\eta}\right)^{l+1},
\end{aligned}$$

da $|Y|/\eta \geqq 1$.

---

* Diese Abschätzung lässt sich mit geringer Mühe zu

$$\left|\sum_{h=0}^{q-1} \rho^{h^k} b_{l+1}\left(\frac{h}{q}-u\right)\right| < \gamma_5(k, l, \epsilon)\, q^{\kappa+\epsilon}$$

verschärfen. Wird nämlich $u-[u] = u_0$ gesetzt, so ist

$$\begin{aligned}
\sum_{h=0}^{q-1} \rho^{h^k} b_{l+1}\left(\frac{h}{q}-u\right) &= \sum_{0\leq h/q<u_0} \rho^{h^k} b_{l+1}\left(\frac{h}{q}+1-u_0\right) + \sum_{u_0\leq h/q<1} \rho^{h^k} b_{l+1}\left(\frac{h}{q}-u_0\right) \\
&= \sum_{\mu=0}^{l+1}\left\{p_{l,\mu}(1-u_0) \sum_{0\leq h/q<u_0} \rho^{h^k}\left(\frac{h}{q}\right)^{\mu} + p_{l,\mu}(-u_0) \sum_{u_0\leq h/q<1} \rho^{h^k}\left(\frac{h}{q}\right)^{\mu}\right\},
\end{aligned}$$

wo $p_{l,\mu}(x)$ ein Polynom $(l+1-\mu)$-ten Grades mit allein von l und μ abhängenden Coefficienten bedeutet. Hieraus folgt die Behauptung wegen

$$\operatorname*{Max}_{0\leq u_0<1} \sum_{\mu=0}^{l+1} \{|p_{l,\mu}(1-u_0)| + |p_{l,\mu}(-u_0)|\} = \gamma_6(l)$$

und (2.12).—Macht man von dieser Verschärfung Gebrauch, so kann man in der rechten Seite von (5.3) $\gamma_3(k, l)\, q$ durch $\gamma_4(k, l, \epsilon)\, q^{\kappa+\epsilon}$ und hernach in der rechten Seite von (5.4) $\gamma_{11}(k, l)$ durch $\gamma_{12}(k, l, \epsilon)\, q^{-(1/K)+\epsilon}$ ersetzen.

SATZ 8. *Auf dem major arc gilt für jedes ganze $l \geqq 0$ und jedes $\epsilon > 0$ die Abschätzung*

$$(5.4) \qquad |f(x)-\phi_\rho(x)| < \gamma_{10}(k, l, \epsilon)\, q^{\kappa+\epsilon} + \gamma_{11}(k, l)\, n^{-\delta l-\alpha}.$$

Beweis. Aus Satz 6, Formel (5.2), Satz 7 folgt mit Rücksicht auf (2.3)

$$|f(x)-\phi_\rho(x)| < \gamma_3\left(k, \left[\frac{l}{k}\right], \epsilon\right) q^{\kappa+\epsilon} + \gamma_4(k, l)\, q \cdot q^l n^{-\alpha l} \cdot 8^{l+1} q^{-(l+1)} n^{\alpha(l+1)}$$

$$= \gamma_{10}(k, l, \epsilon)\, q^{\kappa+\epsilon} + \gamma_{11}(k, l)\, n^{-\delta l+\alpha}.$$

Wenn $\delta > 0$ ist, folgt aus Satz 8, indem l als Function von k und δ so festgelegt wird, dass $l > \alpha/\delta$ wird, sofort die Abschätzung (1.3); dagegen reicht (5.4) nicht aus, um im Falle $\delta = 0$ (1.2) zu beweisen.

6. *Schärfere Abschätzung von* $|f(x)-\phi_\rho(x)|$.

Ersetzt man in (4.10) $b_{l+1}\{(h/q)-u\}$ durch seine Fourierreihe (3.3) und vertauscht Summation und Integration, so entsteht die Formel

$$(6.1) \qquad R_r = \lim_{M\to\infty} \sum_{m=-M}^{M}{}' \frac{S_{\rho, m}}{(2\pi i m)^{l+1}} \int_0^\infty e^{-Yu^k - 2\pi i m u}\, P_{l+1}(u, Y)\, du,$$

also nach (4.3), vorbehaltlich der Convergenz von $\lim_{M\to\infty} \ldots$ für alle vorkommenden μ,

$$(6.2) \quad R_r = \sum_{\alpha(l+1) \leq \mu \leq l+1} \beta_3(k, l+1, \mu)\, Y^\mu \lim_{M\to\infty} \sum_{m=-M}^{M}{}' \frac{S_{\rho, m}}{(2\pi i m)^{l+1}} \times \int_0^\infty e^{-Yu^k - 2\pi i m u}\, u^{k\mu-l-1}\, du.$$

Im Falle $l > 0$ ist die Vertauschung von Summation und Integration mittels La. Vorl. Satz 232 leicht zu rechtfertigen; in dem besonders interessirenden Falle $l = 0$ ist die Rechtfertigung aber schwieriger; man hat dann Satz 11 heranzuziehen. Deshalb scheint es mir einfacher, wie es in P.N. I geschieht, von der Fourierentwickelung der Function $g(Y, t)$ statt von der Euler'schen Summenformel auszugehen und die Integrale, durch welche die Fouriercoefficienten dargestellt sind, durch $(l+1)$-malige partielle Integration umzuformen. So erhält man für jedes ganze $l \geqq 0$

$$g(Y, t) = a_t + \int_0^\infty e^{-Yu^k}\, du + \sum_{\lambda=0}^{l} P_\lambda(0, Y) \lim_{M\to\infty} \sum_{m=-M}^{M}{}' \frac{e^{2\pi i m t}}{(2\pi i m)^{\lambda+1}}$$

$$+ \lim_{M\to\infty} \sum_{m=-M}^{M}{}' \frac{e^{2\pi i m t}}{(2\pi i m)^{l+1}} \int_0^\infty e^{-Yu^k - 2\pi i m u}\, P_{l+1}(u, Y)\, du,$$

wo
$$a_t = \begin{cases} 0 & \text{für} \quad 0 < t < 1, \\ \frac{1}{2} & \text{für} \quad t = 0. \end{cases}$$

Hieraus ergibt sich auf Grund von (3 . 3), (4 . 1) und Satz 4 :

SATZ 9. *Unter den Voraussetzungen von Satz* 6 *gelten für jedes ganze* l *mit* $rk \leqq l < (r+1)k$ *die Formeln* (6 . 1) *und mit dem genannten Vorbehalt* (6 . 2).

SATZ 10. *Es sei* $\xi \geqq 0$, $H > 0$, γ *reell*, $0 < \epsilon < 1$; *dann ist*

(6 . 3)
$$\left| \int_0^\infty e^{-v} v^\xi e^{\pm i(Hv^a + \gamma v)} dv \right| < \gamma_{13}(k, \xi, \epsilon) H^{\frac{1}{2}\epsilon - \frac{1}{2}}.$$

Beweis. Nach dem Landau- van der Corput'schen Hilfssatz (vgl. z.B. La. Vorl. Satz 413) ist für $0 \leqq v_1 < v_2$

$$\left| \int_{v_1}^{v_2} e^{\pm i(Hv^a + \gamma v)} dv \right| \leqq \frac{4}{\sqrt{\{a(1-a)\}}} H^{-\frac{1}{2}} v_2^{1-\frac{1}{2}a}.$$

Da die Function $e^{-v} v^\xi$ aus ein oder zwei monotonen Stücken besteht und ein allein von ξ abhängendes Maximum hat, ergibt die zwei- bezw. viermalige Anwendung des zweiten Mittelwertzatzes für $\epsilon_1 > 0$

$$\left| \int_0^{H^{\epsilon_1}} e^{-v} v^\xi e^{\pm i(Hv^a + \gamma v)} dv \right| \leqq \gamma_{14}(k, \xi) H^{-\frac{1}{2} + \epsilon_1(1-\frac{1}{2}a)}.$$

Ferner ist

$$\left| \int_{H^{\epsilon_1}}^\infty e^{-v} v^\xi e^{\pm i(Hv^a + \gamma v)} dv \right| \leqq \int_{H^{\epsilon_1}}^\infty e^{-v} v^\xi dv < e^{-\frac{1}{2}H^{\epsilon_1}} \int_0^\infty e^{-\frac{1}{2}v} v^\xi dv$$
$$= \gamma_{15}(\xi) e^{-\frac{1}{2}H^{\epsilon_1}},$$

und es gilt, sofern $\epsilon_1 \leqq 1/(2-a)$ ist,

$$e^{-\frac{1}{2}H^{\epsilon_1}} < \gamma_{16}(k, \epsilon_1) H^{-\frac{1}{2} + \epsilon_1(1-\frac{1}{2}a)}.$$

Somit folgt für $0 < \epsilon_1 \leqq 1/(2-a)$

$$\left| \int_0^\infty e^{-v} v^\xi e^{\pm i(Hv^a + \gamma v)} dv \right| < \gamma_{17}(k, \xi, \epsilon_1) H^{-\frac{1}{2} + \epsilon_1(1-\frac{1}{2}a)},$$

d.h. die Behauptung, wenn $\epsilon_1 = \epsilon/(2-a)$ genommen wird.

SATZ 11. *Es sei* $l \geqq 0$ *ganz*, $\mu \geqq al+1$ *ganz*, m *ganz* $0 < \epsilon < 1$. *Dann ist*

(6 . 4)
$$\left| \int_0^\infty e^{-Yu^k - 2\pi i m u} u^{k\mu - l - 1} du \right| < \gamma_{18}(k, l, \mu, \epsilon) |m|^{-\frac{1}{2} + \frac{1}{2}\epsilon} \eta^{-\mu + a(l + \frac{1}{2} - \frac{1}{2}\epsilon)}.$$

Beweis. Es ist

$$\int_0^\infty e^{-Yu^k-2\pi imu}\,u^{k\mu-l-1}\,du = \int_0^\infty e^{-\eta u^k}\,u^{k\mu-l-1}\,e^{-i(2\pi mu+\zeta u^k)}\,du$$

$$= a\eta^{-\mu+al}\int_0^\infty e^{-v}\,v^{\mu-al-1}\,e^{\pm i\{2\pi|m|\eta^{-a}v^{-a}+(\zeta/\eta)v\}}\,dv.$$

Wegen der Voraussetzung $\mu \geqq al+1$ ist der vorangehende Satz anwendbar mit $\xi = \mu-al-1$, $H = 2\pi|m|\eta^{-a}$, $\gamma = \pm(\zeta/\eta)$.

SATZ 12. *Unter den Voraussetzungen von Satz 6 gilt, $l = kr$ gesetzt, für $0<\epsilon<1$ die Abschätzung*

$$|R_r| < \gamma_{19}(k, r, \epsilon)\,q^{\kappa+\epsilon}\,\eta^{\alpha(l+\frac{1}{2}-\frac{1}{2}\epsilon)}\left(\frac{|Y|}{\eta}\right)^{l+1} \tag{6.5}$$

$$= \gamma_{19}(k, r, \epsilon)\,q^{\kappa+\epsilon}\,\eta^{r+\frac{1}{2}\alpha-\frac{1}{2}\alpha\epsilon}\left(\frac{|Y|}{\eta}\right)^{kr+1}.$$

Beweis. In (6.2) darf insbesondere $l = kr$ angenommen werden; dann wird der kleinste in der Summe vorkommende Wert von μ gleich $al+1$, sodass Satz 11 für jeden Summanden angewandt werden kann. Wegen der Convergenz der Reihe $\sum_{m=0}^{\infty} \frac{1}{m^{l+\frac{3}{2}-\frac{1}{2}\epsilon}}$ und der Abschätzung (2.1) folgt daher

$$|R_r| < q^{\kappa+\epsilon}\,\eta^{\alpha(l+\frac{1}{2}-\frac{1}{2}\epsilon)} \sum_{\mu=al+1}^{l+1} \gamma_{20}(k, l, \mu, \epsilon)\left(\frac{|Y|}{\eta}\right)^{\mu},$$

und hieraus die Behauptung wegen $(|Y|/\eta) \geqq 1$.

SATZ 13. *Auf dem major arc gilt für jedes ganze $r \geqq 0$ und für jedes $\epsilon > 0$ die Abschätzung*

$$|f(x)-\phi_\rho(x)| < \gamma_3(k, r, \epsilon)\,q^{\kappa+\epsilon}+\gamma_{21}(k, r, \epsilon)\,q^{\kappa-\frac{1}{2}+\epsilon}\,n^{-\delta(kr+\frac{1}{2})+\frac{1}{2}\alpha+\frac{1}{2}\alpha\epsilon}. \tag{6.6}$$

Beweis. Ohne Beschränkung der Allgemeinheit darf ich $\epsilon < 1$ annehmen. Aus Satz 9, (5.2), Satz 12 und (2.3) folgt dann

$$f(x)-\phi_\rho(x)| < \gamma_3(k, r, \epsilon)\,q^{\kappa+\epsilon}$$

$$+\gamma_{19}(k, r, \epsilon)\,q^{\kappa+\epsilon}\cdot q^{l+\frac{1}{2}-\frac{1}{2}\epsilon}\,n^{-\alpha(l+\frac{1}{2}-\frac{1}{2}\epsilon)}\cdot 8^{l+1}\,q^{-(l+1)}\,n^{\alpha(l+1)}$$

$$= \gamma_3(k, r, \epsilon)\,q^{\kappa+\epsilon}+\gamma_{21}(k, r, \epsilon)\,q^{\kappa-\frac{1}{2}+\frac{1}{2}\epsilon}\,n^{-\delta(l+\frac{1}{2})+\frac{1}{2}\alpha+\frac{1}{2}\alpha\epsilon}.$$

Aus der Abschätzung (6.6), die schärfer ist als (5.4), folgt im Falle $\delta > 0$ durch passende Wahl von r *a fortiori* (1.3), es folgt aber auch im

Falle $\delta = 0$ (1.2). Es genügt dabei $l = 0$ anzunehmen. Specialisirt man die vorangehende Entwickelung von Anfang an auf $\delta = 0$, so erhält man in der Tat einen sehr kurzen und einfachen Beweis von (1.1)*.

7. *Aufstellung der asymptotischen Entwickelung mittels der Mellin'schen Formel und Abschätzungen des Restes.*

Ich setze zur Abkürzung für $\sigma > 1$

$$(7.1)\quad Z(s) = q^s \sum_{m=1}^{\infty} \frac{\rho^m}{m^s} = q^s \sum_{h=1}^{q}{}' \rho^h \sum_{m=0}^{\infty} \frac{1}{(mq+h)^s} = \sum_{h=1}^{q}{}' \rho^h \zeta\left(s, \frac{h}{q}\right).$$

SATZ 14. (1) $Z(s)$ *ist bis auf einen Pol erster Ordnung in* $s = 1$ *mit dem Residuum* S_ρ *in der ganzen s-Ebene regulär.*

(2) *Für* $\sigma > -l$ $(l > 0)$ *ist gleichmässig in* σ $Z(s) = O(|t|^{l+\frac{1}{2}})$.

(3) *Für ganzes* $q \geqq 0$ *ist*

$$(7.2)\quad Z(-g) = a'_g - g! \sum_{h=1}^{q}{}' \rho^h b_{g+1}\left(\frac{h}{q}\right),$$

wo

$$a'_g = \begin{cases} -1 & \text{für } g = 0, \\ 0 & \text{für } g > 0. \end{cases}$$

(4) *Für* $\sigma < 0$ *ist*

$$(7.3)\quad Z(s) = 2\Gamma(1-s)\left\{\sin\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{S^{(c)}_{\rho, m}}{(2\pi m)^{1-s}} + \cos\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{S^{(s)}_{\rho, m}}{(2\pi m)^{1-s}}\right\}.$$

Beweis. Bekanntlich ist $\zeta(s, w)$ $(0 < w \leqq 1)$ bis auf einen Pol erster Ordnung in $s = 1$ mit dem Residuum 1 in der ganzen Ebene regulär, genügt für $\sigma > -l$ $(l > 0)$ gleichmässig in σ der Abschätzung $\zeta(s, w) = O(|t|^{l+\frac{1}{2}})$ und besitzt für $\sigma < 0$, im Falle $w < 1$ auch für $s = 0$ die Darstellung

$$(7.4)\quad \zeta(s, w) = 2\Gamma(1-s)\left\{\sin\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\cos 2\pi m w}{(2\pi m)^{1-s}} + \cos\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\sin 2\pi m w}{(2\pi m)^{1-s}}\right\}.$$

Hieraus folgen die Behauptungen (1), (2), (4). Für ganzes $g > 0$,

* Ich will nicht verschweigen, dass die Abschätzungen (6.5), (6.6) schlechter sind, als die (im Falle $l = 0$) von Hardy-Littlewood in P.N. I mit dem subtilen Werkzeug der Sattelpunktsintegration gewonnenen.

im Falle $w<1$ auch für $g=0$ ergibt sich aus (7 . 4)

$$\zeta(-g,\, w) = 2g!\sum_{m=1}^{\infty}\frac{\sin(2\pi mw-\frac{1}{2}\pi g)}{(2\pi m)^{1+g}}$$

$$= g!\lim_{M\to\infty}\sum_{m=-M}^{M}{}'\frac{e^{2\pi imw}}{(2\pi im)^{1+g}} = -g!\,b_{g+1}(w),$$

während $\zeta(0,\,1) = -1+\zeta(0) = -\frac{1}{2}$ ist. Damit ist auch (7 . 2) bewiesen.

SATZ 15. *Unter den Voraussetzungen von Satz 6 ist für jedes reelle l der Strecke $rk<l<(r+1)k$*

$$R_r = \frac{a}{2\pi i}\int_{(-l)}\Gamma(sa)\,Y^{-sa}\,\mathrm{Z}(s)\,ds, \tag{7 . 5}$$

worin $\int_{(-l)}$ *das Integral längs der verticalen Geraden mit der Abscisse $-l$ von unten nach oben bedeutet.*

Beweis. Nach der Formel von Mellin ist (vgl. La. Mel. S. 89)

$$f(x) = 1+\frac{1}{2\pi i}\int_{(2)} a\Gamma(sa)\,Y^{-sa}\,\mathrm{Z}(s)\,ds.$$

Ich verschiebe den Integrationsweg nach links auf die Gerade $(-l)$; der Integrand besitzt Pole an den Stellen $s=1$ und $s=-\nu k$ $(\nu=0,\,1,\,2,\,\ldots)$ mit den Residuen

bei $s=1$:

$$a\Gamma(a)\,Y^{-a}\,S_\rho = \frac{\Gamma(a+1)\,S_\rho}{qy^a} = \phi_\rho(x);$$

bei $s=-rk$:

$$a\,\frac{(-1)^\nu}{\nu!\,a}\,Y^\nu\,\mathrm{Z}(-\nu k) = a'+\frac{(-1)^{\nu+1}(k\nu)!}{\nu!}\,Y^\nu\sum_{h=1}^{q}\rho^{hk}\,b_{k\nu+1}\Big(\frac{h}{q}\Big);$$

somit wird

$$f(x) = \phi_\rho(x)+\sum_{\nu=1}^{r}\frac{(-1)^{\nu+1}(k\nu)!}{\nu!}\,Y^\nu\sum_{h=1}^{q}\rho^{hk}\,b_{k\nu+1}\Big(\frac{h}{q}\Big) \tag{7 . 6}$$

$$+\frac{a}{2\pi i}\int_{(-l)}\Gamma(sa)\,Y^{-sa}\,\mathrm{Z}(s)\,ds,$$

w.z.b.w.

SATZ 16. *Unter den Voraussetzungen von Satz 6 ist für jedes reelle l der Strecke $rk < l < (r+1)k$ und jedes $\epsilon > 0$*

$$|R_r| < \gamma_{22}(k, l, \epsilon)\, q^{\kappa+\epsilon}\, \eta^{al} \left(\frac{|Y|}{\eta}\right)^{l+1}. \tag{7.7}$$

Beweis. Nach (2.1) ist für $s = -l+ti$ und jedes $\epsilon > 0$

$$\left| \sum_{m=1}^{\infty} \frac{S_{\rho,m}^{(c)}}{(2\pi m)^{1-s}} \right| < \gamma_{23}(k, l, \epsilon)\, q^{\kappa+\epsilon}, \quad \left| \sum_{m=1}^{\infty} \frac{S_{\rho,m}^{(s)}}{(2\pi m)^{1-s}} \right| < \gamma_{23}(k, l, \epsilon)\, q^{\kappa+\epsilon},$$

ferner $|Y^{-sa}| = |Y|^{al}\, e^{at\psi}$, daher folgt aus (7.3)

$$|\Gamma(sa)\, Y^{-sa}\, Z(s)| < 2\gamma_{23}(k, l, \epsilon)\, q^{\kappa+\epsilon}\, e^{at\psi}\, G(s, k), \tag{7.8}$$

wo

$$G(s, k) = \left|\Gamma(sa)\,\Gamma(1-s)\,\sin\frac{\pi s}{2}\right| + \left|\Gamma(sa)\,\Gamma(1-s)\,\cos\frac{\pi s}{2}\right|$$

$$< \gamma_{24}(k, l)\, e^{-\frac{1}{2}\pi a\,|t|}\,(1+|t|)^{(1-a)l}.$$

Aus der letzten Abschätzung folgt

$$\int_{-\infty}^{\infty} G(-l+ti, k)\, e^{at\psi}\, dt < \gamma_{24}(k, l) \int_{-\infty}^{\infty} e^{-a|t|(\frac{1}{2}\pi-\psi)\,\mathrm{sign}\,t}(1+|t|)^{(1-a)l}\, dt$$

$$< \gamma_{24}(k, l)\, e^{a(\frac{1}{2}\pi-\psi)} \int_1^{\infty} e^{-au(\frac{1}{2}\pi-\psi)}\, u^{(1-a)l}\, du$$

$$+ \gamma_{24}(k, l)\, e^{a(\frac{1}{2}\pi+\psi)} \int_1^{\infty} e^{-au(\frac{1}{2}\pi+\psi)}\, u^{(1-a)l}\, du$$

$$< \gamma_{25}(k, l) \left\{ \left(\frac{1}{\frac{1}{2}\pi-\psi}\right)^{(1-a)l+1} + \left(\frac{1}{\frac{1}{2}\pi+\psi}\right)^{(1-a)l+1} \right\},$$

also wegen (2.2)

$$\int_{-\infty}^{\infty} G(-l+ti, k)\, e^{at\psi}\, dt < 2\gamma_{25}(k, l) \left(\frac{|Y|}{\eta}\right)^{(1-a)l+1}. \tag{7.9}$$

Aus (7.5), (7.8), (7.9) folgt die Behauptung.

Die Abschätzung (7.7) stimmt genau überein mit der in Fussnote*, S. 391 gegebenen. Um eine schärfere Abschätzung zu erhalten, kann man auch im Falle $l > 0$ genau den Weg von La. Mel. verfolgen; man hat dabei nur die dortige Formel (3) zu ersetzen durch die etwas allgemeinere, aber ebenso zu beweisende (7.10).

SATZ 17. *Es sei* $\xi \geqq 0$, $H > 0$, γ *reell*, $0 < \epsilon < 1$.

Dann ist

$$(7.10) \qquad \left| \int_0^\infty e^{-u}\, u^{\xi}\, e^{\pm iH(u\log u + \gamma u)}\, du \right| < \gamma_{26}(\xi, \epsilon)\, H^{\frac{1}{2}\epsilon - \frac{1}{2}}.$$

Alsdann ergibt sich

SATZ 18. *Unter den Voraussetzungen von Satz 6 ist für jedes reelle* l *der Strecke* $rk < l < (r+1)k$ *und jedes* $\epsilon > 0$

$$(7.11) \qquad |R_r| < \gamma_{27}(k, l, \epsilon)\, q^{\kappa+\epsilon}\, \eta^{la} \left(\frac{|Y|}{\eta}\right)^{l+\frac{1}{2}+\frac{1}{2}\epsilon}.$$

Da der Vergleich mit (6.5) unbequem ist, specialisire ich auf den major arc; dort wird

$$(7.12) \qquad |R_r| < \gamma_{28}(k, l, \epsilon)\, q^{\kappa-\frac{1}{2}+\frac{1}{2}\epsilon}\, n^{-\delta l+\frac{1}{2}a+\frac{1}{2}(a\epsilon)}.$$

Dies ist im Falle $\delta > 0$ insofern etwas besser als (6.6), als hier $l > kr+\frac{1}{2}$ gewählt werden kann.

8. *Über die Brauchbarkeit der Abschätzung* $|f(x)-\psi_\rho(x)| < C_1 q^{-(1/K)+\epsilon} n^a$.

Es sei auf dem Kreise $|x| = e^{-(1/n)}$ eine Fareyzerschneidung der Ordnung $[n^{1-\alpha}]$ mit einem $\alpha < a$ zu Grunde gelegt. Ich will zeigen, dass bei passender Wahl dieses α die Abschätzungen

$$(8.11) \qquad |f(x)| < C_4\, n^{\alpha-(a/K)+\epsilon} \quad \text{auf} \quad \mathfrak{m},$$

$$(8.12) \qquad |f(x)-\psi_\rho(x)| < C_5\, q^{-(1/K)+\epsilon}\, n^a \quad \text{auf} \quad \mathfrak{M}^*$$

ausreichen, um (1.1) für alle $s \geqq (k-2)K+5$ zu beweisen.

Verfolgt man den Gang des Beweises in La. Vorl. VI. Teil, Kap. 1, § 5, so findet man, dass sämtliche Fehlerglieder mit Ausnahme von zweien bereits für $s \geqq 2K+1$ die verlangte Gestalt $bn^{sa-1-\epsilon}$ haben. Kritisch sind nur die Glieder

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} \frac{\{f(x)\}^s}{x^{n+1}}\, dx \quad \text{und} \quad \sum_{\mathfrak{M}} \int_{\mathfrak{M}} \frac{\{f(x)^s - \{\psi_\rho(x)\}^s}{x^{n+1}}\, dx.$$

* (8.12) ergibt sich z.B. durch eine geringfügige Änderung aus dem Beweis des Hilfssatzes 5 in La. War. bezw. des ersten Teiles des Beweises von Satz 278 in La. Vorl. oder auch aus dem vorangehenden Satz 16 für $l = 0$, bezw. aus Satz 8 für $l = 0$, wenn man ihn entsprechend Fussnote * auf S. 391 modificirt.

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1928.

Liabilities.	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				360	13	0
Life Compositions Fund				3101	7	0
Reserve Fund in 1927	431	9	3			
Appreciation of Investments	115	6	3			
Reserve Fund				546	15	6
Sundry Creditors				23	12	0
Revenue Account, balance in hand				38	9	10
				£4770	4	9

Assets.	£	s.	d.
London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 76¾ ...	295	9	9
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 61⅛	428	1	10
Funding Loan £3000 4 per Cent. at 88¾ ...	2662	10	0
Conversion Loan £1700 3½ per Cent. at 77⅝	1319	12	6
The above-mentioned stocks are valued at the present market prices.			
Cash at Bank	64	10	8
	£4770	4	9

REVENUE ACCOUNT for the year 1st November, 1927, to 31st October, 1928.

	Year 1926-27. £	s.	d.	£	s.	d.
Balance from last year	539	8	2	*566*	*9*	*1*
Dividends on—						
Lord Rayleigh's Fund ...	38	13	0	*38*	*11*	*11*
The Society's Investments	205	1	8	*203*	*16*	*3*
Entrance Fees	31	10	0	*48*	*6*	*3*
Annual Subscriptions	509	8	0	*459*	*6*	*0*
Sales of *Proceedings*	369	12	3	*544*	*1*	*3*
Ditto *Journal*	79	1	9	*59*	*16*	*6*
Advertisements revenue	37	5	0	*47*	*5*	*0*
Grant from Royal Society		—		*150*	*0*	*0*
Mr. H. T. J. Norton's Donation	150	0	0		—	
	£1959	19	10	*2117*	*12*	*3*

	Year 1926-27. £	s.	d.	£	s.	d.
Printing and Circulating *Proceedings* ...	1305	9	0	*967*	*8*	*3*
Ditto *Journal*	426	15	0	*426*	*11*	*0*
General Printing	125	11	3	*107*	*7*	*6*
Purchases and Binding for Library ...	24	1	5	*43*	*11*	*1*
Expenses of Meetings	30	0	0	*30*	*0*	*0*
Postages and Sundries	9	13	4	*9*	*6*	*3*
Balance in hand on 31st October, 1928 ...	38	9	10	*539*	*8*	*2*
	£1959	19	10	*2117*	*12*	*3*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.
Balance in hand 31st October, 1927 ...	3	1	6
Dividends received	2	8	0
	£5	9	6

	£	s.	d.
Cash at Bank	5	9	6
	£5	9	6

ACCOUNT OF LIEUT.-COL. A. J. C. CUNNINGHAM'S TRUST FUND

For Factorisation of Numbers.

	£	s.	d.
Of Lieut.-Col. A. J. C. Cunningham's Executors' Legacy	1000	0	0
	£1000	0	0

	£	s.	d.
Cash at Bank	1000	0	0
	£1000	0	0

STATEMENT OF TRUST FUNDS, 31st October, 1928.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.

£229. 13*s.* 4*d.* 4 per Cent. Funding Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

8th November, 1928.

Signed on behalf of the Council:—

G. B. JEFFERY, *Chairman.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

A. E. JOLLIFFE, *Auditor.*

18th November, 1928.

Die Abschätzung des ersten ergibt [La. Vorl. Formel (321)]

$$\left|\sum_{\mathfrak{m}}\int_{\mathfrak{m}}\frac{\{f(x)\}^s}{x^{n-1}}\,dx\right| < B_1 n^{(s-4)[a-(a/K)]+2a+\epsilon}$$

$$= B_1 n^{sa-1} n^{-(a/K)(s-4)+1-2a+\epsilon}.$$

Wegen $a<a$ wird $\{-a(s-4)/K\}+1-2a+\epsilon$ frühestens negativ für $s=(k-2)K+5$; damit dies (bei passender Wahl von ϵ) wirklich eintritt, ist notwendig und hinreichend, dass

$$-\frac{a}{K}\{(k-2)K+1\}+a(k-2)<0$$

d.h.

$$a > \frac{(k-2)K}{(k-2)K+1}\,a = \left(1-\frac{1}{(k-2)K+1}\right)a \tag{8.2}$$

ist.

Nach La. Vorl. Bd. 1, S. 271, 272 ist, wenn zur Abkürzung $\Psi_\rho(x)=f(x)-\psi_\rho(x)$ geschrieben wird,

$$\left|\sum_{\mathfrak{M}}\int_{\mathfrak{M}}\frac{\{f(x)\}^s-\{\psi_\rho(x)\}^s}{x^{n-1}}\,dx\right| \leqq 2^s e\left\{\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|\Psi|^s d\mathfrak{Y}+\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|\Psi|^{s-1}_{\psi}\,d\mathfrak{Y}\right\}. \tag{8.3}$$

Hierin ist nach (8.12), da zu jedem q höchstens q Bögen $\mathfrak{M}$ gehören und die Länge eines einzelnen $\mathfrak{M}$ höchstens $4\pi/qn^{1-a}$ beträgt,

$$2^s e\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|\Psi|^s d\mathfrak{Y} < B_2\sum_{q\leq n^a} q\,.\,q^{-(s/K)+\epsilon}n^{as}\,.\,q^{-1}n^{a-1}$$

$$= B_2 n^{sa-1}\,.\,n^{-\delta s+a}\sum_{q\leq n^a} q^{-(s/K)+\epsilon}.$$

Also gilt, wenn $\epsilon=1/(2K)$ gewählt wird, für $s\geqq K+1$

$$2^s e\sum_{\mathfrak{M}}\int_{\mathfrak{M}}|\Psi|^s d\mathfrak{Y} < b_2 n^{sa-1}\,.\,n^{-\delta s+a}. \tag{8.4}$$

Damit $-\delta s+a=-as+a(s+1)$ für alle $s\geqq(k-2)K+5$ negativ wird, ist notwendig und hinreichend, dass

$$-\{(k-2)K+5\}a+\{(k-2)K+6\}a<0,$$

d.h.

$$a < \frac{(k-2)K+5}{(k-2)K+6}\,a = \left(1-\frac{1}{(k-2)K+6}\right)a \tag{8.5}$$

ist.

Weiter ist mit Rücksicht auf (8 . 12) und

$$\int_{\mathfrak{M}} |\psi|^{s-1} d\mathfrak{F} < B_3 n^{s\alpha-1-\alpha+\frac{1}{2}\epsilon} q^{-(s-1/K)}$$

(vgl. La. Vorl. S. 272)

$$2^s e \sum_{\mathfrak{M}} \int_{\mathfrak{M}} |\Psi|\,|\psi|^{s-1} d\mathfrak{F} < B_4 n^{s\alpha+1+\alpha-\alpha+\frac{1}{2}\epsilon} \sum_{\eta \leq n^\alpha} q \,.\, q^{-(1/K)+\epsilon} \,.\, q^{-(s-1)/K}$$

$$= B_4 n^{s\alpha-1-\delta+\frac{1}{2}\epsilon} \sum_{\eta \leq n^\alpha} q^{-(s/K)+1+\epsilon},$$

also gilt, wenn $\epsilon \leqq 1/(2K)$ gewählt wird, für $s \geqq 2K+1$

$$2^s e \sum_{\mathfrak{M}} \int_{\mathfrak{M}} |\Psi|\,|\psi|^{s-1} d\mathfrak{F} < B_5 n^{s\alpha-1} \,.\, n^{-\delta+\frac{1}{2}\epsilon}. \tag{8.6}$$

Die Bedingungen (8 . 2) und (8 . 5) sind verträglich. Infolgedessen kann α als Function von k so fixirt werden, dass beide erfüllt sind, und dann in (8 . 6) ϵ als Function von k so fixirt werden, dass gleichzeitig $\epsilon \leqq 1/(2K)$ und $\epsilon/2 < \delta$ ist. Alsdann folgt, dass für alle $s \geqq (k-2)K+5$

$$\left| \sum_{\mathfrak{m}} \int_{\mathfrak{m}} \frac{\{f(x)\}^s}{x^{n+1}} dx \right| + \left| \sum_{\mathfrak{M}} \int_{\mathfrak{M}} \frac{\{f(x)\}^s - \{\psi_\rho(x)\}^s}{x^{n+1}} dx \right| < b_3 n^{s\alpha-1-c_2}$$

gilt. w.z.b.w.

z.Zt. Halle a.S., den 20. März 1928.

A GENERALIZATION OF THE FOURIER-BESSEL INTEGRAL TRANSFORM

By C. Fox.

[Received 3 June, 1927.—Read 9 June, 1927.—Revised 14 June, 1928.]

1. *Introduction and notation.*

1 . 1. The object of this paper is to generalize the well known integral transform, namely that, with appropriate restrictions :

(1 . 11) *If*
$$f(y) = \int_0^\infty F(x)\, J_\nu(xy)\, x\, dx,$$

then
$$F(z) = \int_0^\infty f(y)\, J_\nu(yz)\, y\, dy.$$

The generalization consists in replacing the Bessel functions first by a generalized hypergeometric function, and subsequently by a linear combination of such functions. The hypergeometric functions are defined by the formula

(1 . 12) $${}_pf_q(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_q;\ x)$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)\ldots\Gamma(\alpha_p+n)}{\Gamma(\rho_1+n)\,\Gamma(\rho_2+n)\ldots\Gamma(\rho_q+n)}\,\frac{x^n}{n!},$$

and they are integral functions of x if $q \geqslant p$.

We shall restrict ourselves to the case $q = p+1$, for this is the only case in which the functions (1 . 12) have asymptotic expansions of a type which seems to be essential for the existence of theorems such as (1 . 11).

The theorems which we shall prove are enunciated in §§ 1 . 3 and 1 . 5, and the notation which will be used throughout the paper is explained in §§ 1 . 2, 1 . 4, and 1 . 6.

1.2. *Notation.* We shall write

$$\phi_0 = \sum_{r=1}^{p} a_r - \sum_{r=1}^{p+1} \rho_r + \tfrac{1}{2},$$

$$\Phi_0(x) = (\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}\, {}_pf_{p+1}(a_1, a_2, \ldots, a_p; \rho_1, \rho_2, \ldots, \rho_{p+1}; -\tfrac{1}{4}x^2), \tag{1.21}$$

$${}_ra_s = 1-a_s+\rho_r \quad (s = 1, 2, 3, \ldots, p; r = 1, 2, 3, \ldots, p+1),$$

$${}_r\rho_r = \rho_r \quad (r = 1, 2, 3, \ldots, p+1),$$

$${}_r\rho_s = 1-\rho_s+\rho_r \quad (s = 1, 2, 3, \ldots, p+1; r = 1, 2, 3, \ldots, p+1; r \neq s),$$

$$\psi_r = \sum_{s=1}^{p} {}_ra_s - \sum_{s=1}^{p+1} {}_r\rho_s + \tfrac{1}{2} = 1-2\rho_r-\phi_0,$$

$$\Psi_r(x) = (\tfrac{1}{2}x)^{-\psi_r-\frac{1}{2}}\, {}_pf_{p+1}({}_ra_1, {}_ra_2, \ldots, {}_ra_p; {}_r\rho_1, {}_r\rho_2, \ldots, {}_r\rho_{p+1}; -\tfrac{1}{4}x^2) \quad (r = 1, 2, 3, \ldots, p+1).$$

We shall define the numbers $B_r (r = 1, 2, 3, \ldots, p+1)$ as a set satisfying the p equations

$$\sum_{r=1}^{p+1} \frac{B_r}{\sin(\rho_r-a_s)\pi} = 0 \quad (s = 1, 2, 3, \ldots, p), \tag{1.22}$$

and, finally, we shall suppose throughout that x, y, and z are positive real variables.

1.3. Theorem 1. *If*

$$f(y) = \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx,$$

then

$$\tfrac{1}{2}\{F(z+0)+F(z-0)\} \sum_{r=1}^{p+1} B_r \sin(\phi_0+\rho_r)\pi = \int_0^\infty f(y) \sum_{r=1}^{p+1} B_r \Psi_r(yz)\, y\, dy,$$

provided that

(i) $\int_0^\infty |F(x)| \sqrt{x}\, .\, dx$ *exists.*

(ii) $F(x)$ *is of bounded variation near* $x = z$.

(iii) $\phi_0 \leqq 0$.

(iv) $2\rho_r \geq -\phi_0 \quad (r = 1, 2, 3, \ldots, p+1)$.

(v) *No two of the* $a_r \quad (r = 1, 2, 3, \ldots, p)$ *are equal.*

(vi) *There is a number* ν *such that* $0 \leqq \nu \leqq 1$ *and* $\nu < \phi_0+2a_r < \nu+2$ $(r = 1, 2, 3, \ldots, p)$: *except that, for one value of* r, *we may have the equality* $\phi_0+2a_r = \nu+2$.

The converse of this theorem is also true, subject to the following conditions :—

(i) *and* (ii) *as above, except that* $F(x)$ *is replaced by* $f(x)$ *and* z *by* y.

(iii) $\psi_r \leqslant 0 \quad (r = 1, 2, 3, \ldots, p+1)$.

(iv) $\phi_0 < 1$.

(v) *and* (vi) *as above, except that, for one value of* r *we may have the equality* $\phi_0 + 2a_r = 2$; *but for no value of* r *may we have* $\phi_0 + 2a_r = \nu + 2$.

The theorem contains, as special cases, the Fourier-Bessel theorem and other integral transforms which will be examined in § 5.

1 . 4. *Notation.* The functions $\Psi_r(x)$ $(r = 1, 2, 3, \ldots, p+1)$ will be called the *integral conjugates* of $\Phi_0(x)$.

We next define the function $\Psi_0(x)$ and the set of functions $\Phi_r(x)$ $(r = 1, 2, 3, \ldots, p+1)$. If $\psi_0 = -1-\phi_0$, then

$$\Psi_0(x) = (\tfrac{1}{2}x)^{-\psi_0-\frac{1}{2}} \times {}_pf_{p+1}(2-a_1, 2-a_2, \ldots, 2-a_p; 2-\rho_1, 2-\rho_2, \ldots, 2-\rho_{p+1}; -\tfrac{1}{4}x^2).$$

It should be noticed that, if we introduce the notation

$$\rho_0 = 1 = {}_0\rho_0,$$
$${}_0a_s = 1-a_s+\rho_0 = 2-a_s \quad (s = 1, 2, 3, \ldots, p),$$
$${}_0\rho_s = 1-\rho_s+\rho_0 = 2-\rho_s \quad (s = 1, 2, 3, \ldots, p+1),$$

then this definition is analogous to the definition of the other functions $\Psi_r(x)$, for, obviously,

$$\Psi_0(x) = (\tfrac{1}{2}x)^{-\psi_0-\frac{1}{2}} {}_pf_{p+1}({}_0a_1, {}_0a_2, \ldots, {}_0a_p; {}_0\rho_1, {}_0\rho_2, \ldots, {}_0\rho_{p+1}; -\tfrac{1}{4}x^2).$$

Next let

$$\phi_r = \sum_{s=1}^{p}(2-{}_ra_s) - \sum_{s=1}^{p+1}(2-{}_r\rho_s)+\tfrac{1}{2} = -2+2\rho_r+\phi_0 \quad (r = 1, 2, 3, \ldots, p+1),$$

and write

$$\Phi_r(x) = (\tfrac{1}{2}x)^{-\phi_r-\frac{1}{2}} \times {}_pf_{p+1}(2-{}_ra_1, 2-{}_ra_2, \ldots, 2-{}_ra_p; 2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2)$$
$$(r = 1, 2, 3, \ldots, p+1).$$

The definition of $\Phi_0(x)$ is analogous to this definition, since $\rho_0 = 1$, $a_s = 2-{}_0a_s$, and $\rho_s = 2-{}_0\rho_s$.

The functions $\Phi_r(x)$ $(r = 1, 2, 3, \ldots, p+1)$ will be called the *integral associates* of $\Phi_0(x)$.

From the equations of § 1 . 2, it is easily seen that the integral conjugates of the function $\Phi_r(x)$ are the set of functions $\Psi_s(x)$ $(s = 0, 1, 2, \ldots, p+1\,;\, s \neq r)$, from which $\Psi_r(x)$ is omitted.

Conversely, the integral conjugates of the function $\Psi_r(x)$ are the set of functions $\Phi_s(x)$ $(s = 0, 1, 2, \ldots, p+1\,;\, s \neq r)$, from which $\Phi_r(x)$ is omitted.

Hence to every $\Phi_r(x)$ corresponds a function $\sum_{s=0}^{p+1} B_s \Psi_s(x)$ $(B_r = 0)$; and it is easy to deduce from (1 . 22) and the other equations of § 1 . 2 that the B_s are the same set of numbers for all the values of r, though they do not necessarily occur in the same order for different values of r.

Again, to every $\Psi_t(x)$ corresponds in the same way a function $\sum_{s=0}^{p+1} A_s \Phi_s(x)$ $(A_t = 0)$; and it is easy to deduce from (1 . 2) that the A_s are the same set of numbers as the B_s, though they do not necessarily occur in the same order. They do occur in the same order when $t = r$.

As illustrations, take

$$\text{(i) } \Phi_0(x) = J_\nu(x), \quad \text{and} \quad \text{(ii) } \Phi_0(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_\mu(\tfrac{1}{2}x) J_\nu(\tfrac{1}{2}x).$$

In the first case the integral associates and conjugates are

$$\Phi_0(x) = J_\nu(x), \qquad \Psi_0(x) = J_{-\nu}(x),$$
$$\Phi_1(x) = J_{-\nu}(x), \quad \Psi_1(x) = J_\nu(x).$$

To discuss the second illustration, we need the theorem* that

$$\pi^{\frac{1}{2}} J_\mu(x) J_\nu(x)$$
$$= x^{\mu+\nu}\, {}_2f_3[\tfrac{1}{2}(\mu+\nu+1),\ \tfrac{1}{2}(\mu+\nu+2);\ \mu+\nu+1,\ \mu+1,\ \nu+1;\ -x^2],$$

whence we obtain the following results :—

$$\Phi_0(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_\mu(\tfrac{1}{2}x) J_\nu(\tfrac{1}{2}x), \qquad \Psi_0(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \frac{d}{dx}\left\{\tfrac{1}{2}x J_{-\mu}(\tfrac{1}{2}x) J_{-\nu}(\tfrac{1}{2}x)\right\},$$

$$\Phi_1(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{-\mu}(\tfrac{1}{2}x) J_{-\nu}(\tfrac{1}{2}x), \quad \Psi_1(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \frac{d}{dx}\left\{\tfrac{1}{2}x J_\mu(\tfrac{1}{2}x) J_\nu(\tfrac{1}{2}x)\right\},$$

$$\Phi_2(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{-\mu}(\tfrac{1}{2}x) J_\nu(\tfrac{1}{2}x), \qquad \Psi_2(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \frac{d}{dx}\left\{\tfrac{1}{2}x J_\mu(\tfrac{1}{2}x) J_{-\nu}(\tfrac{1}{2}x)\right\},$$

$$\Phi_3(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_\mu(\tfrac{1}{2}x) J_{-\nu}(\tfrac{1}{2}x), \qquad \Psi_3(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \frac{d}{dx}\left\{\tfrac{1}{2}x J_{-\mu}(\tfrac{1}{2}x) J_\nu(\tfrac{1}{2}x)\right\}.$$

Returning to the general case, we observe that $\Phi_0(x)$ and all its integral associates are $p+2$ independent solutions of the differential

* Cf. G. N. Watson, *Theory of Bessel functions* (1922), 147, Equation 5.

equation

$$(1.41)\quad \Big[x(\vartheta+2\alpha_1)(\vartheta+2\alpha_2)\dots(\vartheta+2\alpha_p)+\frac{d}{dx}(\vartheta+2\rho_1-2)(\vartheta+2\rho_2-2)\dots \dots(\vartheta+2\rho_{p+1}-2\Big](x^{\frac{1}{2}+\phi_0}y)=0,$$

where, as usual, ϑ denotes $x(d/dx)$; also the $p+2$ functions $\Psi_r(x)$ $(r=0, 1, 2, \dots, p+1)$ satisfy the differential equation which is obtained from (1.41) by replacing α_r, ρ_r, and ϕ_0 by $2-\alpha_r$, $2-\rho_r$, and ψ_0.

The next theorem is the generalization of Theorem 1 which arises when $\Phi_0(x)$ is replaced by a linear combination of the functions $\Phi_r(x)$ $(r=0, 1, 2, \dots, p+1)$, so that a special solution of (1.41) is replaced by a general solution. Before carrying out this generalization, it is necessary to define additional functions of the types Φ and Ψ in the following manner.

If $\alpha_{p+1}=\rho_{p+2}$ we may evidently write

$$(1.42)\quad \Phi_0(x)=(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}} \times {}_{p+1}f_{p+2}(\alpha_1, \alpha_2, \dots, \alpha_p, \alpha_{p+1}; \rho_1, \rho_2, \dots, \rho_{p+1}, \rho_{p+2}; -\tfrac{1}{4}x^2).$$

Now let ${}_{p+2}\alpha_s=1-\alpha_s+\rho_{p+2}$ $(s=1, 2, 3, \dots, p+1)$, so that, in particular, ${}_{p+2}\alpha_{p+1}=1=2-{}_{p+2}\alpha_{p+1}$, and also let

$${}_{p+2}\rho_s=1-\rho_s+\rho_{p+2} \quad (s=1, 2, 3, \dots, p+1;\ s\neq p+2),$$

$${}_{p+2}\rho_{p+2}=\rho_{p+2}.$$

We now define $\Psi_{p+2}(x)$ after the manner of § 1.2 and $\Phi_{p+2}(x)$ after the manner of the definition of $\Phi_r(x)$ $(r\leqslant p+1)$ given in the present section.

We have consequently constructed two functions:

$$\Psi_{p+2}(x)=(\tfrac{1}{2}x)^{-\psi_{p+2}-\frac{1}{2}} \times {}_{p+1}f_{p+2}({}_{p+2}\alpha_1, {}_{p+2}\alpha_2, \dots, {}_{p+2}\alpha_{p+1}; {}_{p+2}\rho_1, {}_{p+2}\rho_2, \dots, {}_{p+2}\rho_{p+2}; -\tfrac{1}{4}x^2),$$

$$\Phi_{p+2}(x)=(\tfrac{1}{2}x)^{-\phi_{p+2}-\frac{1}{2}} \times {}_{p+1}f_{p+2}(2-{}_{p+2}\alpha_1, 2-{}_{p+2}\alpha_2, \dots, 2-{}_{p+2}\alpha_{p+1}; 2-{}_{p+2}\rho_1, 2-{}_{p+2}\rho_2, \dots, 2-{}_{p+2}\rho_{p+2}; -\tfrac{1}{4}x^2),$$

which possess the properties that the integral conjugates of $\Phi_{p+2}(x)$ are the group of functions $\Psi_s(x)$ $(s=0, 1, 2, \dots, p+1)$, and the integral conjugates of $\Psi_{p+2}(x)$ consist of the group $\Phi_s(x)$ $(s=0, 1, 2, \dots, p+1)$.

For example, when $\Phi_0(x)=J_\nu(x)$ (the case in which $p=0$), we take $\alpha_1=\rho_2$ $(=\sigma$, say), and we then get

$$\Phi_2(x)=(\tfrac{1}{2}x)^{2+\nu-\sigma}{}_1f_2(1; 2+\nu-\sigma, 2-\sigma; -\tfrac{1}{4}x^2),$$

$$\Psi_2(x)=(\tfrac{1}{2}x)^{2-\nu+2\sigma}{}_1f_2(1; \sigma-\nu, \sigma; -\tfrac{1}{4}x^2).$$

These functions occur in the integral transform* of

$$f(y) = \int_0^\infty F(x)\{A_1 J_\nu(xy) + A_2 J_{-\nu}(xy)\}\, x\, dx.$$

Returning to the general case, we extend the process just explained by taking

$$a_{p+r} = \rho_{p+r+1},$$

where r is any positive integer. We then replace ρ_{p+2} in (1 . 42) by ρ_{p+3}, ρ_{p+4}, ..., ρ_{p+r} in turn, and so we define the functions

$$\Phi_{p+3}(x), \Phi_{p+4}(x), \ldots, \Phi_{p+r}(x) \quad \text{and} \quad \Psi_{p+3}(x), \Psi_{p+4}(x), \ldots, \Psi_{p+r}(x);$$

these functions possess properties analogous to those of

$$\Phi_{p+2}(x) \quad \text{and} \quad \Psi_{p+2}(x).$$

We are now in a position to describe the second theorem.

1 . 51. *Introduction to Theorem* 2.

It is now required to find an integral transform of

$$f(y) = \int_0^\infty F(x)\Big[\sum_{r=0}^{\lambda} A_r \Phi_r(xy)\Big]\, x\, dx, \tag{1 . 51}$$

where some of the A_r may be zero. If $\lambda > p+1$, the functions $\Phi_r(xy)$ $(r = p+2,\ p+3,\ \ldots,\ \lambda)$ are to be constructed in the manner explained in § 1 . 4.

There is no loss of generality in assuming that $\lambda \geqslant p+1$, for the case $\lambda < p+1$ is equivalent to the case in which there are $p+1$ functions with $A_r (r = \lambda+1,\ \lambda+2,\ \ldots,\ p+1)$ equal to zero.

If we suppose that the number of the A_r which are *not* zero is μ, then the equation in a,

$$\sum_{s=0}^{\lambda} \frac{A_s}{\sin(\rho_s - a)\pi} = 0, \tag{1 . 52}$$

has its roots of the form†

$$a = m + \beta_r \quad (r = 1,\ 2,\ 3,\ \ldots,\ \mu-1;\ m = \ldots,\ -3,\ -2,\ 1,\ 0,\ 1,\ 2,\ 3,\ \ldots).$$

We have to assume that no two of the β_r are equal or differ by an integer. When this assumption is incorrect, the methods of this paper fail to give an integral transform of (1 . 52) [cf. condition (v) below].

* This transform was discovered by G. H. Hardy, *Proc. London Math. Soc.* (2), 23 (1925), lxi (*Records* for June 12, 1924). See § 5 below.

† The equation is easily reduced to an equation of degree $\mu-1$ in $e^{2\pi i a}$.

Now, corresponding to each root β_r, choose a value* of m (say m_r) such that

$$\nu < \phi_0 + 2m_r + 2\beta_r < \nu + 2,$$

where ν is the number specified in condition (vi) of Theorem 1. This choice of the m_r, when possible, is unique.

Let $$\rho_{\lambda+r} = \alpha_{\lambda+r-1} = m_r + \beta_r \quad (r = 1, 2, 3, \ldots, \mu-1),$$

and, writing $\alpha_{\lambda+r-1}$ for α_{p+1} in (1.42), define the set of functions $\Psi_{\lambda+r}(x)$ $(r = 1, 2, 3, \ldots, \mu-1)$ in the way in which $\Psi_{p+2}(x)$ is defined in § 1.4.

Next solve the $\lambda+\mu-1$ equations †,

$$A_r B_r = 0 \quad (r = 0, 1, 2, \ldots, \lambda), \tag{1.53}$$

$$\sum_{r=0}^{\lambda+\mu-1} \frac{B_r}{\sin(\rho_r - \alpha_n)\pi} = 0 \quad (n = 1, 2, 3, \ldots, \lambda-1), \tag{1.54}$$

for the ratios of the $\lambda+\mu$ numbers $B_0, B_1, B_2, \ldots, B_{\lambda+\mu-1}$.

Before enunciating Theorem 2, we observe that, since $\alpha_{\lambda+r-1}$ $(r = 1, 2, 3, \ldots, \mu-1)$ satisfies (1.52), we may write (1.52) in the form

$$\sum_{s=0}^{\lambda} \frac{A_s}{\sin(\rho_s - \alpha_n)\pi} = 0 \quad (n = \lambda, \lambda+1, \ldots, \lambda+\mu-2). \tag{1.55}$$

We thereby make the equations satisfied by the A_s and the B_s assume a symmetrical form.

1.52. THEOREM 2. *If*

$$f(y) = \int_0^\infty F(x)\left[\sum_{r=0}^{\lambda} A_r \Phi_r(xy)\right] x\,dx,$$

then

$$-\tfrac{1}{2}\{F(z+0)+F(z-0)\}\sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1} A_r B_s \sin(\phi_0+\rho_r+\rho_s)\pi = \int_0^\infty f(y)\left[\sum_{s=0}^{\lambda+\mu-1} B_s \Psi_s(yz)\right] y\,dy,$$

where the constants B_s and the functions $\Psi_s(z)$ for $s > p+1$ are defined as in § 1.51, provided that

(i) $\int_0^\infty |F(x)|\sqrt{x}\,dx$ *exists.*

(ii) $F(x)$ *is of bounded variation near* $x = z$.

* If $\phi_0 + 2\alpha_r \neq \nu + 2$ for any of the values $1, 2, 3, \ldots, p$ of r, then we may have $\phi_0 + 4m_r + 2\beta_r = \nu + 2$ for *one* value of r.

† Since μ only of the A_r do not vanish, (1.53) merely prescribes the vanishing of μ of the B_r, and so it is equivalent to μ equations.

(iii) $\phi_r \leqslant 0$ *for all* r *for which* $A_r \neq 0$.

(iv) $2\rho_r > -\phi_0$ $(r = 1, 2, 3, \ldots, p+1, p+2, \ldots, \lambda+\mu-2)$.

(v) *No two of the* $\alpha_r (r = 1, 2, 3, \ldots, p+1, p+2, \ldots, \lambda+\mu-2)$ *are equal.*

(vi) *There is a number* ν *such that* $0 \leqslant \nu \leqslant 1$ *and* $\nu < \phi_0+2\alpha_r < \nu+2$ $(r = 1, 2, 3, \ldots, \lambda+\mu-2)$; *except that, for one value of* r, *we may have the equality* $\phi_0+2\alpha_r = \nu+2$.

In certain circumstances these restrictions can be lightened; for example, if, for a certain value of r, the coefficient of $x^{-\phi_0-\frac{1}{2}-2\alpha_r}$ vanishes in the asymptotic expansion* of $\sum_{r=0}^{\lambda} A_r \Phi_r(x)$, and if the corresponding α_r does not differ from any of the other α_s by an integer, then, for this particular value of r, the condition $\phi_0+\alpha_r > \nu$ may be replaced by the condition $\phi_0+2\alpha_r > \nu-2$.

Again, if the B_s are such that, for a certain value of r, the coefficient of $x^{\phi_0-\frac{1}{2}+2\alpha_r}$ vanishes in the asymptotic expansion of $\sum_{s=0}^{\lambda+\mu-1} B_s \Psi_s(x)$, then, for this particular value of r, the condition $\phi_0+2\alpha_r < \nu+2$ may replaced by the condition $\phi_0+2\alpha_r < \nu+4$.

1.6. *Further notation, and principles of the proofs.*

For brevity we shall write

$$\chi_0(x) = \sum_{r=1}^{p+1} B_r \Psi_r(x), \tag{1.61}$$

so that Theorem 1 assumes the form :—

If
$$f(y) = \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx,$$

then $\frac{1}{2}\{F(z+0)+F(z-0)\} \sum_{r=1}^{p+1} B_r \sin(\phi_0+\rho_r)\pi = \int_0^\infty f(y)\, \chi_0(yz)\, y\, dy.$

The principles of the proof are briefly as follows :—

Condition (v) is necessary to ensure that the functions $\Phi_0(x)$ and $\Psi_r(x)$ have asymptotic expansions of the forms given in § 2.1. Conditions (ii) and (iv), together with the equations (1.22), are then used to prove

* This asymptotic expansion can be deduced from results given in § 2.1.

that

$$(1.62)\quad \int_0^\infty \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx = \lim_{T\to\infty} \int_0^\infty F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy.$$

With the help of the asymptotic expansions, combined with condition (vi) and the equations (1.22), it is then possible, though somewhat troublesome, to establish theorems of the type

$$\int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy = K\, \frac{\sin T(x-z)}{x-z} + P(x, z) \cos Tx + Q(x, z) \sin Tx + MT^\lambda,$$

where M and K are independent of T and are bounded for all pairs of values of x and z, and $\lambda < 0$. An application of the Riemann-Lebesgue theorem then reduces the right-hand side of (1.62) to

$$\lim_{T\to\infty} K \int_0^\infty F(x)\, \frac{\sin T(x-z)}{x-z}\, dx,$$

and the required result follows immediately.

I shall prove Theorem 1 only when $\phi_0+2a_r < \nu+2$ $(r = 1, 2, 3, \ldots, p)$; the modifications necessary to deal with the cases in which one of the a_r is such that $\phi_0+2a_r = \nu+2$ will be indicated in foot-notes.

The proof of Theorem 1 is contained in §§ 2 and 3, and an outline of the proof of Theorem 2 follows in § 4; the details of the proof of Theorem 2 will be given in full only when the necessary arguments are appreciably different from those of Theorem 1.

In § 5 I shall deal with special cases and, in particular, I shall give some new integral transforms involving products of Bessel functions.

2. *Lemmas preliminary to the proof of Theorem* 1.

2.1. LEMMA 1. *The asymptotic expansions of* $\Phi_0(x)$ *and* $\Psi_r(x)$ $(r = 1, 2, 3, \ldots, p+1)$ *when* x *tends to infinity.*

The asymptotic expansion of $\Phi_0(x)$ *is given by the formula*

$$(2.101)\quad \Phi_0(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left[\cos(x+\tfrac{1}{2}\phi_0\pi)\left\{1+\sum_{s=1}^{M} C_s x^{-2s}\right\} + \frac{H}{x}\sin(x+\tfrac{1}{2}\phi_0\pi)\left\{1+\sum_{s=1}^{N} D_s x^{-2s}\right\}\right] - (\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}} \sum_{r=1}^{p} K_r x^{-2a_r}\left\{1+\sum_{s=1}^{L_r} E_s^{(r)} x^{-2s}\right\} + x^{\frac{1}{2}-m} R_{mr}$$

where

(i) H, C_s, D_s, K_r, *and* $E_s^{(r)}$ *are constants independent of* x.

(ii) m *is any positive integer greater than* ϕ_0-1.

(iii) $|R_m| < \mathbf{B}x^{-\gamma}$, *where* $\mathbf{B}$ *and* γ *are positive and are independent of* x, *and also* $\gamma<1$; *the values of* $\mathbf{B}$ *and* γ *are given by equations* (2 . 122) *and* (2 . 121).

(iv) M, N, *and* L_r $(r=1, 2, 3, \ldots, p)$ *are the greatest integers such that*

$$2M<m, \quad 2N+1<m, \quad \phi_0+2a_r+2L_r<m-1+\gamma.$$

(v) *The value of* H *is given by the formula*

$$H=\tfrac{1}{2}\{(1-\phi_0)^2-(1-\phi_0)+\tfrac{1}{6}\} + \sum_{s=1}^{p}(a_s^2-a_s+\tfrac{1}{6})-\sum_{s=1}^{p+1}(\rho_s^2-\rho_s+\tfrac{1}{6})-\tfrac{1}{6}. \tag{2 . 102}$$

(vi) *If no two of the* a_r *are equal*, then*

$$K_r=2^{2a_r}\Gamma(a_r)\frac{\prod_{\sigma=1}^{p}{}' \Gamma(a_\sigma-a_r)}{\prod_{\sigma=1}^{p+1}\Gamma(\rho_\sigma-a_r)} \quad (r=1, 2, 3, \ldots, p), \tag{2 . 103}$$

the dash indicating that the factor for which $\sigma=r$ *is to be omitted from the product.*

Before establishing (2 . 101), we observe that the corresponding asymptotic expansion of $\Psi_r(x)$ is given by the formula

$$\Psi_r(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\cos(x+\tfrac{1}{2}\psi_r\pi)\left\{1+\sum_{s=1}^{M}C_{r,s}x^{-2s}\right\} + \frac{H_r}{x}\sin(x+\tfrac{1}{2}\psi_r\pi)\left\{1+\sum_{s=1}^{N}D_{r,s}x^{-2s}\right\}\right] + (\tfrac{1}{2}x)^{-\psi_r-\frac{1}{2}}\sum_{s=1}^{p}K_{r,s}x^{-2\,{}_r a_s}\left\{1+\sum_{t=1}^{L_s}E_{r,t}^{(s)}x^{-2t}\right\}+x^{\frac{1}{2}-m}R_{r,m},$$

where the various constants and $R_{r,m}$ satisfy conditions resembling the conditions given above.

The values of H_r and $K_{r,s}$ are, in fact, obtained by substituting ψ_r, ${}_r a_s$ and ${}_r\rho_s$ for ϕ_0, a_s and ρ_s respectively in (2 . 102) and (2 . 103). By using the definitions of ψ_r, ${}_r a_s$, and ${}_r\rho_s$ given in § 1 . 2, we can express

* When two of the a_r are equal or differ by an integer, the constants K_r must, in general, be replaced by expressions involving powers of $\log x$; it is unnecessary to specify here that no two of the a_r may differ by an integer, because the possibility of that is already precluded by condition (vi) in the enunciation of Theorem 1, namely $\nu<\phi+2a_r<\nu+2$.

H_r and $K_{r,s}$ in terms of ϕ_0, α_s, and ρ_s; and we then find that

$$H_r = H \quad (r = 1, 2, 3, \ldots, p+1). \tag{2.104}$$

We now establish (2.101) by considering the integral

$$\frac{1}{2\pi i}\int_C \{G(w)-H(w)\}\, x^{2w}\, dw,$$

where

$$G(w) = 2^{-2w}\Gamma(-w)\,\frac{\Gamma(\alpha_1+w)\,\Gamma(\alpha_2+w)\ldots\Gamma(\alpha_p+w)}{\Gamma(\rho_1+w)\,\Gamma(\rho_2+w)\ldots\Gamma(\rho_{p+1}+w)},$$

$$H(w) = \cos(w-\phi_0)\pi \sum_{r=0}^{m} \nu_r \Gamma(-2w+\phi_0-r),$$

x is positive, m is an arbitrary positive integer, and the contour C consists of the arc of the circle $|w| = \lambda$ which lies on the right of the straight line $R(w) = -a$ $(0 < a < \lambda)$, together with the segment of this straight line whose ends are on the circle. The contour is to be so chosen that none of the poles of $G(w)$ and $H(w)$ lie on it, and, when λ is made to tend to infinity, it is supposed that λ passes through such a sequence of values that the lower bound of the distance of C from the poles of $G(w)$ and $H(w)$ is positive (not zero).

Suitable values for the coefficients ν_r will be determined presently, and certain inequalities satisfied by a will also be obtained.

2.11. LEMMA 1.1. *The coefficients ν_r can be chosen so that the integral along the circular arc of C tends to zero as $\lambda \to \infty$, and the integral along the line $R(w) = -a$ is absolutely convergent.*

At all points of C, with the exception of a finite segment of the line $R(w) = -a$, we have

$$|\arg w| < \pi-\delta, \quad |\arg(\alpha_r+w)| < \pi-\delta \quad (\alpha = 1, 2, 3, \ldots, p),$$

where δ is an arbitrary (small) positive number. Hence, when $|w|$ is large and w is on the contour,

$$\log\Gamma(\alpha_r+w) = (\alpha_r+w-\tfrac{1}{2})\log w - w + \tfrac{1}{2}\log(2\pi) + \varpi(w)$$

where*

$$\varpi(w) = \frac{b_1}{w} + \frac{b_2}{w^2} + \ldots + \frac{b_j}{w^j} + O\left(\frac{1}{w^{j+\frac{1}{2}}}\right), \tag{2.111}$$

the coefficients b_j being independent of w. In particular,

$$b_1 = \tfrac{1}{2}(\alpha_r^2 - \alpha_r + \tfrac{1}{6}).$$

* Cf. Whittaker and Watson, *Modern analysis* (1920), 278.

In $G(w)$ replace $\Gamma(-w)$ by

$$-\frac{\pi}{\sin w\pi\,\Gamma(w+1)},$$

and it is then easy to see that, when $|w|$ is large,

$$G(w) = -\frac{\pi}{\sin w\pi}\,\frac{w^{-2w+\phi_0-\frac{1}{2}}}{2\pi\,.\,2^{2w}}\,e^{2w+\chi(w)},$$

where $$\chi(w) = \frac{c_1}{w}+\frac{c_2}{w^2}+\ldots+\frac{c_j}{w^j}+O\left(\frac{1}{w^{j+\frac{1}{2}}}\right),$$

and

$$c_1 = \tfrac{1}{2}\left[\sum_{s=1}^{p}(\alpha_s^2-\alpha_s+\tfrac{1}{6})-\sum_{s=1}^{p+1}(\rho_s^2-\rho_s+\tfrac{1}{6})-\tfrac{1}{6}\right]. \tag{2.112}$$

Similarly

$$H(w) = -\frac{\pi\cos(w-\phi_0)\pi}{\sin(2w-\phi_0)\pi}\,\frac{w^{-2w+\phi_0-\frac{1}{2}}}{\sqrt{(2\pi)}\,.\,2^{2w}}\,e^{2w}\sum_{r=0}^{m}(-1)^r\,\frac{\nu_r\,e^{\varpi_r(w)}}{2^{r+\frac{1}{2}-\phi_0}\,w^r},$$

where the functions $\varpi_r(w)$ are of the same type as $\varpi(w)$.

Hence, on reduction,

$$G(w)-H(w) = -\frac{\pi}{\sin w\pi}\,\frac{w^{-2w+\phi_0-\frac{1}{2}}}{2^{2w}}\,e^{2s}$$
$$\times\left[\frac{e^{\chi(w)}}{2\pi}-\frac{1}{2\sqrt{(2\pi)}}\sum_{r=0}^{m}(-1)^r\,\frac{\nu_r\,e^{\varpi_r(w)}}{2^{r+\frac{1}{2}-\phi_0}\,w^r}\right.$$
$$\left.-\frac{1}{2\sqrt{(2\pi)}}\,\frac{\sin\phi_0\pi}{\sin(2w-\phi_0)\pi}\sum_{r=0}^{m}(-1)^r\,\frac{\nu_r\,e^{\varpi_r(w)}}{2^{r+\frac{1}{2}-\phi_0}\,w^r}\right].$$

Now, from (2.111), we may write

$$e^{\varpi_r(w)} = 1+\frac{d_{r,1}}{w}+\frac{d_{r,2}}{w^2}+\ldots+\frac{d_{r,m}}{w^m}+O\left(\frac{1}{w^{m+\frac{1}{2}}}\right),$$

and hence the coefficients ν_r can be successively chosen so that

$$\frac{e^{\chi(w)}}{2\pi}-\frac{1}{2\sqrt{(2\pi)}}\sum_{r=0}^{m}(-1)^r\,\frac{\nu_r\,e^{\varpi_r(w)}}{2^{r+\frac{1}{2}-\phi_0}\,w^r} = O\left(\frac{1}{w^{m+\frac{1}{2}}}\right). \tag{2.113}$$

We shall, for the future, assume that the coefficients ν_r are chosen in this manner, and, in particular, that

$$\nu_0 = 2^{1-\phi_0}\pi^{-\frac{1}{2}}, \tag{2.114}$$

$$\nu_1 = -2^{1-\phi_0}\pi^{-\frac{1}{2}}\left[\tfrac{1}{2}\{(1-\phi_0)^2-(1-\phi_0)+\tfrac{1}{6}\}\right.$$
$$\left.+\sum_{s=1}^{p}(\alpha_s^2-\alpha_s+\tfrac{1}{6})-\sum_{s=1}^{p+1}(\rho_s^2-\rho_s+\tfrac{1}{6})-\tfrac{1}{6}\right]. \tag{2.115}$$

Again, at all points of C for which $|w|$ is large, we have

$$-\frac{1}{2\sqrt{2\pi}}\frac{\sin\phi_0\pi}{\sin(2w-\phi_0)\pi}\sum_{r=0}^{m}(-1)^r\frac{\nu_r e^{\varpi_r(w)}}{2^{r+\frac{1}{2}-\phi_0}w^r}=O(1),$$

and, finally, if $w=\lambda e^{i\theta}$, we have

$$\left|\frac{w^{-2w}e^{2w}}{2^{2w}\sin w\pi}\right|\leqslant H[-2\lambda(\log\lambda-1+\log 2-\log x)\cos\theta+2\lambda(\theta\mp\tfrac{1}{2}\pi)\sin\theta],$$

since x is positive, H being a constant (independent of $\arg w$), and the upper or lower sign being taken according as θ is positive or negative.

From these results we deduce that

$$|\{G(w)-H(w)\}x^{2w}|\leqslant\{H_1\lambda^{-m-1+\phi_0}+H_2\lambda^{\phi_0-\frac{1}{2}}e^{\mp 2\pi\lambda\sin\theta}\}$$
$$\times\exp[-2\lambda(\log\lambda-1+\log 2-\log x)\cos\theta+2\lambda(\theta\mp\tfrac{1}{2}\pi)\sin\theta],$$

where H_1 and H_2 are constants, and the ambiguous sign is determined as above.

Since $m+1-\phi_0>0$, it follows from Jordan's lemma* that

$$\int\{G(w)-H(w)\}x^{2w}\,dw\to 0$$

as $\lambda\to\infty$, when the path of integration is the part of C which lies on the right of the imaginary axis ($-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi$).

Consider next the two arcs of C between the imaginary axis and $R(w)=-a$. For points $\lambda e^{i\theta}$ on these arcs we have

$$-a\leqslant\lambda\cos\theta\leqslant 0,$$

and, for points on the upper arc ($\theta\geqslant\frac{1}{2}\pi$),

$$\lambda(\theta-\tfrac{1}{2}\pi)\sin\theta<\lambda\tan(\theta-\tfrac{1}{2}\pi)\sin\theta=-\lambda\cos\theta\leqslant a,$$

with a similar result for the lower arc ($\theta\leqslant-\frac{1}{2}\pi$).

Hence, on both arcs,

$$\exp[-2\lambda(\log\lambda-1+\log 2-\log x)\cos\theta+2\lambda(\theta\mp\tfrac{1}{2}\pi)\sin\theta]$$
$$\leqslant H_3\exp(2a\log\lambda)$$
$$=H_3\lambda^{2a},$$

where H_3 is independent of λ and θ. Consequently, at all points of

* Cf. Whittaker and Watson, *loc. cit.*, 115–116.

these arcs of C we have

$$|\{G(w)-H(w)\}\, x^{2w}| \leqslant H_4 \lambda^{-m-1+\phi_0-2a} + H_5 \lambda^{\phi_0-\frac{1}{2}+2a}\, e^{\mp 2\pi\lambda \sin\theta},$$

(where H_4 and H_5 are independent of w) with the same convention concerning the ambiguous sign.

We now choose a so as to satisfy the conditions already imposed on the contour C, with the additional condition

$$-m+\phi_0+2a < 0. \tag{2.116}$$

It then follows (i) that

$$\int \{G(w)-H(w)\}\, x^{2w}\, dw \to 0$$

as $\lambda \to \infty$, when the path of integration is either of the circular arcs of C on the left of the imaginary axis, and (ii) that

$$\int_{-a-\infty i}^{-a+\infty i} \{G(w)-H(w)\}\, x^{2w}\, dw$$

is absolutely convergent when the path of integration is the straight line $R(w) = -a$.

This completes the proof of Lemma 1.1.

2.12. From Cauchy's theorem it follows that the sum of the residues of $\{G(w)-H(w)\}x^{2w}$ at its poles on the right of the line $R(w) = -a$ is equal to

$$-\frac{1}{2\pi i}\int_{-a-\infty i}^{a+\infty i} \{G(w)-H(w)\}\, x^{2w}\, dw\,;$$

the absolute value of this does not exceed

$$\frac{1}{2\pi x^{2a}}\int_{-\infty}^{\infty} |G(-a+it)-H(-a+it)|\, dt,$$

and, as has been shown in § 2.11, the last integral is absolutely convergent.

Now we can choose a, consistently with (2.116), so that

$$2a > m-\phi_0-1,$$

and hence, if we define γ and $\mathbf{B}$ by the equations*

$$\gamma = 2a-m+\phi_0+1 > 0, \tag{2.121}$$

$$\mathbf{B} = \frac{2^{\phi_0+\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} |G(-a+it)-H(-a+it)|\, dt, \tag{2.122}$$

* It also follows from (2.116) that $\gamma < 1$.

then the sum of the residues of $\{G(w)-H(w)\}x^{2w}$ at its poles on the right of the line $R(w) = -a$ is equal to

$$2^{-\phi_0-\frac{1}{2}}x^{-m+\phi_0+1}R_m,$$

where $|R_m| \leqslant \mathbf{B}x^{-\gamma}$, as stated in condition (iii) following equation (2 . 101).

The sum of the residues of $G(w)x^{2w}$ at the poles of $\Gamma(-w)$ is

$${}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2).$$

When no two of the a_r are equal or differ by an integer, the sum of the residues of $G(w)$ at those poles of $\prod_{r=1}^{p}\Gamma(a_r+w)$ which are on the right of the line $R(w) = -a$ is

$$\sum_{r=1}^{p} K_r x^{-2a_r}\left(1+\sum_{s=1}^{L_r} E_s^{(r)}x^{-2s}\right),$$

where L_r is the greatest integer such that $a_r+L_r < a$, *i.e.* such that

$$\phi_0+2a_r+2L_r < m-1+\gamma.$$

The r-th term of the sum by which $H(w)x^{2w}$ is defined is

$$\nu_r \cos(w-\phi_0)\pi\,\Gamma(-2w+\phi_0-r)\,x^{2w};$$

all the poles of this function are on the right of the line $R(w) = -a$, and the sum of the residues at them is

$$\tfrac{1}{2}\nu_r x^{\phi_0-r}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\cos(\tfrac{1}{2}n-\phi_0-r)\pi = \tfrac{1}{2}\nu_r x^{\phi_0-r}\cos(x+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}r\pi).$$

Combining these results, we get

$$\begin{aligned}(2.123)\quad &{}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2)\\ &+\sum_{r=1}^{p}K_r x^{-2a_r}\left(1+\sum_{s=1}^{L_r}E_s^{(r)}x^{-2s}\right)-\tfrac{1}{2}x^{\phi_0}\sum_{r=0}^{m}\nu_r x^{-r}\cos(x+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}r\pi)\\ &= 2^{-\phi_0-\frac{1}{2}}x^{-m+\phi_0+1}R_m,\end{aligned}$$

and on multiplying this through by $(\frac{1}{2}x)^{-\phi_0-\frac{1}{2}}$ we at once obtain (2 . 101).

2 . 2. LEMMA 2. *With the notation explained in* §§ 1 . 2, 1 . 4, *and* 1 . 6

$$(2.21)\quad \int_0^T \chi_0(yz)\,y\,dy\int_0^\infty F(x)\,\Phi_0(xy)\,x\,dx = \int_0^\infty F(x)\,x\,dx\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy,$$

when the conditions enunciated in Theorem 1 are satisfied. Also

$$(2.22)\quad \int_0^\infty \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx = \lim_{T\to\infty} \int_0^\infty F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy,$$

provided that the limit on the right exists.

From (2.101) it follows that $(xy)^{\frac{1}{2}}\Phi_0(xy)$ is a bounded function of x and y when $\phi_0 \leqslant 0$ and $\phi_0 + 2a_s > \nu \geqslant 0$ [cf. condition (vi)]. Hence, by condition (i),

$$\int_0^\infty F(x)\, x^{\frac{1}{2}} (xy)^{\frac{1}{2}} \Phi_0(xy)\, dx$$

converges uniformly with respect to y.

Now the inequality $\phi_0 + 2a_s \leqslant \nu + 2$ is equivalent to the inequality $\psi_r + 2\,{}_r a_s \geqslant 1 - \nu \geqslant 0$, as may be seen from the equations of § 1.2; and so, z being fixed, condition (vi) also ensures that $(yz)^{\frac{1}{2}}\chi_0(yz)$ is a bounded function of y when y lies in the interval (Δ, T), where $\Delta > 0$. Hence

$$(2.23)\quad \int_\Delta^T \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx = \int_0^\infty F(x)\, x\, dx \int_\Delta^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy.$$

If the upper bound of $|(xy)^{\frac{1}{2}}\Phi_0(xy)|$ is M, then

$$\left| \int_0^\Delta \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx \right| \leqslant M \int_0^\Delta |\chi_0(yz)\, y^{\frac{1}{2}}|\, dy \int_0^\infty |F(x)|\, x^{\frac{1}{2}}\, dx\,;$$

and since $\psi_r < 1$, because $2\rho_r \geqslant -\phi_0$ $(r = 1, 2, 3, \ldots, p+1)$ by condition (iv), the integral on the right is convergent, and so the integral on the left is convergent at its lower limit.

Now, given ϵ, we can choose δ, independent of T, so that

$$M \int_0^\delta |\chi_0(yz)\, y^{\frac{1}{2}}|\, dy \int_0^\infty |F(x)|\, x^{\frac{1}{2}} dx < \tfrac{1}{2}\epsilon,$$

and hence

$$\left| \int_0^\delta \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx \right| < \tfrac{1}{2}\epsilon,$$

while

$$\left| \int_0^\infty F(x)\, x\, dx \int_0^\delta \Phi_0(xy)\, \chi_0(yz)\, y\, dy \right| \leqslant M \int_0^\infty |F(x)\, x^{\frac{1}{2}}|\, dx \int_0^\delta |\chi_0(yz)|\, y^{\frac{1}{2}}\, dy < \tfrac{1}{2}\epsilon.$$

It follows from (2.23), by taking $\Delta = \delta$, that

$$\left| \int_0^T \chi_0(yz)\, y\, dy \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx - \int_0^\infty F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy \right| < \epsilon$$

for all values of T. Since ϵ is arbitrary, the expression on the left must vanish, and this establishes (2 . 21).

If we make $T\to\infty$ in (2 . 21), we obtain (2 . 22).

2 . 3. LEMMA 3. *With the notation of* § 1 *for hypergeometric functions,*

$$(\tfrac{1}{2}x)^{\phi_0+\frac{1}{2}}(\tfrac{1}{2}z)^{\psi_1+\frac{1}{2}}\left(1-\frac{x^2}{z^2}\right)\int\Phi_0(xy)\,\Psi_1(yz)\,y\,dy,$$

which is evidently equal to

$$\left(1-\frac{x^2}{z^2}\right)\int y^{2\rho_1-1}{}_pf_{p+1}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p;\ \rho_1,\ \rho_2,\ \ldots,\ \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times{}_pf_{p+1}({}_1\alpha_1,\ {}_1\alpha_2,\ \ldots,\ {}_1\alpha_p;\ {}_1\rho_1,\ {}_1\rho_2,\ \ldots,\ {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2)\,dy,$$

is expressible in the form $E_1+E_2+E_3+E_4$, *where*

$$E_1=\tfrac{1}{2}y^{2\rho_1}{}_pf_{p+1}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p;\ \rho_1,\ \rho_2,\ \ldots,\ \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times{}_pf_{p+1}({}_1\alpha_1,\ {}_1\alpha_2,\ \ldots,\ {}_1\alpha_p;\ \ {}_1\rho_1+1,\ {}_1\rho_2,\ \ldots,\ {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

$$E_2=-\tfrac{1}{8}x^2y^{2\rho_1+2}\sum_{r=0}^{p+1}(-1)^{r+1}{}_pf_{p+1}(\alpha_1+1,\ \alpha_2+1,\ \ldots,\ \alpha_p+1;$$
$$\rho_1+1,\ \rho_2,\ \rho_3,\ \ldots,\ \rho_{n-1},\ \rho_n+1,\ \rho_{n+1}+1,\ \ldots,\ \rho_{p+1}+1;\ -\tfrac{1}{4}x^2y^2)$$
$$\times{}_pf_{p+1}({}_1\alpha_1,\ {}_1\alpha_2,\ \ldots,\ {}_1\alpha_p;$$
$${}_1\rho_1+1,\ {}_1\rho_2+1,\ \ldots,\ {}_1\rho_n+1,\ {}_1\rho_{n+1},\ {}_1\rho_{n+2},\ \ldots,\ {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

$$E_3=\frac{(-1)^r}{2}\frac{x^2}{z^2}\,y^{2\rho_1}{}_pf_{p+1}(\alpha_1+1,\ \alpha_2+1,\ \ldots,\ \alpha_p+1;$$
$$\rho_1+1,\ \rho_2,\ \rho_3,\ \ldots,\ \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times{}_pf_{p+1}({}_1\alpha_1-1,\ {}_1\alpha_2-1,\ \ldots,\ {}_1\alpha_p-1;\ {}_1\rho_1,\ {}_1\rho_2,\ \ldots,\ {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

$$E_4=\frac{(-1)^p}{2}\frac{x^2}{z^2}\,y^{2\rho_1}\sum_{n=1}^{p}(-1)^{n-1}$$
$$\times{}_pf_{p+1}(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ \alpha_{n+1}+1,\ \alpha_{n+2}+1,\ \ldots,\ \alpha_p+1;$$
$$\rho_1,\ \rho_2,\ \ldots,\ \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times{}_pf_{p+1}({}_1\alpha_1,\ {}_1\alpha_2,\ \ldots,\ {}_1\alpha_{n-1},\ {}_1\alpha_n-1,\ {}_1\alpha_{n+1}-1,\ \ldots,\ {}_1\alpha_p-1;$$
$${}_1\rho_1,\ {}_1\rho_2,\ \ldots,\ {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2).$$

There also exist p *results similar to this, which are obtained by replacing* $(\frac{1}{2}z)^{\psi_1+\frac{1}{2}}\Psi_1(yz)$ *by* $(\frac{1}{2}z)^{\psi_r+\frac{1}{2}}\Psi_r(yz)$ $(r=2,\ 3,\ \ldots,\ p+1)$ *and making corresponding changes in* E_1, E_2, E_3, E_4.

It is easy to verify the following relations by direct differentiation :

$$(2.31)\quad \frac{d}{dy}\,{}_pf_{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$

$$= -\tfrac{1}{2}x^2y\,{}_pf_{p+1}(\alpha_1+1, \alpha_2+1, \ldots, \alpha_p+1;\ \rho_1+1, \rho_2+1, \ldots, \rho_{p+1}+1;\ -\tfrac{1}{4}x^2y^2),$$

$$(2.32)\quad \frac{d}{dy}\left[y^{2\alpha_1}\,{}_pf_{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\right.$$

$$\left. = 2y^{2\alpha_1-1}\,{}_pf_{p+1}(\alpha_1+1, \alpha_2, \alpha_3, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\right],$$

$$(2.33)\quad \frac{d}{dy}\left[y^{2\rho_1}\,{}_pf_{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\right.$$

$$\left. = 2y^{2\rho_1-1}\,{}_pf_{p+1}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\right].$$

Let

$$I = \int y^{2\rho_1-1}\,{}_pf_{p+1}(\alpha_1, \alpha_2, \ldots, \alpha_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_pf_{p+1}({}_1\alpha_1, {}_1\alpha_2, \ldots, {}_1\alpha_p;\ {}_1\rho_1, {}_1\rho_2, \ldots {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2)\,dy.$$

Integrate this by parts, using (2.33) to integrate the second hypergeometric function in the integrand, and (2.31) to differentiate the first. It follows that

$$I = E_1+\frac{x^2}{4}\int y^{2\rho_1+1}\,{}_pf_{p+1}(\alpha_1+1, \alpha_2+1, \ldots, \alpha_p+1;$$
$$\rho_1+1, \rho_2+1, \ldots, \rho_{p+1}+1;\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_pf_{p+1}({}_1\alpha_1, {}_1\alpha_2, \ldots, {}_1\alpha_p;\ {}_1\rho_1+1, {}_1\rho_2, {}_1\rho_3, \ldots, {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2)\,dy.$$

Now integrate this integral by parts p times, using (2.33) to increase successively all the parameters ${}_1\rho_s$ $(s = 2, 3, \ldots, p+1)$ by unity and to decrease successively all the parameters ρ_{s+1} $(s = 2, 3, \ldots, p+1)$ by unity; this is possible because, by § 1.2, $\rho_s+{}_1\rho_s = 1+\rho_1$. The result of carrying out this operation is

$$I = E_1+E_2+(-1)^p\frac{x^2}{4}\int y^{2\rho_1+1}\,{}_pf_{p+1}(\alpha_1+1, \alpha_2+1, \ldots, \alpha_p+1;$$
$$\rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_pf_{p+1}({}_1\alpha_1, {}_1\alpha_2, \ldots, {}_1\alpha_p;\ {}_1\rho_1+1, {}_1\rho_2+1, \ldots, {}_1\rho_{p+1}+1;\ -\tfrac{1}{4}y^2z^2)\,dy.$$

By means of (2.31) and (2.33) integrate again by parts so as to decrease the parameter ρ_1+1 by unity; it follows that

$$I = E_1+E_2+E_3+(-1)^p\frac{x^2}{4}\int y^{2\rho_1-1}\,{}_pf_{p+1}(\alpha_1+1, \alpha_2+1, \ldots, \alpha_p+1;$$
$$\rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_pf_{p+1}({}_1\alpha_1-1, {}_1\alpha_2-1, \ldots, {}_1\alpha_p-1;\ {}_1\rho_1, {}_1\rho_2, \ldots, {}_1\rho_{p+1};\ -\tfrac{1}{4}y^2z^2)\,dy.$$

Now integrate this integral by parts p times, using (2.31) to decrease successively all the parameters α_s+1 $(s=1, 2, 3, \ldots, p)$ by unity and to increase successively all the parameters ${}_1\alpha_s-1$ $(s=1, 2, 3, \ldots, p)$ by unity; this is possible because, by § 1.2, $\alpha_s+{}_1\alpha_s=1+\rho_1$. The final result is

$$I = E_1+E_2+E_3+E_4+\frac{x^2}{z^2}I,$$

and this establishes the lemma.

2.4. LEMMA 4. *For every fixed pair of values of x and z, such that*

$$x>0, \quad z>0, \quad x\neq z,$$

$$(xz)^{\frac{1}{2}}\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy = O(1) \quad as \quad T\to\infty.$$

Consider the integral*

$$(2.41)\quad (xz)^{\frac{1}{2}}\int_0^T\Big[\Phi_0(xy)\,\chi_0(yz)\,y - \frac{2}{\pi(xz)^{\frac{1}{2}}}\cos(xy+\tfrac{1}{2}\phi_0\pi)\sum_{r=1}^{p+1}B_r\cos(yz+\tfrac{1}{2}\psi_r\pi)\Big]dy,$$

where the coefficients B_r satisfy the equations (1.22).

On substituting the asymptotic expansions of $\Phi(xy)$ and $\chi_0(xz)$ given in § 2.1, we find that, when $x\neq z$, the integral is convergent when $T\to\infty$, whatever be the values of x and z, so long as $x>0$, $z>0$.

It is also evident that

$$\int_0^T\cos(xy+\tfrac{1}{2}\phi_0\pi)\sum_{r=1}^{p+1}B_r\cos(yz+\tfrac{1}{2}\psi_r\pi)\,dy$$

is bounded for all values of T and for all the specified values of x and z; and the lemma follows immediately.

2.5. LEMMA 5†. *If $x>0$, $z>0$, $T>0$, and $x\neq z$, then*

$$(2.51)\quad \Big(1-\frac{x^2}{z^2}\Big)\int_0^T\Phi_0(xy)\,\chi_0(yz)\,y\,dy$$
$$= V_1(x,z)\cos T(x+z)+V_2(x,z)\sin T(x+z)$$
$$+V_3(x,z)\cos T(x-z)+V_4(x,z)\sin T(x-z)$$
$$+V_5(x,z)\cos Tx+V_6(x,z)\sin Tx+M(x,z,T)\,T^{-\epsilon},$$

* If, when $\nu=1$ [cf. condition vi], we have $\phi_0+2\alpha_s=3$ for one value of α_s, so that $\psi_r+2{}_r\alpha_s=0$ $(r=1, 2, 3, \ldots, p+1)$, it is necessary to subtract also a multiple of

$$(xz)^{-\frac{1}{2}}\cos(xy+\tfrac{1}{2}\phi_0\pi)$$

from the integrand of (2.41).

† The object of this lemma is to show that there are no constant terms (independent of T) on the right-hand side of (2.51). The importance of this will appear in Lemma 6.

where (i) $V_1(x, z)$, $V_2(x, z)$, ..., $V_6(x, z)$ *are bounded functions of* x *and* z, *independent of* T; (ii) $\epsilon > 0$; *and* (iii) $M(x, z, T)$ *is bounded for all positive (non-zero) values of* x, z, *and* T.

Since $\chi_0(yz)$ is a linear combination of the functions $\Psi_r(yz)$ $(r = 1, 2, 3, \ldots, p+1)$ we first consider the contribution to (2 . 51) of

$$\left(1-\frac{x^2}{z^2}\right) B_1 \int_0^T \Phi_0(xy)\, \Psi_1(yz)\, y\, dy. \tag{2 . 52}$$

By Lemma 3, this is equal to

$$B_1(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}(\tfrac{1}{2}z)^{-\psi_1-\frac{1}{2}}\Big[E_1+E_2+E_3+E_4\Big]_0^T.$$

From conditions (iii) and (iv), $2\rho_1 > -\phi_0 \geqq 0$, and so E_1, E_2, E_3, and E_4 all vanish at the lower limit.

To determine the order of magnitude of the terms at the upper limit when T is large, we use the last asymptotic expansion given in § 2 . 1.

Consider, for example, the contribution to (2 . 52) of

$$B_1(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}(\tfrac{1}{2}z)^{-\psi_1-\frac{1}{2}}E_1 \tag{2 . 53}$$

at the upper limit. Save for a factor which is independent of T and is bounded in any bounded range of values of x and z, it is equal to

$$\begin{aligned} T^{2\rho_1+\phi_0}\Big[&\sum_{r=0}^{m}\nu_r(Tx)^{-\nu}\cos(Tx+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}r\pi) \\ &-2(Tx)^{-\phi_0}\sum_{r=1}^{p}K_r(Tx)^{-2\alpha_r}\Big\{1+\sum_{s=1}^{L_r}E_s^{(r)}(Tx)^{-2s}\Big\}+2^{\frac{1}{2}-\phi_0}(Tx)^{1-m}R_m\Big] \\ \times T^{\psi_1-1}\Big[&\sum_{r=0}^{m'}\nu_r'(Tz)^{-\nu}\cos(Tz+\tfrac{1}{2}\psi_1\pi-\tfrac{1}{2}\pi+\tfrac{1}{2}r\pi) \\ &-2(Tz)^{1-\psi_1}\sum_{r=1}^{p}K_r'(Tz)^{-2\alpha_r}\Big\{1+\sum_{s=1}^{L_r'}E_s^{(r)\prime}(Tz)^{-2s}\Big\} \\ &+2^{\frac{3}{2}-\psi_1}(Tz)^{1-m'}R_{m'}'\Big], \end{aligned}$$

where (i) m is any integer greater than ϕ_0-1, (ii) $|R_m| < \mathbf{B}(Tx)^{-\gamma}$ $(0 < \gamma < 1)$, where $\mathbf{B}$ is a positive constant (independent of x and T), so that R_m is a bounded function of x and T, (iii) the integers L_r are the greatest integers such that $\phi_0+2\alpha_r+2L_r < m-1+\gamma$ $(r = 1, 2, 3, \ldots, p)$, and (iv) ν_r, K_r, and $E_s^{(r)}$ are all constants (independent of x and T).

The dashed letters in the last factor of the product are subject to similar conditions, and, in particular, $R_{m'}'$ is a bounded function of z and T.

From § 1 . 2 it follows that

$$2\rho_1+\phi_0+\psi_1-1=0, \tag{2 . 54}$$

and so the powers of T outside the square brackets cancel.

When we multiply out the expressions inside the square brackets the terms of the resulting sum can be divided into five groups.

Group 1. These are the terms of the product

$$\sum_{r=1}^{m} \nu_r (Tx)^{-r} \cos(Tx+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}r\pi) \times \sum_{r=1}^{m'} \nu_r' (Tz)^{-r} \cos(Tz+\tfrac{1}{2}\psi_1\pi-\tfrac{1}{2}\pi+\tfrac{1}{2}r\pi),$$

all of which evidently are comprised in the expression on the right of (2 . 51).

Group 2. These are the terms of the product

$$-2\sum_{l=0}^{m} \nu_l (Tx)^{-l} \cos(Tx+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}l\pi) \sum_{r=1}^{p} K_r' (Tz)^{1-\psi_1-2_1a_r} \left\{1+\sum_{s=1}^{L_r'} E_s^{(r)\prime} (Tz)^{-2s}\right\}.$$

Consider the term

$$-2\nu_l E_s^{(r)\prime} x^{-l} z^{1-\psi_1-2_1a_r-2s} T^{-l+1-\psi_1-2_1a_r-2s} \cos(Tx+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}l\pi).$$

When $-l+1-\psi_1-2_1a_r-2s \leqslant 0$, this term is comprised in the expression on the right of (2 . 51).

When $-l+1-\psi_1-2_1a_r-2s > 0$, this term cannot be comprised in the expression on the right of (2 . 51), for if it were, Lemma 4 would be contradicted. Such a term can occur in (2 . 53), but, if so, it follows from Lemma 4 that it must cancel with similar terms arising from one or more of the other constituents of the left-hand side of (2 . 51), namely the expressions

$$\left(1-\frac{x^2}{z^2}\right) B_r \int_0^T \Phi_0(xy)\, \Psi_r(yz)\, y\, dy \quad (r=2, 3, \ldots, p+1). \tag{2 . 55}$$

Group 3. These are the terms of the product

$$-2\sum_{l=0}^{m'} \nu_l' \cos(Tz+\tfrac{1}{2}\psi_1\pi-\tfrac{1}{2}\pi+\tfrac{1}{2}l\pi) \times \sum_{r=1}^{p} K_r (Tx)^{-\phi_0-2a_r} \left\{1+\sum_{s=1}^{L_r} E_s^{(r)} (Tx)^{-2s}\right\}.$$

From condition (vi) of § 1 . 2, we have $\phi_0+2a_r > \nu \geqslant 0$, and so all these terms are comprised in the expression on the right of (2 . 51).

Group 4. These are the terms of the product

$$4\sum_{r=1}^{p} K_r (Tx)^{-\phi_0-2a_r} \left\{1+\sum_{s=1}^{L_r} E_s^{(r)} (Tx)^{-2s}\right\} \times \sum_{l=1}^{p} K_l' (Tz)^{1-\psi_1-2_1a_l} \left\{1+\sum_{n=1}^{L_l'} E_n^{(r)\prime} (Tz)^{-2n}\right\}.$$

From § 1 . 2 it follows that $1-\psi_1-2\,{}_1a_l = -2+\phi_0+2a_l < \nu$; also $-\phi_0-2a_r < -\nu$, and so every term of this group contains T in the form T^{-j} where $j>0$. Hence the terms of this group are comprised in the expression on the right of (2 . 51).

Group 5. These are the terms which involve R_m or $R'_{m'}$ or both.

Since we may assign to m and m' any fixed values as large as we please, and since, when m and m' have been fixed, R_m and $R'_{m'}$ are both bounded functions of x, z, and t, we may choose m and m' so large that all terms which involve R_m or $R'_{m'}$ or both are comprised in the expression $M(x, z, T)T^{-\epsilon}$ which occurs on the right of (2 . 51).

The contributions of E_2 and E_3 can be treated in a similar manner with the same result.

The contribution of E_4, namely

$$B_1(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}(\tfrac{1}{2}z)^{-\psi_1-\frac{1}{2}}E_4,$$

must be examined in greater detail. Save for a factor which is independent of T and is bounded in any bounded range of values of x and z, it is equal to

$$\begin{aligned}
T^{2\rho_1}\sum_{n=1}^{p} T^{\phi_0+p-n}\Bigg[&\sum_{r=0}^{m_n}\nu_{r,n}(Tx)^{-r}\cos(Tx+\tfrac{1}{2}\phi_0\pi+\tfrac{1}{2}p\pi-\tfrac{1}{2}n\pi+\tfrac{1}{2}\nu\pi)\\
&-2(Tx)^{-\phi_0-p+n}\sum_{r=1}^{n}K_{r,n}(Tx)^{-2\alpha_r}\Big\{1+\sum_{s=1}^{L_{r,n}}E^{(r)}_{s,n}(Tx)^{-2s}\Big\}\\
&-2(Tx)^{-\phi_0-p+n}\sum_{r=n+1}^{p}K_{r,n}(Tx)^{-2\alpha_r-2}\Big\{1+\sum_{s=1}^{L_{r,n}}E^{(r)}_{s,n}(Tx)^{-2s}\Big\}\\
&+2^{\frac{1}{2}-\phi_0-p+n}(Tx)^{-m_n+1}R_{m,n}\Bigg]\\
\times T^{\psi_1-p+n-1}\Bigg[&\sum_{r=0}^{m'_n}\nu'_{r,n}(Tz)^{-r}\cos(Tz+\tfrac{1}{2}\psi_1\pi-\tfrac{1}{2}p\pi+\tfrac{1}{2}n\pi-\tfrac{1}{2}\pi+\tfrac{1}{2}r\pi)\\
&-2(Tz)^{-\psi_1+p-n+1}\sum_{r=1}^{n-1}K'_{r,n}(Tz)^{-2\,{}_1\alpha_s}\Big\{1+\sum_{s=1}^{L'_{r,n}}E^{(r)\prime}_{s,n}(Tz)^{-2s}\Big\}\\
&-2(Tz)^{-\psi_1+p-n+1}\sum_{r=n}^{p}K'_{r,n}(Tz)^{-2\,{}_1\alpha_s+2}\Big\{1+\sum_{s=1}^{L'_{r,n}}E^{(r)\prime}_{s,n}(Tz)^{-2s}\Big\}\\
&+2^{\frac{1}{2}-\psi_1+p-n}(Tz)^{1-m'_n}R'_{m'_n}\Bigg],
\end{aligned}$$

where the significance of the various symbols $\nu_{r,n}$, ... is evident by analogy.

It follows from (2 . 54) that the powers of T outside the square brackets cancel. We divide up the terms of the product of the expressions inside the square brackets into five groups, as before; and the terms of Groups 1, 2, 3, and 5 can be discussed in the same way as before, with the same conclusions. The terms of Group 4 are now the terms of the product

$$4\Big[\sum_{r=1}^{n} K_{r,n}(Tx)^{-\phi_0-p+n-2a_r}\Big\{1+\sum_{s=1}^{L_{r,n}} E^{(r)}_{s,n}(Tx)^{-2s}\Big\} + \sum_{r=n+1}^{p} K_{r,n}(Tx)^{-\phi_0-p+n-2a_r-2}\Big\{1+\sum_{s=1}^{L_{r,n}} E^{(r)}_{s,n}(Tx)^{-2s}\Big\}\Big]$$

$$\times\Big[\sum_{m=1}^{n-1} K'_{m,n}(Tz)^{-\psi_1+p-n+1-2_1a_m}\Big\{1+\sum_{s=1}^{L'_{m,n}} E^{(m)\prime}_{s,n}(Tz)^{-2s}\Big\} + \sum_{m=n}^{p} K'_{m,n}(Tz)^{-\psi_1+p-n+3-2_1a_m}\Big\{1+\sum_{s=1}^{L'_{m,n}} E^{(m)\prime}_{s,n}(Tz)^{-2s}\Big\}\Big].$$

Any term of this product which involves T^j, where $j>0$, may occur in (2 . 53), but it is not comprised in the terms on the right of (2 . 51), for if it were, Lemma 4 would be contradicted. It must consequently cancel with similar terms, arising from (2 . 55).

Terms which involve T^j, where $j<0$, are comprised in the terms on the right of (2 . 51), and so it remains to consider only those terms which are independent of T.

Since, by conditions (v) and (vi), no two of the a_r are equal or differ by an integer, it follows that the terms which are independent of T are of the form

$$4K_{n,n}K'_{n,n}x^{-\phi_0-p+n-2a_n}z^{-\psi_1+p-n+1-2_1a_n} \quad (n=1,\ 2,\ 3,\ \ldots,\ p).$$

The sum of these terms in the part of (2 . 52) which contains the value of E_n at the upper limit is

$$-B_1(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}(\tfrac{1}{2}z)^{-\psi_1-\frac{1}{2}}(-1)^p x^2\sum_{n=1}^{p}\frac{K_{n,n}K'_{n,n}}{x^{2a_n}z^{2_1a_n+2}},$$

and the sum of all these terms in

$$\Big(1-\frac{x^2}{z^2}\Big)\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy$$

is consequently

$$-(-1)_p(\tfrac{1}{2}x)^{-\phi_0-\frac{1}{2}}\sum_{r=1}^{p+1}B_r\sum_{n=1}^{p}\frac{K_{n,n}K'_{n,n}}{x^{2a_n}z^{2_1a_n-2}}. \tag{2 . 56}$$

Now $K_{s,n}$ and $K'_{s,n}$ can be calculated from (2 . 13); their values are

$$K_{s,n} = 2^{2a_s}\Gamma(a_s)\frac{\overset{n}{\underset{\sigma=1}{\Pi'}}\Gamma(a_\sigma - a_s)\overset{p}{\underset{\sigma=n+1}{\Pi}}\Gamma(a_\sigma + 1 - a_s)}{\overset{p+1}{\underset{\sigma=1}{\Pi}}\Gamma(\rho_\sigma - a_s)} \qquad (s = 1, 2, 3, \ldots, n),$$

$$K'_{s,n} = 2^{2\,{}_1a_s - 2}\Gamma({}_1a_s - 1)\frac{\overset{n}{\underset{\sigma=1}{\Pi}}\Gamma({}_1a_\sigma - {}_1a_s + 1)\overset{p}{\underset{\sigma=n}{\Pi'}}\Gamma({}_1a_\sigma - a_s)}{\overset{p+1}{\underset{\sigma=1}{\Pi}}\Gamma({}_1\rho_\sigma - {}_1a_s + 1)} \qquad (s = n, n+1, \ldots, p),$$

the dash indicating that the factor for which $\sigma = s$ is to be omitted, and the values of $K_{s,n}$ $(s > n)$ and of $K'_{s,n}$ $(s < n)$ not being required.

Substituting these values in (2 . 56) and changing the order of the summations, we find that (2 . 56) is a multiple of

$$\sum_{n=1}^{p}\frac{\overset{p}{\underset{\sigma=1}{\Pi}}\sin(\rho_\sigma - a_n)\pi}{\overset{p}{\underset{\sigma=1}{\Pi'}}\sin(a_\sigma - a_n)\pi}\left(\frac{x}{z}\right)^{\frac{3}{2}-\phi_0-2a_n}\sum_{r=1}^{p+1}\frac{B_r}{\sin(\rho_r - a_n)\pi} = 0,$$

by equations (1 . 22).

Hence all the terms obtained by evaluating the integral on the left of (2 . 51) are comprised in the expression on the right.

2 . 6. Lemma 6. *If z is not a point of the interval (a, b), (where a may be zero, or b may be infinite), then*

$$\int_0^\infty \chi_0(yz)\, y\, dy \int_a^b F(x)\,\Phi_0(xy)\, x\, dx = 0.$$

We divide the investigation of the lemma into three cases.

2 . 61. *Case* 1. $a \neq 0$ *and* b *finite.*

As in Lemma 2, we can show that

$$\int_0^\infty \chi_0(yz)\, y\, dy \int_a^b F(x)\,\Phi_0(xy)\, x\, dx = \lim_{T\to\infty}\int_a^b F(x)\, x\, dx \int_0^T \Phi_0(xy)\,\chi_0(yz)\, y\, dy,$$

provided that the limit on the right exists.

When we substitute for the inner integral on the right by means of (2 . 51), we express this limit as a sum of multiples of expressions of the

type

$$\lim_{T\to\infty}\int_a^b \frac{z^2 x\,F(x)}{z^2-x^2}\,P(x,z)\begin{matrix}\sin\\ \cos\end{matrix} T(x\pm z)\,dx, \quad \lim_{T\to\infty}\int_a^b \frac{z^2 x\,F(x)}{z^2-x^2}\,Q(x,z)\begin{matrix}\sin\\ \cos\end{matrix} Tx\,dx,$$

$$\lim_{T\to\infty}\int_a^b \frac{z^2 x\,F(x)}{z^2-x^2}\,M(x,z,T)\,T^{-\epsilon}\,dx,$$

where $P(x, z)$ and $Q(x, z)$ are bounded functions of x and z $(x>0, z>0)$, while $M(x, z, T)$ is bounded for all values of x, z and T, and $\epsilon>0$.

If z is neither an internal point nor an end point of the interval (a, b), it follows from the Riemann-Lebesgue theorem* that the first two limits are zero, and the third limit is evidently zero.

2.62. *Case* 2. $a\neq 0$, b *infinite.*

When b is infinite and z is neither an internal point nor an end point of the interval (a, ∞), it follows from Lemma 4 that

$$\left|x^{\frac{1}{2}}\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy\right| < K,$$

however large x and T may be, K being independent of both T and x.

By condition (i) of § 1.3, if ϵ is an arbitrary positive number, we can now choose c (independent of T) so that

$$\int_c^\infty |F(x)|\,x^{\frac{1}{2}}\,dx < \frac{\epsilon}{2K},$$

and then, by the analysis of Case 1, we have

$$\left|\int_a^c F(x)\,x\,dx\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy\right| < \tfrac{1}{2}\epsilon.$$

for all sufficiently large values of T.

Hence, given ϵ, we have

$$\left|\int_a^\infty F(x)\,x\,dx\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy\right| \leqslant \left|\int_a^c F(x)\,x\,dx\int_0^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy\right| + \int_c^\infty |F(x)|\,x^{\frac{1}{2}}\,dx\,.\left|\int_0^T x^{\frac{1}{2}}\Phi_0(xy)\,\chi_0(yz)\,y\,dy\right| < \epsilon,$$

for all sufficiently large values of T; this establishes Case 2 of the lemma.

2.63. *Case* 3. $a=0$, b *finite.*

If we define K, as in § 2.62, it follows from condition (i) of § 1.3 that, if ϵ is an arbitrary positive number, we can choose a positive number

* Cf. Whittaker and Watson, *Modern analysis* (1920), 172.

c' (independent of T) so that

$$\int_0^{c'} |F(x)|\, x^{\frac{1}{2}}\, dx < \frac{\epsilon}{2K}.$$

Hence, as in Case 2,

$$\left|\int_0^b F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy\right| \leqslant \int_0^{c'} |F(x)|\, x^{\frac{1}{2}}\, dx \,.\, \left|\int_0^T x^{\frac{1}{2}} \Phi_0(xy)\, \chi_0(yz)\, y\, dy\right|$$
$$+ \left|\int_{c'}^b F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy\right|$$
$$< \epsilon,$$

for all sufficiently large values of T; and this establishes Case 3 of the lemma.

2.7. Lemma 7. *The limit*

$$\lim_{T\to\infty} \int_a^b x^{\frac{1}{2}}\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy \tag{2.701}$$

exists, provided that either

(i) *a and b are positive constants (independent of x and T), or*

(ii) *$a = a(T)$ and $b = b(T)$ are functions of T which lie in a fixed interval (c_1, c_2), where $0 < c_1 < c_2$, and both $a(T)$ and $b(T)$ tend to limits as $T \to \infty$. Further, either or both of these limits may be equal to z.*

From conditions (iii) and (iv) of Theorem 1, we have $2\rho_r > 0$ $(r = 1, 2, 3, \ldots, p+1)$, and so the inner integral converges uniformly at its lower limit near any pair of values of x and z such that $x > 0$, $z > 0$. It is consequently sufficient to consider the integral taken between the limits c and T, where $c > 0$ and c is independent of T.

On using the asymptotic expansions of Lemma 1, we have

$$y\Phi_0(xy)\, \chi_0(yz) \tag{2.702}$$
$$= \frac{2}{\pi(xz)^{\frac{1}{2}}}\left[\cos(xy + \tfrac{1}{2}\phi_0\pi)\left\{1 + \sum_{s=1}^{M} C_s(xy)^{-2s}\right\}\right.$$
$$+ \frac{H}{xy}\sin(xy + \tfrac{1}{2}\phi_0\pi)\left\{1 + \sum_{s=1}^{N} D_s(xy)^{-2s}\right\}$$
$$\left. - (\tfrac{1}{2}xy)^{-\phi_0} \sum_{r=1}^{p} K_r(xy)^{-2a_r}\left\{1 + \sum_{s=1}^{L_r} E_s^{(r)}(xy)^{-2s}\right\} + (xy)^{\frac{1}{2}-m} R_m\right]$$
$$\times \sum_{n=1}^{p+1} B_n\left[\cos(yz + \tfrac{1}{2}\psi_n\pi)\left\{1 + \sum_{s=1}^{M_n} C_{n,s}(yz)^{-2s}\right\}\right.$$
$$+ \frac{H_n}{yz}\sin(yz + \tfrac{1}{2}\psi_n\pi)\left\{1 + \sum_{s=1}^{N_n} D_{n,s}(yz)^{-2s}\right\}$$
$$\left. - (\tfrac{1}{2}yz)^{-\psi_n} \sum_{r=1}^{p} K_{n,r}(yz)^{-2_n a_r}\left\{1 + \sum_{s=1}^{L_{n,r}} E_{n,s}^{(r)}(yz)^{-2s}\right\} + (yz)^{\frac{1}{2}-m_n} R_{n,m_n}\right],$$

where the constants satisfy the conditions specified in § 2 . 1, m and m_n are any two integers greater that ϕ_0-1 and ψ_n-1 respectively, while

$$|R_m| < \mathbf{B}(xy)^{-\gamma}, \quad |R_{n,m_n}| < \mathbf{B}_n(yz)^{-\gamma_n} \quad (0<\gamma<1;\ 0<\gamma_n<1),$$

$\mathbf{B}$ and $\mathbf{B}_n$ being independent of x, y and z.

From conditions (vi) of § 1 . 3 we have $\phi_0+2a_r \geqslant \nu \geqslant 0$, while from the equations of § 1 . 2 we have

$$\psi_n+2_na_r = 3-\phi_0-2a_r > 1-\nu \geqslant 0.$$

Hence, when m and m_n ($n=1, 2, 3, \ldots, p$) are chosen sufficiently large, it follows that

$$\int_c^T \Phi_0(xy)\,\chi_0(yz)\,y\,dy$$

is expressible as an integral which is uniformly convergent with respect to both x and z, together with some terms of the form*

$$(2.703)\quad \int_c^T \frac{2}{\pi(xz)^{\frac{1}{2}}}\left[\cos(xy+\tfrac{1}{2}\phi_0\pi)+\frac{H}{xy}\sin(xy+\tfrac{1}{2}\phi_0\pi)\right] \times \sum_{r=1}^{p+1} B_r\left[\cos(yz+\tfrac{1}{2}\psi_r\pi)+\frac{H_r}{yz}\sin(yz+\tfrac{1}{2}\psi_r\pi)\right]dy.$$

When the integrand is multiplied out, the integral of those terms in the result which contain two sines converges uniformly with respect to x and z. Hence, to establish the lemma, we have merely to prove that the limits

$$\lim_{T\to\infty}\int_a^b dx\int_c^T \sum_{r=1}^{p+1} B_r\cos(xy+\tfrac{1}{2}\phi_0\pi)\cos(yz+\tfrac{1}{2}\psi_r\pi)\,dy,$$

$$\lim_{T\to\infty}\int_a^b dx\int_c^T \frac{1}{xy}\sum_{r=1}^{p+1} B_rH_r\cos(xy+\tfrac{1}{2}\phi_0\pi)\sin(yz+\tfrac{1}{2}\psi_r\pi)\,dy,$$

$$\lim_{T\to\infty}\int_a^b dx\int_c^T \frac{1}{xy}\sum_{r=1}^{p+1} B_r\sin(xy+\tfrac{1}{2}\phi_0\pi)\cos(yz+\tfrac{1}{2}\psi_r\pi)\,dy,$$

---

* If $\psi_n+2_na_r=0$ [*i.e.* from (1.21) $\phi_0+2a_r=3$ (cf. condition vi, § 1.3)] there occurs the term $-2^{\psi_n}K_{n,r}$ in the second square bracket of (2.702). We must consequently add to (2.703) the term

$$-\int_c^T \frac{2}{\pi(xz)^{\frac{1}{2}}}[\cos(xy+\tfrac{1}{2}\phi_0\pi)]\sum_{r=1}^{p+1} B_r\,2^{\psi_r}K_{n,r}\,dy,$$

and so, in addition to the three limits below we must demonstrate the existence of

$$\lim_{T\to\infty}\int_a^b dx\int_c^T \cos(xy+\tfrac{1}{2}\phi_0\pi)\,dy.$$

An application of the Riemann-Lebesgue theorem shows immediately that this limit exists.

all exist. We write $I_1(T)$, $I_2(T)$, $I_3(T)$ respectively for the integrals whose limits require investigation.

2.71. *Proof of the existence of* $\lim_{T\to\infty} I_1(T)$.

We have

$$2I_1(T) = \int_a^b dx \int_c^T \sum_{r=1}^{p+1} B_r \left[\cos\left\{y(x+z)+\tfrac{1}{2}(\phi_0+\psi_r)\pi\right\} + \cos\left\{y(x-z)+\tfrac{1}{2}(\phi_0-\psi_r)\pi\right\}\right] dy$$

$$= \int_a^b \sum_{r=1}^{p+1} B_r \frac{\sin\left\{T(x+z)+\tfrac{1}{2}(\phi_0+\psi_r)\pi\right\}}{x+z}\, dx$$

$$- \int_a^b \sum_{r=1}^{p+1} B_r \frac{\sin\left\{c(x+z)+\tfrac{1}{2}(\phi_0+\psi_r)\pi\right\}}{x+z}\, dx$$

$$+ \left\{\sum_{r=1}^{p+1} B_r \cos\tfrac{1}{2}(\phi_0-\psi_r)\pi\right\} \int_a^b \frac{\sin T(x-z)-\sin c(x-z)}{x-z}\, dx$$

$$+ \left\{\sum_{r=1}^{p+1} B_r \sin\tfrac{1}{2}(\phi_0-\psi_r)\pi\right\} \int_a^b \frac{\cos T(x-z)-\cos c(x-z)}{x-z}\, dx.$$

From the conditions in the enunciation of Lemma 7 it is easy to see that the first three of these four expressions tend to limits as $T\to\infty$, but the fourth expression does *not* of necessity tend to a limit, unless

$$\sum_{r=1}^{p+1} B_r \sin\tfrac{1}{2}(\phi_0-\psi_r)\pi = 0, \tag{2.711}$$

that is, unless

$$\sum_{r=1}^{p+1} B_r \cos(\phi_0+\rho)\pi = 0, \tag{2.712}$$

as is seen by substituting for ψ_r by means of § 2.1.

To establish the existence of $\lim_{T\to\infty} I_1(T)$ it is consequently adequate to prove that (2.712) is true; and, in view of (1.22), this will be the case if

$$\begin{vmatrix} \cos(\phi_0+\rho_1)\pi, & \cos(\phi_0+\rho_2)\pi, & \dots, & \cos(\phi_0+\rho_{p+1})\pi \\ \operatorname{cosec}(\rho_1-\alpha_1)\pi, & \operatorname{cosec}(\rho_2-\alpha_1)\pi, & \dots, & \operatorname{cosec}(\rho_{p+1}-\alpha_1)\pi \\ \operatorname{cosec}(\rho_1-\alpha_2)\pi, & \operatorname{cosec}(\rho_2-\alpha_2)\pi, & \dots, & \operatorname{cosec}(\rho_{p+1}-\alpha_2)\pi \\ \dots & \dots & \dots & \dots \\ \operatorname{cosec}(\rho_1-\alpha_p)\pi, & \operatorname{cosec}(\rho_2-\alpha_p)\pi, & \dots, & \operatorname{cosec}(\rho_{p+1}-\alpha_p)\pi \end{vmatrix} = 0. \tag{2.713}$$

Now take the elements of any column in this determinant and multiply the first element by*

$$-\prod_{r=1}^{p}\prod_{s=1}^{p}{}' \sin\pi(a_s-a_r)$$

and the $(n+1)$-th element by

$$\prod_{m=1}^{p+1}\sin(\rho_m-a_r)\pi\prod_{s=1}^{p}{}'\sin(a_s-a_r)\pi \quad (r=1, 2, 3, \ldots, p),$$

then if, for each column, the sum of the products so obtained is zero, it is evident that (2 . 713) is satisfied.

The sum for the first column is

$$\sum_{r=1}^{p}\prod_{m=2}^{p+1}\sin(\rho_m-a_r)\pi\prod_{s=1}^{p}{}'\sin(a_s-a_r)\pi-\cos(\phi_0+\rho_1)\pi\prod_{r=1}^{p}\prod_{s=1}^{p}{}'\sin(a_s-a_r)\pi,$$

and the sums for the other columns are obtained by interchanging ρ_1 in this sum with $\rho_2, \rho_3, \ldots, \rho_{p+1}$ in turn.

If we write $e^{\rho_m\pi i}=\mathbf{P}_m$, $e^{a_s\pi i}=\mathbf{A}_s$ and substitute for ϕ_0 from § 1 . 2, we see that the sum obtained from the first column will vanish if

$$\sum_{r=1}^{p}\prod_{m=2}^{p+1}(\mathbf{P}_m^2-\mathbf{A}_r^2)\prod_{s=1}^{p}{}'(\mathbf{A}_s^2-\mathbf{A}_r^2)\,\mathbf{A}_r^{-2p}=(\mathbf{P}_2^2\mathbf{P}_3^2\ldots\mathbf{P}_{p+1}^2-\mathbf{A}_1^2\mathbf{A}_2^2\ldots\mathbf{A}_p^2)\times\prod_{r=1}^{p}\prod_{s=1}^{p}{}'(\mathbf{A}_s^2-\mathbf{A}_r^2)\,\mathbf{A}_r^{-1}\mathbf{A}_s^{-1};$$

this is an algebraic identity, which is easily verified. By symmetry, the sums obtained from the other columns vanish, and so the determinant vanishes; and we have seen that this is sufficient to establish the existence of $\lim_{T\to\infty} I_1(T)$.

2 . 72. *Proofs of the existence of* $\lim_{T\to\infty} I_2(T)$ *and* $\lim_{T\to\infty} I_3(T)$.

We have

$$2I_2(T)=\int_a^b dx\int_c^T\frac{1}{xy}\sum_{r=1}^{p+1}B_rH_r[\sin\{y(x+z)+\tfrac{1}{2}(\phi_0+\psi_r)\pi\}+\sin\{y(x-z)+\tfrac{1}{2}(\phi_0-\psi_r)\pi]\,dy.$$

The integrals with respect to y of those terms which contain $x+z$ are uniformly convergent with respect to both x and z, and so these integrals tend to limits as $T\to\infty$.

* The dashes indicate that factors for which $r=s$ are to be omitted.

Since, by 2.104, $H_r = H$ $(r = 1, 2, 3, \ldots, p+1)$, it follows from (2.711) that the remaining terms reduce to

$$H\left\{\sum_{r=1}^{p+1} B_r \cos \tfrac{1}{2}(\phi_0 - \psi_r)\,\pi\right\} \int_a^b dx \int_c^T \frac{\sin y(x-z)}{xy}\, dy.$$

Writing $y = u/(x-z)$, we see that this is a constant multiple of

$$\int_a^b \frac{dx}{x} \int_{c(x-z)}^{T(x-z)} \frac{\sin u}{u}\, du,$$

and this evidently tends to a limit as $T \to \infty$, whether z is in the interval (a, b) or not.

Hence $\lim_{T\to\infty} I_2(T)$ exists, and the existence of $\lim_{T\to\infty} I_3(T)$ can be established in the same way. This completes the proof of Lemma 7, and we now can proceed to the proof of Theorem 1.

3. *Proof of Theorem* 1.

3.1. If $F(x)$ is of bounded variation near $x = z$, so is $F(x)\sqrt{x}$, and we may write

$$F(x)\sqrt{x} = G_1(x) - G_2(x),$$

where $G_1(x)$ and $G_2(x)$ are positive increasing functions of x.

Given ϵ, we can choose a number δ so small that both (i) $F(x)$ is of bounded variation in the interval $(z-\delta, z+\delta)$, and also (ii)

$$G_1(z+\delta) - G_1(z+0) < \epsilon, \qquad G_1(z-0) - G_1(z-\delta) < \epsilon,$$
$$G_1(z+\delta) - G_2(z+0) < \epsilon, \qquad G_2(z-0) - G_2(z-\delta) < \epsilon.$$

By the second mean-value theorem there exists a number ξ between 0 and δ such that

$$(3.11)\quad \int_z^{z+\delta} G_1(x)\sqrt{x}\, dx \int_0^T \Phi_0(xy)\,\chi_0(yz)\, y\, dy$$
$$= G_1(z+0) \int_z^{z+\delta} \sqrt{x}\, dx \int_0^T \Phi_0(xy)\,\chi_0(yz)\, y\, dy$$
$$+ \left\{G_1(z+\delta) - G_1(z+0)\right\} \int_{z+\xi}^{z+\delta} \sqrt{x}\, dx \int_0^T \Phi_0(xy)\,\chi_0(yz)\, y\, dy.$$

From Lemma 7 of § 2.7, the first term on the right tends to a limit as $T \to \infty$, when δ is fixed, while the absolute value of the second term remains less than a constant multiple of ϵ.

Hence the limit of the integral on the left of (3 . 11), when $T \to \infty$, is

$$G_1(z+0) \lim_{T\to\infty} \int_z^{z+\delta} \sqrt{x}\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy = C_1(z)\, G_1(z+0)/\sqrt{z},$$

where $C_1(z)$ is a function of z and of the parameters a and ρ which occur in $\Phi_0(x)$, but is independent of the form of $F(z)$.

The function $G_2(z)$ can be treated in the same manner, and similar reasoning can evidently be applied to the interval $(z-\delta, z)$.

Collecting our results, we get

$$\lim_{T\to\infty} \int_{z-\delta}^{z+\delta} F(x)\, x\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy = C_1(z)\, F(z+0) + C_2(z)\, F(z-0),$$

where
$$C_2(z) = \lim_{T\to\infty} \int_{z-\delta}^{z} \sqrt{x}\, dx \int_0^T \Phi_0(xy)\, \chi_0(yz)\, y\, dy\,;$$

$C_2(z)$, like $C_1(z)$, is a function of z and of the parameters a and ρ which occur in $\Phi_0(x)$, but is independent of the form of $F(z)$.

Combining these results with Lemmas (2) and (6) we get the theorem :

If
$$f(y) = \int_0^\infty F(x)\, \Phi_0(xy)\, x\, dx,$$

then
$$C_1(z)\, F(z+0) + C_2(z)\, F(z-0) = \int_0^\infty f(y)\, \chi_0(yz)\, y\, dy,$$

where $C_1(z)$ and $C_2(z)$ are independent of the form of $F(z)$.

3 . 2. *The evaluation of $C_1(z)$ and $C_2(z)$.*

By direct differentiation we have

$$\frac{d}{dx}\left[x^{2\rho_1}\, {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\right]$$
$$= 2x^{2\rho_1-1}\, {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2).$$

Hence, after a slight rearrangement,

$$(3.21)\quad 2^{-\frac{1}{2}+\phi_0}\, y^{-\phi_0-\frac{3}{2}}\, v^{2\rho_1}\, {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\tfrac{1}{4}v^2y^2)$$
$$= \int_0^v x^{2\rho_1+\phi_0-\frac{3}{2}} (\tfrac{1}{2}xy)^{-\phi_0-\frac{1}{2}}\, {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1, \rho_2, \ldots, \rho_{p+1};\ -\tfrac{1}{4}x^2y^2)\, x\, dx$$
$$= \int_0^v x^{2\rho_1+\phi_0-\frac{3}{2}}\, \Phi_0(xy)\, x\, dx.$$

We now apply the theorem of § 3 . 1 with

$$F(z) = \begin{cases} x^{2\rho_1+\phi_0-\frac{3}{2}} & (x < v), \\ 0 & (x > v), \end{cases}$$

$f(y) = 2^{\phi_0-\frac{1}{2}} y^{-\phi_0-\frac{1}{2}} v^{2\rho_1} {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\frac{1}{4}v^2y^2)$,

conditions (i) and (ii) of § 1 . 3 being evidently satisfied.

The application of the theorem gives on reduction

$$\int_0^\infty v^{1-\phi_0} y^{-\phi_0-\frac{1}{2}} {}_pf_{p+1}(a_1, a_2, \ldots, a_p;\ \rho_1+1, \rho_2, \rho_3, \ldots, \rho_{p+1};\ -\tfrac{1}{4}v^2y^2)\chi_0(yz)\, y\, dy$$

$$= \begin{cases} \{C_1(z)+C_2(z)\} \dfrac{z^{2\rho_1+\phi_0-\frac{3}{2}}}{2^{\phi_0-\frac{1}{2}} v^{2\rho_1+\phi_0-1}} & (v > z), \\ C_2(z) \dfrac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}} & (v = z), \\ 0 & (v < z). \end{cases}$$

Denoting the integrand by J, we have

$$(3\,.\,22) \quad \int_0^\infty \left\{ J - \frac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}\pi} \frac{\sin y(v+z)}{y} \sum_{r=1}^{p+1} B_r \cos \tfrac{1}{2}(\phi_0-\psi_r)\pi \right\} dy$$

$$= \begin{cases} \{C_1(z)+C_2(z)\} \dfrac{z^{2\rho_1+\phi_0-\frac{3}{2}}}{2^{\phi_0-\frac{1}{2}} v^{2\rho_1+\phi_0-1}} - \dfrac{z^{-\frac{1}{2}}}{2^{\phi_0+\frac{1}{2}}} \displaystyle\sum_{r=1}^{p+1} B_r \cos \tfrac{1}{2}(\phi_0-\psi_r)\pi & (v > z), \\ C_2(z) \dfrac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}} & (v = z), \\ \dfrac{z^{-\frac{1}{2}}}{2^{\phi_0+\frac{1}{2}}} \displaystyle\sum_{r=1}^{p+1} B_r \cos \tfrac{1}{2}(\phi_0-\psi_r)\pi. & (v < z). \end{cases}$$

The asymptotic expansion of J can be found from § 2 . 1. Save for a factor which is independent of y, v, and z, it is derived from (2 . 72) by multiplying throughout by $v^{\frac{1}{2}}y^{-1}$, changing ϕ_0 to ϕ_0-1, and changing x to v. We then find, by using condition (vi) of § 1 . 3, that the asymptotic expansion of the integrand of (3 . 22) is of the form*

$$(3\,.\,23) \quad \frac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}\pi} \sum_{r=1}^{p+1} B_r \frac{\sin\{y(v+z)+\frac{1}{2}(\phi_0+\psi_r)\pi\}}{y} + y^{-1-\sigma} M(v, z, y),$$

---

* If $\nu = 1$, and $\phi_0 + 2\alpha_t = 3$ for some value of t, we have $\psi_r + 2_r\alpha_t = 0$ for this value of t; we must consequently include in (3 . 23) a term which is a multiple of

$$\frac{\sin(yv + \frac{1}{2}\phi_0\pi)}{y},$$

but the argument is otherwise unaffected.

where $\sigma > \nu \geqq 0$ by condition (vi) of § 1 . 3, and $M(v, z, y)$ is a bounded function of v, z, y for all values of v, z, y which are greater than 0.

Hence the integral on the left in (3 . 22) converges uniformly with respect to both v and z when these have any values greater than zero, and so the integral is a continuous function of v and z. Consequently, when $v \to z$, the limits of the first and third expressions on the right in (3 . 22) must both be equal to the second expression, that is to say

$$\{C_1(z)+C_2(z)\}\frac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}}-\frac{z^{-\frac{1}{2}}}{2^{\phi_0+\frac{1}{2}}}\sum_{r=1}^{p+1}B_r\cos\tfrac{1}{2}(\phi_0-\psi_r)\pi$$
$$= C_2(z)\frac{z^{-\frac{1}{2}}}{2^{\phi_0-\frac{1}{2}}} = \frac{z^{-\frac{1}{2}}}{2^{\phi_0+\frac{1}{2}}}\sum_{r=1}^{p+1}B_r\cos\tfrac{1}{2}(\phi_0-\psi_r)\pi.$$

From the equations $\psi_r = 1-2\rho_r-\phi_0$ $(r = 1, 2, 3, \ldots, p+1)$, it now follows at once that

$$C_1(z) = C_2(z) = \tfrac{1}{2}\sum_{r=1}^{p+1}B_r\sin(\phi_0+\rho_r)\pi,$$

and this completes the proof of Theorem 1.

4. *Proof of Theorem 2.*

4 . 1. We require the same system of lemmas as were used in the proof of Theorem 1. Let

$$\Phi(xy) = \sum_{r=0}^{\lambda}A_r\Phi_r(xy), \qquad \chi(xy) = \sum_{s=0}^{\lambda+\mu-1}B_s\Psi_s(yz).$$

The asymptotic expansions of the functions $\Phi_r(x)$ and $\Psi_r(x)$ are given in Lemma 1 of § 2, and, just as in Lemma 2 of § 2, we can prove the following lemma :—

If $\phi_r \leqslant 0$ for every r for which $A_r \neq 0$, and if $2_s\rho_r > -\phi_r$ (so that, by § 1 . 2 and § 1 . 4, $2\rho_s > -\phi_0$) then

$$\int_0^T\chi(yz)\,y\,dy\int_0^\infty F(x)\,\Phi(xy)\,x\,dx = \int_0^\infty F(x)\,x\,dx\int_0^T\Phi(xy)\,\chi(yz)\,y\,dy.$$

Also

$$\int_0^\infty\chi(yz)\,y\,dy\int_0^\infty F(x)\,\Phi(xy)\,x\,dx = \lim_{T\to\infty}\int_0^\infty F(x)\,x\,dx\int_0^T\Phi(xy)\,\chi(yz)\,y\,dy,$$

provided that the limit on the right exists.

4 . 2. We now investigate

$$\left(1-\frac{x^2}{z^2}\right)(\tfrac{1}{2}x)^{\phi_r+\frac{1}{2}}(\tfrac{1}{2}z)^{\psi_s+\frac{1}{2}}\int \Phi_r(xy)\,\Phi_s(yz)\,y\,dy \doteq I_{r,s}, \quad \text{say.}$$

When $r \neq s$ and $r \leqslant p+1$, $s \leqslant p+1$, we have

$$\begin{aligned} I_{r,s} &= \left(1-\frac{x^2}{z^2}\right)\int y^{2\rho_s-2\rho_r+1}\,{}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p\,; \\ &\qquad 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1}\,;\ -\tfrac{1}{4}x^2y^2) \\ &\qquad \times {}_pf_{p+1}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p\,;\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1}\,;\ -\tfrac{1}{4}y^2z^2)\,dy \\ &= \left(1-\frac{x^2}{z^2}\right)\int y^{2\rho_s-2\rho_r+1}\,{}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p\,; \\ &\qquad 2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \cdots * \cdots,\ 2-{}_r\rho_{p+1}\,;\ -\tfrac{1}{4}x^2y^2) \\ &\qquad \times {}_pf_{p+1}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p\,;\ {}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \cdots * \cdots,\ {}_s\rho_{p+1}\,;\ -\tfrac{1}{4}y^2z^2)\,dy\,; \end{aligned}$$

in the last integral, $2-{}_r\rho_s$ and ${}_s\rho_r$ have been placed first in the groups of parameters to which they belong, and the asterisks denote, both here and below, that the parameters involving ${}_r\rho_s$ and ${}_s\rho_r$ are to be omitted from their natural positions.

Compare the last integral with $I_{0,1}$, which was the particular case of $I_{r,s}$ required for Theorem 1, and which was evaluated in detail in § 2 . 3. The success of the method used to evaluate $I_{0,1}$ was a consequence of the relations

$$ {}_1\rho_1+1 = \alpha_t+{}_1\alpha_t = \rho_\tau+{}_1\rho_\tau = \rho_1+1$$

$$(t = 1, 2, 3, \ldots, p\,;\ \tau = 2, 3, \ldots, p+1).$$

Now, from §§ 1 . 2 and 1 . 4, we have

$$3-{}_r\rho_s = 2-{}_r\alpha_t+{}_s\alpha_t = 2-{}_r\rho_\tau+{}_s\rho_\tau = {}_s\rho_r+1$$

$$(t = 1, 2, 3, \ldots, p\,;\ \tau = 1, 2, 3, \ldots, p+1\,;\ \tau \neq r)$$

and also $$2-{}_r\rho_r+{}_s\rho_s = {}_s\rho_r+1.$$

These are precisely the relations which are necessary for the success of the method of § 2 . 3 when applied to evaluate $I_{r,s}$. The result of carrying out the method step by step is

$$I_{r,s} = F_1+F_2+F_3+F_4 \quad (r \leqslant p+1,\ s \leqslant p+1,\ r \neq s), \tag{4 . 21}$$

where

$$F_1 = \tfrac{1}{2} y^{2\rho_s - 2\rho_r + 2}\, {}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p;$$
$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2 \ldots * \ldots,\ 2-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p;\ {}_s\rho_r+1,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots * \ldots,\ {}_s\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

$$F_2 = -\tfrac{1}{8}x^2 y^{2\rho_s - 2\rho_r + 4} \sum_{\substack{n=1 \\ n \neq r}}^{p+1} (-1)^{\xi}\, {}_pf_{p+1}(3-{}_r\alpha_1,\ 3-{}_r\alpha_2,\ \ldots,\ 3-{}_r\alpha_p;$$
$$3-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots * \ldots,$$
$$2-{}_r\rho_{n-1},\ 3-{}_r\rho_n,\ \ldots,\ 3-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y)$$
$$\times {}_pf_{p+1}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p;\ {}_s\rho_r+1,\ {}_s\rho_1+1,\ {}_s\rho_2+1,\ \ldots,\ {}_s\rho_n+1,\ {}_s\rho_{n+1},$$
$$\ldots * \ldots,\ {}_s\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

where $\xi = n-1$ when $n < r$ and $\xi = n-2$ when $n > r$.

$$F_3 = \frac{(-1)^p}{2}\,\frac{x^2}{z^2}\, y^{2\rho_s - 2\rho_r + 2}\, {}_pf_{p+1}(3-{}_r\alpha_1,\ 3-{}_r\alpha_2,\ \ldots,\ 3-{}_r\alpha_p;$$
$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2 \ldots * \ldots,\ 2-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y^2),$$
$$\times {}_pf_{p+1}({}_s\alpha_1-1,\ {}_s\alpha_2-1,\ \ldots,\ {}_s\alpha_p-1;\ {}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots * \ldots,\ {}_s\rho_{p+1};\ -\tfrac{1}{4}y^2z^2),$$

$$F_4 = \frac{(-1)^p}{2}\,\frac{x^2}{z^2}\, y^{2\rho_s - 2\rho_r + 2} \sum_{n=1}^{p} (-1)^{n-1}$$
$$\times {}_pf_{p-1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_n,\ 3-{}_r\alpha_{n+1},\ \ldots,\ 3-{}_r\alpha_p;$$
$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots * \ldots,\ 2-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_{n-1},\ {}_s\alpha_n-1,\ \ldots,\ {}_s\alpha_p-1;$$
$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots * \ldots,\ {}_s\rho_{p+1};\ -\tfrac{1}{4}y^2z^2).$$

When $r \leqslant p+1$ and $s > p+1$, we have to evaluate

$$I_{r,s} = \left(1-\frac{x^2}{z^2}\right) \int y^{2\rho_s - 2\rho_r + 1}\, {}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p;$$
$$2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\alpha_{s-1};\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)$$
$$= \left(1-\frac{x^2}{z^2}\right) \int y^{2\rho_s - 2\rho_r + 1}\, {}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p;$$
$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots * \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\alpha_{s-1};$$
$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots * \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)\,dy\ ;$$

in the last integral, $2-{}_r\rho_r$ has been placed last and ${}_s\rho_r$ has been placed first in the groups of parameters to which they belong, and the asterisks now denote that the parameters involving ${}_r\rho_r$ and ${}_s\rho_r$ are to be omitted from their natural positions.

We now have

$$\begin{aligned} 3-{}_r\rho_s &= 2-{}_ra_t+{}_sa_t \quad (t=1, 2, 3, \ldots, p) \\ &= 2-{}_r\rho_\tau+{}_s\rho_\tau \quad (\tau=1, 2, 3, \ldots, p+1;\ \tau\neq r) \\ &= 2-{}_r\rho_r+{}_s\rho_s = 1+{}_s\rho_r = 2-{}_r\rho_s+{}_sa_{s-1}, \end{aligned}$$

since, by § 1 . 4, ${}_sa_{s-1}=1$.

Hence the integral now under consideration can be evaluated by the same method as the previous integral, the only difference being that functions of the type ${}_pf_{p+1}$ are replaced by functions of the type ${}_{p+1}f_{p+2}$. The result of carrying out the method is

$$I_{r,s} = G_1+G_2+G_3+G_4 \quad (r\leqslant p+1,\ s>p+1), \tag{4.22}$$

where

$$\begin{aligned} G_1 = {} & \tfrac{1}{2}y^{2\rho_s-2\rho_r+2}\,{}_{p+1}f_{p+2}(2-{}_ra_1,\ 2-{}_ra_2,\ \ldots,\ 2-{}_ra_p,\ 2-{}_r\rho_s; \\ & 2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2) \\ & \times {}_{p+1}f_{p+2}({}_sa_1,\ {}_sa_2,\ \ldots,\ {}_sa_p,\ {}_sa_{s-1}; \\ & {}_s\rho_r+1,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots *\ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2), \\ G_2 = {} & -\tfrac{1}{8}x^2y^{2\rho_s-2\rho_r+4}\sum_{\substack{n=1\\(n\neq r)}}^{p+1}(-1)^{\xi}\,{}_{p+1}f_{p+2}(3-{}_ra_1,\ 3-{}_ra_2,\ \ldots,\ 3-{}_ra_p,\ 3-{}_r\rho_s; \\ & 3-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots, \\ & 2-{}_r\rho_{n-1},\ 3-{}_r\rho_n,\ \ldots,\ 3-{}_r\rho_{p+1},\ 3-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2) \\ & \times {}_{p+1}f_{p+2}({}_sa_1,\ {}_sa_2,\ \ldots,\ {}_sa_p,\ {}_sa_{s-1}; \\ & {}_s\rho_r+1,\ {}_s\rho_1+1,\ \ldots *\ldots,\ {}_s\rho_n+1,\ {}_s\rho_{n+1},\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2) \\ & + \frac{(-1)^{p+1}}{8}x^2y^{2\rho_s-2\rho_r+4}\,{}_{p+1}f_{p+2}(3-{}_ra_1,\ 3-{}_ra_2,\ \ldots,\ 3-{}_ra_p,\ 3-{}_r\rho_s; \\ & 3-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 3-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2) \\ & \times {}_{p+1}f_{p+2}({}_sa_1,\ {}_sa_2,\ \ldots,\ {}_sa_p,\ {}_sa_{s-1}; \\ & \rho_r+1,\ {}_s\rho_1+1,\ {}_s\rho_2+1,\ \ldots *\ldots,\ {}_s\rho_{p+1}+1,\ {}_s\rho_s+1;\ -\tfrac{1}{4}y^2z^2), \end{aligned}$$

where $\xi = n-1$ when $n < r$ and $\xi = n-2$ when $n > r$.

$$G_3 = \frac{(-1)^p}{2}\frac{x^2}{z^2}\,y^{2\rho_s-2\rho_r+2}\,{}_{p+1}f_{p+2}(3-{}_r\alpha_1,\ 3-{}_r\alpha_2,\ \ldots,\ 3-{}_r\alpha_p,\ 3-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+1}f_{p+2}({}_s\alpha_1-1,\ {}_s\alpha_2-1,\ \ldots,\ {}_s\alpha_p-1,\ {}_s\alpha_{s-1}-1;$$

$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots *\ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2),$$

$$G_4 = \frac{(-1)^{p+1}}{2}\frac{x^2}{z^2}\,y^{2\rho_s-2\rho_r+2}\sum_{n=1}^{p}(-1)^{n-1}\,{}_{p+1}f_{p+2}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_n,$$

$$3-{}_r\alpha_{n+1},\ \ldots,\ 3-{}_r\alpha_p,\ 3-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+1}f_{p+2}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_{n-1},\ {}_s\alpha_n-1,\ \ldots,\ {}_s\alpha_p-1,\ {}_s\alpha_{s-1}-1;$$

$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots *\ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)$$

$$-\frac{x^2}{2z^2}\,y^{2\rho_s-2\rho_r+2}\,{}_{p+1}f_{p+2}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p,\ 2-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+1}f_{p+2}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\alpha_{s-1}-1;$$

$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots *\ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2).$$

It must be observed, however, that ${}_s\alpha_{s-1}-1=0$, and that, whenever ${}_s\alpha_{s-1}-1$ occurs as a parameter in a ${}_{p+1}f_{p+2}$, the first term in the series (1.12) by which it is defined is to be omitted†.

The evaluation of $I_{r,s}$, when $r > p+1$ and $s \leqslant p+1$, is almost identical with the problem just discussed. In this case we write

$$I_{r,s} = \left(1-\frac{x^2}{z^2}\right)\int y^{2\rho_s-2\rho_r+1}\,{}_{p+1}f_{p+2}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p,\ 2-{}_r\alpha_{r-1};$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2\ldots{}_r\rho_2,\ \ldots *\ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+1}f_{p+2}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\rho_r;\ {}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots *\ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ \tfrac{1}{4}y^2z^2)\,dy,$$

where $2-{}_r\rho_s$ has been placed first and ${}_s\rho_s$ has been placed last in the groups of parameters to which they belong, and the asterisks now denote that the parameters involving ${}_r\rho_s$ and ${}_s\rho_s$ are to be omitted from their natural positions.

The result of the evaluation is evidently

$$I_{r,s} = G_1'+G_2'+G_3'+G_4' \quad (r > p+1,\ s \leqslant p+1), \tag{4.23}$$

† The necessity for this modification is due to the notation, which automatically introduces superfluous constants involving $\Gamma(0)$ in the course ot the integrations.

where G_t is obtained from G_t $(t = 1, 2, 3, 4)$ by replacing ${}_r\rho_s$ and ${}_s\alpha_{s-1}$ where they occur in G_t by $2-{}_r\alpha_{r-1}$ and ${}_s\rho_r$ respectively.

When $r > p+1$, $s > p+1$, and $r \neq s$, we write

$$I_{r,s} = \left(1-\frac{x^2}{z^2}\right)\int y^{2\rho_s-2\rho_r+1}\,{}_{p+2}f_{p+3}(2-{}_r\alpha_1, 2-{}_r\alpha_2, \ldots, 2-{}_r\alpha_p, 2-{}_r\alpha_{r-1}, 2-{}_r\rho_s;$$
$$2-{}_r\rho_s, 2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{r+1}, 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_{p+2}f_{p+3}({}_s\alpha_1, {}_s\alpha_2, \ldots, {}_s\alpha_p, {}_s\rho_r, {}_s\alpha_{s-1};\ {}_s\rho_r, {}_s\rho_1, {}_s\rho_2, \ldots, {}_s\rho_{p+1}, {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)\,dy,$$

where $2-{}_r\rho_s$ has been inserted in both sets of parameters of the first hypergeometric function and ${}_s\rho_r$ has been inserted in both sets of parameters of the second. Since

$$2-{}_r\alpha_{r-1} = {}_s\alpha_{s-1} = 1,$$

we have

$$\begin{aligned} 1+{}_s\rho_r &= 3-{}_r\rho_s = 2-{}_r\alpha_t+{}_s\alpha_t \quad (t = 1, 2, 3, \ldots, p) \\ &= 2-{}_r\alpha_{r-1}+{}_s\rho_r = {}_s\alpha_{s-1}+2-{}_r\rho_s \\ &= 2-{}_r\rho_\tau+{}_s\rho_\tau \quad (\tau = 1, 2, 3, \ldots, p+1) \\ &= 2-{}_r\rho_r+{}_s\rho_s, \end{aligned}$$

and so $I_{r,s}$ can be evaluated by the same method as before. The result is

$$I_{r,s} = G_1''+G_2''+G_3''+G_4'' \quad (r > p+1,\ s > p+1;\ r \neq s), \tag{4.24}$$

where

$$G_1'' = \tfrac{1}{2}y^{2\rho_s-2\rho_r+2}\,{}_{p+2}f_{p+3}(2-{}_r\alpha_1, 2-{}_r\alpha_2, \ldots, 2-{}_r\alpha_p, 2-{}_r\alpha_{r-1}, 2-{}_r\rho_s;$$
$$2-{}_r\rho_s, 2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}, 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_{p+2}f_{p+3}({}_s\alpha_1, {}_s\alpha_2, \ldots, {}_s\alpha_p, {}_s\rho_r, {}_s\alpha_{s-1};\ {}_s\rho_r+1, {}_s\rho_1, {}_s\rho_2, \ldots, {}_s\rho_{p+1}, {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2),$$

$$G_2'' = -\tfrac{1}{8}x^2y^{2\rho_s-2\rho_r+4}\sum_{n=1}^{p+1}(-1)^{n-1}$$
$$\times\,{}_{p+2}f_{p+3}(3-{}_r\alpha_1, 3-{}_r\alpha_2, \ldots, 3-{}_r\alpha_p, 3-{}_r\alpha_{r-1}, 3-{}_r\rho_s;$$
$$3-{}_r\rho_s, 2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{n-1}, 3-{}_r\rho_n, \ldots, 3-{}_r\rho_{p+1}, 3-{}_r\rho_r;$$
$$-\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_{p+2}f_{p+3}({}_s\alpha_1, {}_s\alpha_2, \ldots, {}_s\alpha_p, {}_s\rho_r, {}_s\alpha_{s-1};$$
$${}_s\rho_r+1, {}_s\rho_1+1, {}_s\rho_2+1, \ldots, {}_s\rho_n+1, {}_s\rho_{n+1}, \ldots, {}_s\rho_{p+1}, {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)$$
$$+\frac{(-1)^p}{8}x^2y^{2\rho_s-2\rho_r+4}\,{}_{p+2}f_{p+3}(3-{}_r\alpha_1, 3-{}_r\alpha_2, \ldots, 3-{}_r\alpha_p, 3-{}_r\alpha_{r-1}, 3-{}_r\rho_s;$$
$$3-{}_r\rho_s, 2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}, 3-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times\,{}_{p+2}f_{p+3}({}_s\alpha_1, {}_s\alpha_2, \ldots, {}_s\alpha_p, {}_s\rho_r, {}_s\alpha_{s-1};$$
$${}_s\rho_r+1, {}_s\rho_1+1, {}_s\rho_2+1, \ldots, {}_s\rho_{p+1}+1, {}_s\rho_s+1;\ -\tfrac{1}{4}y^2z^2),$$

$$G_3'' = \frac{(-1)^{p+1}}{2}\frac{x^2}{z^2}\,y^{2\rho_s-2\rho_r+2}$$

$$\times {}_{p+2}f_{p+3}(3-{}_s\alpha_1,\ 3-{}_s\alpha_2,\ \ldots,\ 3-{}_s\alpha_p,\ 3-{}_s\alpha_{r-1},\ 3-{}_s\rho_s;$$

$$3-\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+2}f_{p+3}({}_s\alpha_1-1,\ {}_s\alpha_2-1,\ \ldots,\ {}_s\alpha_p-1,\ {}_s\rho_r-1,\ {}_s\alpha_{s-1}-1;$$

$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2),$$

$$G_4'' = \frac{(-1)^{p+2}}{2}\frac{x^2}{z^2}\,y^{2\rho_s-2\rho_r+2}\sum_{n=1}^{p}(-1)^{n-1}$$

$$\times {}_{p+2}f_{p+3}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_n,\ 3-{}_r\alpha_{n+1},\ \ldots,\ 3-{}_r\alpha_p,\ 3-{}_r\alpha_{r-1},\ 3-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+2}f_{p+3}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_{n-1},\ {}_s\alpha_n-1,\ \ldots,\ {}_s\alpha_p-1,\ {}_s\rho_r-1,\ {}_s\alpha_{s-1}-1;$$

$${}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)$$

$$+\frac{x^2}{2z^2}\,y^{2\rho_s-2\rho_r+2}\,{}_{p+2}f_{p+3}(2-\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p,\ 2-{}_r\alpha_{r-1},\ 3-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+2}f_{p+3}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\rho_r-1,\ {}_s\alpha_{s-1}-1;\ {}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2)$$

$$-\frac{x^2}{2z^2}\,y^{2\rho_s-2\rho_r+2}\,{}_{p+2}f_{p+3}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p,\ 2-{}_r\alpha_{r-1},\ 2-{}_r\rho_s;$$

$$2-{}_r\rho_s,\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$

$$\times {}_{p+2}f_{p+3}({}_s\alpha_1,\ {}_s\alpha_2,\ \ldots,\ {}_s\alpha_p,\ {}_s\rho_r,\ {}_s\alpha_{s-1}-1;\ {}_s\rho_r,\ {}_s\rho_1,\ {}_s\rho_2,\ \ldots,\ {}_s\rho_{p+1},\ {}_s\rho_s;\ -\tfrac{1}{4}y^2z^2),$$

and any hypergeometric function containing the parameter ${}_s\alpha_{s-1}-1$ is to be modified, as previously, by the omission of its first term.

It is not possible to obtain the formula for the special case $s=r$ from these results by putting $s=r$ in these formulae, because the equations for ${}_r\rho_s$ are symmetrical in form only when $s\neq r$ (cf. § 1.2). The value of $I_{r,r}$ can, however, be found by the same methods. When $r\leqslant p+1$, the result is

$$I_{r,r} = P_1+P_2+P_3 \quad (r\leqslant p+1), \tag{4.25}$$

where

$$P_1 = \frac{y^2}{2} \sum_{n=1}^{p} (-1)^{n-1} {}_pf_{p+1}(3-{}_r\alpha_1, 3-{}_r\alpha_2, \ldots, 3-{}_r\alpha_{n-1}, 2-{}_r\alpha_n, \ldots, 2-{}_r\alpha_p;$$
$$2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_r\alpha_1-1, {}_r\alpha_2-1, \ldots, {}_r\alpha_n-1, {}_r\alpha_{n+1}, \ldots, {}_r\alpha_p;$$
$${}_r\rho_1, {}_r\rho_2, \ldots, {}_r\rho_{p+1}; -\tfrac{1}{4}y^2z^2),$$

$$P_2 = (-1)^p \frac{y^2}{2} \sum_{n=1}^{p+1} (-1)^{n-1} {}_pf_{p+1}(3-{}_r\alpha_1, 3-{}_r\alpha_2, \ldots, 3-{}_r\alpha_p;$$
$$3-{}_r\rho_1, 3-{}_r\rho_2, \ldots, 3-{}_r\rho_n, 2-{}_r\rho_{n+1}, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_r\alpha_1-1, {}_r\alpha_2-1, \ldots, {}_r\alpha_p-1;$$
$${}_r\rho_1-1, {}_r\rho_2-1, \ldots, {}_r\rho_{n-1}-1, {}_r\rho_n, \ldots, {}_r\rho_{p+1}; \tfrac{1}{4}y^2z^2),$$

$$P_3 = \frac{2}{x^2} {}_pf_{p+1}(2-{}_r\alpha_1, 2-{}_r\alpha_2, \ldots, 2-{}_r\alpha_p;$$
$$2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_r\alpha_1-1, {}_r\alpha_2-1, \ldots, {}_r\alpha_p-1;$$
$${}_r\rho_1-1, {}_r\rho_2-1, \ldots, {}_r\rho_{p+1}-1; -\tfrac{1}{4}y^2z^2).$$

To prove this, let $I_{r,r} = \left(1-\frac{x^2}{z^2}\right) J_r,$

so that

$$J_r = \int y\, {}_pf_{p+1}(2-{}_r\alpha_1, 2-{}_r\alpha_2, \ldots, 2-{}_r\alpha_p;$$
$$2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_r\alpha_1, {}_r\alpha_2, \ldots, {}_r\alpha_p; {}_r\rho_1, {}_r\rho_2, \ldots, {}_r\rho_{p+1}; -\tfrac{1}{4}y^2z^2)\, dy.$$

Integrate this by parts p times, using (2 . 32), so that at the s-th partial integration ($s = 1, 2, 3, \ldots, p$) the parameter $2-{}_r\alpha_s$ in the first hypergeometric function is increased by unity and the parameter ${}_r\alpha_s$ in the second hypergeometric function is decreased by unity; we thus get

$$J_r = P_1 + (-1)^p \int y\, {}_pf_{p+1}(3-{}_r\alpha_1, 3-{}_r\alpha_2, \ldots, 3-{}_r\alpha_p;$$
$$2-{}_r\rho_1, 2-{}_r\rho_2, \ldots, 2-{}_r\rho_{p+1}; -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_r\alpha_1-1, {}_r\alpha_2-1, \ldots, {}_r\alpha_p-1; {}_r\rho_1, {}_r\rho_2, \ldots, {}_r\rho_{p+1}; -\tfrac{1}{4}y^2z^2)\, dy.$$

Now integrate by parts $p+1$ times, using (2 . 33), so that at the s-th partial integration ($s = 1, 2, 3, \ldots, p+1$) the parameter $2-{}_r\rho_s$ in the first hypergeometric function is increased by unity and the parameter ${}_r\rho_s$ in the second hypergeometric function is decreased by unity; we thus get

$$J_r = P_1+P_2-\int y\, {}_pf_{p+1}(3-{}_ra_1,\ 3-{}_ra_2,\ \ldots,\ 3-{}_ra_p:$$
$$3-{}_r\rho_1,\ 3-{}_r\rho_2,\ \ldots,\ 3-{}_r\rho_{p+1};\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_pf_{p+1}({}_ra_1-1,\ {}_ra_2-1,\ \ldots,\ {}_ra_p-1;\ {}_r\rho_1-1,\ {}_r\rho_2-1,\ \ldots,\ {}_r\rho_{p+1}-1;\ -\tfrac{1}{4}y^2z^2)\,dy.$$

Finally integrate this by parts, using (2 . 31), so that all the parameters in the first hypergeometric function are decreased by unity and all those in the second are increased by unity; we thus get

$$J_r = P_1+P_2+P_3+\frac{x^2}{z^2}J_r,$$

and this completes the evaluation of $I_{r,r}$ when $r \leqslant p+1$.

When $r > p+1$, we have

$$I_{r,r} = \left(1-\frac{x^2}{z^2}\right)\int y\, {}_{p+1}f_{p+2}(2-{}_ra_1,\ 2-{}_ra_2,\ \ldots,\ 2-{}_ra_p,\ 2-{}_ra_{r-1};$$
$$2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_ra_1,\ {}_ra_2,\ \ldots,\ {}_ra_p,\ {}_ra_{r-1};\ {}_r\rho_1,\ {}_r\rho_2,\ \ldots,\ {}_r\rho_{p+1},\ {}_r\rho_r;\ -\tfrac{1}{4}y^2z^2)\,dy.$$

This can evidently be evaluated by the method used for (4 . 23). The result is

$$I_{r,r} = Q_1+Q_2+Q_3 \quad (r > p+1), \tag{4 . 26}$$

where

$$Q_1 = \frac{y^2}{2}\sum_{n=1}^{p}(-1)^{n-1}\,{}_{p+1}f_{p+2}(3-{}_ra_1,\ 3-{}_ra_2,\ \ldots,\ 3-{}_ra_{n-1},\ 2-{}_ra_n,\ \ldots,\ 2-{}_ra_p,$$
$$2-{}_ra_{r-1};\ 2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_ra_1-1,\ {}_ra_2-1,\ \ldots,\ {}_ra_n-1,\ {}_ra_{n+1},\ \ldots,\ {}_ra_p,\ {}_ra_{r-1};$$
$${}_r\rho_1,\ {}_r\rho_2,\ \ldots,\ {}_r\rho_{p+1},\ {}_r\rho_r;\ -\tfrac{1}{4}y^2z^2)$$
$$+\frac{(-1)^p}{2}\,y^2\,{}_{p+1}f_{p+2}(3-{}_ra_1,\ 3-{}_ra_2,\ \ldots,\ 3-{}_ra_p,\ 2-{}_ra_{r-1};$$
$$2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_ra_1-1,\ {}_ra_2-1,\ \ldots,\ {}_ra_p-1,\ {}_ra_{r-1}-1;$$
$${}_r\rho_1,\ {}_r\rho_2,\ \ldots,\ {}_r\rho_{p+1},\ {}_r\rho_r;\ -\tfrac{1}{4}y^2z^2),$$

$$Q_2 = \frac{(-1)^{p+1}}{2} y^2 \sum_{n=1}^{p+1} (-1)^{n-1} {}_{p+1}f_{p+2}(3-{}_r a_1,\ 3-{}_r a_2,\ \ldots,\ 3-{}_r a_p,\ 3-{}_r a_{r-1};$$
$$3-{}_r\rho_1,\ 3-{}_r\rho_2,\ \ldots,\ 3-{}_r\rho_n,\ 2-{}_r\rho_{n+1},\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_r a_1-1,\ {}_r a_2-1,\ \ldots,\ {}_r a_p-1,\ {}_r a_{r-1};$$
$${}_r\rho_1-1,\ {}_r\rho_2-1,\ \ldots,\ {}_r\rho_{n-1}-1,\ {}_r\rho_n,\ \ldots,\ {}_r\rho_{p+1},\ {}_r\rho_r;\ -\tfrac{1}{4}y^2z^2)$$
$$+ \frac{y^2}{2}\ {}_{p+1}f_{p+2}(3-{}_r a_1,\ 3-{}_r a_2,\ \ldots,\ 3-{}_r a_p,\ 3-{}_r a_{r-1};$$
$$3-{}_r\rho_1,\ 3-{}_r\rho_2,\ \ldots,\ 3-{}_r\rho_{p+1},\ 3-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_r a_1-1,\ {}_r a_2-1,\ \ldots,\ {}_r a_p-1,\ {}_r a_{r-1}-1;$$
$${}_r\rho_1-1,\ {}_r\rho_2-1,\ \ldots,\ {}_r\rho_{p+1}-1,\ {}_r\rho_r;\ -\tfrac{1}{4}y^2z^2),$$

$$Q_3 = \frac{2}{x^2}\ {}_{p+1}f_{p+2}(2-{}_r a_1,\ 2-{}_r a_2,\ \ldots,\ 2-{}_r a_p,\ 2-{}_r a_{r-1};$$
$$2-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r;\ -\tfrac{1}{4}x^2y^2)$$
$$\times {}_{p+1}f_{p+2}({}_r a_1-1,\ {}_r a_2-1,\ \ldots,\ {}_r a_p-1,\ {}_r a_{r-1}-1;$$
$${}_r\rho_1-1,\ {}_r\rho_2-1,\ \ldots,\ {}_r\rho_{p+1}-1,\ {}_r\rho_r-1;\ -\tfrac{1}{4}y^2z^2).$$

It is again to be noted that ${}_r a_{r-1}=1$, and that any hypergeometric function containing the parameter ${}_r a_{r-1}-1$ is to be modified, as previously, by the omission of its first term.

4.3. The lemma corresponding to Lemma 4 of § 2.4 is as follows :—

For every fixed pair of values of x and z such that

$$x > 0, \quad z > 0, \quad x \neq z,$$

$$(xz)^{\frac{1}{2}} \int_0^T \Phi(xy)\,\chi(yz)\,y\,dy = O(1) \text{ as } T \to \infty.$$

The method of proof is the same as that of Lemma 4; no new difficulties are introduced by the increase in generality.

4.4. Next, corresponding to Lemma 5 of § 2.5, we have the lemma

If $x>0$, $z>0$, $T>0$ and $x \neq z$, then

$$(4.41)\quad \left(1-\frac{x^2}{z^2}\right)\int_0^T \Phi(xy)\,\chi(yz)\,y\,dy$$
$$= V_1(x,z)\cos T(x+z) + V_2(x,z)\sin T(x+z)$$
$$+ V_3(x,z)\cos T(x-z) + V_4(x,z)\sin T(x-z)$$
$$+ V_5(x,z)\cos Tx + V_6(x,z)\sin Tx + M(x,z,T)\,T^{-\epsilon}$$

where (i) $V_1(x, z)$, $V_2(x, z)$, ..., $V_6(x, z)$ *are bounded functions of* x *and* z, *independent of* T, (ii) $\epsilon > 0$, *and* (iii) $M(x, z, T)$ *is bounded for all positive (non-zero) values of* x, z *and* T.

The method of proof is identical with the method of proving Lemma 5 of § 2 . 5, but there are several important differences of detail.

For example, consider the contribution of

$$A_r B_s \left(1-\frac{x^2}{z^2}\right) \int_0^T \Phi_r(xy)\, \Psi_s(yz)\, y\, dy = A_r B_s (\tfrac{1}{2}x)^{-\phi_r-\frac{1}{2}} (\tfrac{1}{2}z)^{-\psi_r-\frac{1}{2}} I_{r,s}$$

to the integral on the left of (4 . 41).

When $r \neq s$, the value of this expression is given by (4 . 21), (4 . 22), (4 . 23), or (4 . 24). At the lower limit the terms involving F_n, G_n, G'_n, and G''_n $(n = 1, 2, 3, 4)$ vanish; for, by condition (iv), $2\rho_r > -\phi_0$ $(r = 1, 2, 3, \ldots, \lambda+\mu-2)$, and therefore, by §§ 1 . 2 and 1 . 4, we have

$$2_s\rho_r = 2(\rho_s-\rho_r+1) > -(-2+2\rho_r+\phi_0),$$

so that

$$2(\rho_s-\rho_r+1) = 2_s\rho_r > -\phi_r.$$

Now, by condition (iii), $\phi_r < 0$ for all values of r for which $A_r \neq 0$, and so, for all relevant values of r, $\rho_s-\rho_r+1 > 0$.

When $r = s$, it can be seen from (4 . 25) and (4 . 26) that the contribution of $A_r B_r I_{r,r}$ at the lower limit consists of constant multiples of

$$A_r B_r (\tfrac{1}{2}x)^{-\phi_r-\frac{1}{2}} (\tfrac{1}{2}z)^{-\psi_r-\frac{1}{2}}.$$

Since such terms are not comprised in the expression on the right of (4 . 41), it is necessary to choose the B_r so that they vanish for each value of r $(r = 0, 1, 2, \ldots, \lambda+\mu-1)$ for which $A_r \neq 0$. That is to say, we must have

$$A_r B_r = 0 \quad (r = 0, 1, 2, \ldots, \lambda+\mu-1), \tag{4 . 42}$$

and this is the reason for postulating the set of equations (1 . 53).

The investigation of the contribution of

$$A_r B_s (\tfrac{1}{2}x)^{-\phi_r-\frac{1}{2}} (\tfrac{1}{2}z)^{-\psi_s-\frac{1}{2}} I_{r,s} \tag{4 . 43}$$

to (4 . 41) at the upper limit is similar to the corresponding investigation in § 2 . 5. An appeal to (4 . 42) settles the case when $s = r$, and we can find the contribution when $s \neq r$ by substituting the asymptotic expansions of the hypergeometric functions in (4 . 21), (4 . 22), (4 . 23), and (4 . 24).

A slight difficulty occurs when one (or both) of r and s is greater than $p+1$. We then have to omit the first terms of certain of the hypergeometric functions. For example, a factor of G_4 in (4 . 22) is

$$(4.44)\quad {}_{p+1}f_{p+2}({}_s\alpha_1, {}_s\alpha_2, \ldots, {}_s\alpha_p, {}_s\alpha_{s-1}-1\,;$$

$${}_s\rho_r, {}_s\rho_1, {}_s\rho_2, \ldots * \ldots\,;\ {}_s\rho_{p+1}, {}_s\rho_s\,;\ -\tfrac{1}{4}y^2z^2),$$

in which the first term is to be omitted because ${}_s\alpha_{s-1}-1=0$. Evidently the result of omitting the term is

$$-\tfrac{1}{4}y^2z^2\,{}_{p+2}f_{p+3}({}_s\alpha_1+1, {}_s\alpha_2+1, \ldots, {}_s\alpha_p+1, {}_s\alpha_{s-1}, 1\,;$$

$${}_s\rho_r+1, {}_s\rho_1+1, {}_s\rho_2+1, \ldots * \ldots, {}_s\rho_{p+1}+1, {}_s\rho_s+1, 2\,;\ -\tfrac{1}{4}y^2z^2),$$

and the asymptotic expansion of this function when $y\to\infty$ is obtainable from (2 . 123). The difficulty is overcome in a similar manner wherever it occurs.

When we substitute the asymptotic expansions in the right-hand sides of (4 . 21), (4 . 22), (4 . 23) and (4 . 24), we find, as in § 2 . 5, that the resulting terms can be divided into three groups.

Group 1. Terms which are evidently comprised in the expression on the right of (4 . 41).

Group 2. Terms of the forms $T^j\cos xT$, $T^j\sin xT$, $T^j\,(j\geq 0)$ multiplied by expressions independent of T.

Group 3. Terms which are independent of T.

Group 1 needs no discussion. Terms of Group 2, although they may occur in (4 . 43), cannot occur in

$$(4.45)\quad \left(1-\frac{x^2}{z^2}\right)\int_0^T \Phi(xy)\,\chi(yz)\,y\,dy = \sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1} A_rB_s(\tfrac{1}{2}x)^{-\phi_r-\frac{1}{2}}(\tfrac{1}{2}z)^{-\psi_s-\frac{1}{2}}I_{r,s},$$

for, if they did occur, the lemma of § 4 . 3 would be contradicted. Such terms must consequently cancel with similar terms arising from one or more of the constituents of the expression on the right of (4 . 45), and so they contribute nothing to the value of the integral in (4 . 41). The terms of Group 3 can be calculated just as in the corresponding case of Theorem 1. Their aggregate is a linear combination of multiples of

$$(\tfrac{1}{2}x)^{\frac{3}{2}-\phi_0-2\alpha_n}\sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1}\frac{A_rB_s}{\sin(\rho_r-\alpha_n)\pi\,\sin(\rho_s-\alpha_n)\pi}$$

$$(n=1, 2, 3, \ldots, \lambda+\mu-2),$$

and, if these terms are to make no contribution to the value of the integral in (4 . 41), we must have

$$\sum_{r=0}^{\lambda} \sum_{s=0}^{\lambda+\mu-1} \frac{A_r B_s}{\sin(\rho_r - a_n)\pi \sin(\rho_s - a_n)\pi} = 0 \tag{4.46}$$

$$(n = 1, 2, 3, \ldots, \lambda+\mu-2).$$

The expressions on the left factorize, so that (4 . 46) will certainly be satisfied if

$$\sum_{r=0}^{\lambda} \frac{A_r}{\sin(\rho_r - a_n)\pi} = 0 \quad (n = \lambda, \lambda+1, \ldots, \lambda+\mu-2). \tag{4.47}$$

and

$$\sum_{s=0}^{\lambda+\mu-1} \frac{B_s}{\sin(\rho_s - a_n)\pi} = 0 \quad (n = 1, 2, 3, \ldots, \lambda-1); \tag{4.48}$$

and these are precisely the equations by which

a_n $(n = \lambda, \lambda+1, \ldots, \lambda+\mu-2)$ and B_s $(s = 0, 1, 2, \ldots, \lambda+\mu-1)$

are defined in § 1 . 51. This disposes of the terms of Group 3.

It is to be observed that no other choice of the B_s would ensure the truth of the lemma.

4 . 5. LEMMA. *If z is not a point of the interval (a, b), (where a may be zero or b may be infinite), then*

$$\int_0^\infty \chi(yz)\, y\, dy \int_a^b F(x)\, \Phi(xy)\, x\, dx = 0.$$

The proof is identical with the proof of Lemma 6 (§ 2 . 6).

4 . 6. LEMMA. *The limit*

$$\lim_{T\to\infty} \int_a^b x^{\frac{1}{2}} dx \int_0^T \Phi(xy)\, \chi(yz)\, y\, dy \tag{4.61}$$

exists, provided that either

(i) *a and b are positive constants (independent of x and T), or*

(ii) *$a = a(T)$ and $b = b(T)$ are functions of T which lie in a fixed interval (c_1, c_2) where $0 < c_1 < c_2$, and both $a(T)$ and $b(T)$ tend to limits as $T \to \infty$. Further, either or both of these limits may be equal to z.*

To prove this, substitute the asymptotic expansion of the integrand, and proceed as in § 2 . 7. The only modification is that the equation, corresponding to (2 . 712), which must be established in order to ensure

the existence of the limit (4 . 61), is

$$(4.62) \qquad \sum_{r=0}^{\lambda} \sum_{s=0}^{\lambda+\mu-1} A_r B_s \cos(\phi_0+\rho_r+\rho_s)\pi = 0.$$

The truth of this equation is a consequence of (4 . 47) and (4 . 48), as we now proceed to show. Let

$$\phi' = \sum_{r=1}^{\mu-1} \alpha_{\lambda+r-1} - \sum_{r=1}^{\lambda} \rho_r + \tfrac{1}{2}, \qquad \phi'' = \sum_{r=1}^{\lambda-1} \alpha_r - \sum_{r=1}^{\mu} \rho_{\lambda+r} + \tfrac{1}{2},$$

so that $\phi'+\phi''-\frac{1}{2} = \phi_0$, since $\alpha_{\mu+r} = \rho_{\mu+r-1}$ when $r \geqslant 1$.

It follows that (4 . 62) is true if

$$(4.63) \qquad \sum_{s=0}^{\lambda+\mu-1} B_s \sin(\phi''+\rho_s)\pi \sum_{r=0}^{\lambda} A_r \cos(\phi'+\rho_r)\pi + \sum_{r=0}^{\lambda} A_r \sin(\phi'+\rho_r)\pi \sum_{s=0}^{\lambda+\mu-1} B_s \cos(\phi''+\rho_s)\pi = 0,$$

and this, in its turn, is certainly true if both

$$(4.64) \qquad \sum_{r=0}^{\lambda} A_r \cos(\phi'+\rho_r)\pi = 0$$

and

$$(4.65) \qquad \sum_{s=0}^{\lambda+\mu-1} B_s \cos(\phi''+\rho_s)\pi = 0.$$

To establish the truth of (4 . 64), observe that only μ of the A_r are not zero, and that the determinant obtained by eliminating the A_r between (4 . 64) and the $\mu-1$ equations contained in (4 . 47) vanishes by arguments of the type used in dealing with the determinant of (2 . 713), with which the determinant now under discussion is identical in form.

Similarly, since, by (4 . 42), only λ of the B_s are not zero, we can deduce the truth of (4 . 65) from the $\lambda-1$ equations contained in (4 . 48).

We have therefore established (4 . 62), and have consequently proved that the limit (4 . 61) exists.

4 . 7. Just as in § 3 . 1, we can now prove the following theorem :—

If
$$f(y) = \int_0^{\infty} F(x)\,\Phi(xy)\,x\,dx$$

then
$$D_1(z)\,F(z+0)+D_2(z)\,F(z-0) = \int_0^{\infty} f(y)\,\chi(yz)\,y\,dy,$$

where $D_1(z)$ and $D_2(z)$ depend upon z and the parameters α_r and ρ_s ($r = 1, 2, 3, \ldots, \lambda-1$; $s = 1, 2, 3, \ldots, \lambda$), but not on the form of $F(x)$.

4.8. To determine $D_1(z)$ and $D_2(z)$, we use the method of § 3.2. We can prove an analogue of (3.21), namely

$$2^{\phi_r-\frac{1}{2}}\, y^{-\phi_r-\frac{1}{2}}\, v^{2(2-{}_r\rho_1)}\, {}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p\ ;$$
$$3-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1}\ ;\ -\tfrac{1}{4}v^2y^2)$$
$$= \int_0^v x^{2(2-{}_r\rho_1)+\phi_r-\frac{3}{2}}\, \Phi_r(xy)\, x\, dx.$$

Also, since $4-2_r\rho_1+\phi_r = 2\rho_1+\phi_0$ by §§ 1.2 and 1.4, we have

$$\sum_{r=0}^{p+1} A_r\, 2^{\phi_r-\frac{1}{2}}\, y^{-\phi_r-\frac{1}{2}}\, v^{2(2-{}_r\rho_1)}\, {}_pf_{p+1}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p\ ;$$
$$3-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1}\ ;\ -\tfrac{1}{4}v^2y^2)$$
$$+\sum_{r=k+2}^{\lambda} A_r\, 2^{\phi_r-\frac{1}{2}}\, y^{-\phi_r-\frac{1}{2}}\, v^{2(2-{}_r\rho_1)}\, {}_{p+1}f_{p+2}(2-{}_r\alpha_1,\ 2-{}_r\alpha_2,\ \ldots,\ 2-{}_r\alpha_p,\ 2-{}_r\alpha_{r-1}\ ;$$
$$3-{}_r\rho_1,\ 2-{}_r\rho_2,\ \ldots,\ 2-{}_r\rho_{p+1},\ 2-{}_r\rho_r\ ;\ -\tfrac{1}{4}v^2y^2)$$
$$= \int_0^v x^{2\rho_1+\phi_0-\frac{3}{2}} \sum_{r=0}^{\lambda} A_r\Phi_r(xy)\, x\, dx$$
$$= \int_0^v x^{2\rho_1+\phi_0-\frac{3}{2}}\, \Phi(xy)\, x\, dx.$$

Writing $W(v,\ y)$ for the expression on the left, we now apply the theorem of § 4.7 with

$$F(x) = \begin{cases} x^{2\rho_1+\phi_0-\frac{3}{2}} & (x < v), \\ 0 & (x > v), \end{cases}$$
$$f(y) = W(v,\ y)\,;$$

we then get

$$\int_0^\infty W(v,\ y)\, \chi(yz)\, y\, dy = \begin{cases} \{D_1(z)+D_2(z)\}\, z^{2\rho_1+\phi_0-\frac{3}{2}} & (v > z), \\ D_2(z)\, z^{2\rho_1+\phi_0-\frac{3}{2}} & (v = z), \\ 0 & (v < z)\,; \end{cases}$$

$$\int_0^\infty \left\{W(v,\ y)\, \chi(yz)\, y - \frac{v^{2\rho_1+\phi_0-1}\sin y(v-z)}{\pi y z^{\frac{1}{2}}} \sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1} A_r B_s \cos\tfrac{1}{2}(\phi_r-\psi_s)\,\pi\right\} dy$$
$$= \begin{cases} \{D_1(z)+D_2(z)\}\, z^{2\rho_1+\phi_0-\frac{3}{2}} - \dfrac{v^{2\rho_1+\phi_0-1}}{2z^{\frac{1}{2}}} \displaystyle\sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1} A_r B^s \cos\tfrac{1}{2}(\phi_r-\psi_s)\,\pi & (v > z), \\ D_2(z)\, z^{2\rho_1+\phi_0-\frac{3}{2}} & (v = z), \\ \dfrac{v^{2\rho_1+\phi_0-1}}{2z^{\frac{1}{2}}} \displaystyle\sum_{r=0}^{\lambda}\sum_{s=0}^{\lambda+\mu-1} A_r B_s \cos\tfrac{1}{2}(\phi_r-\psi_s)\,\pi & (v < z). \end{cases}$$

As in § 3 . 2, it can be shown that the integral on the left converges uniformly with respect to both v and z when these have any values greater than zero. Hence the expressions on the right must be continuous functions of v and z, and consequently, since there is no discontinuity at $v = z$,

$$D_1(z) = D_2(z) = \tfrac{1}{2} \sum_{r=0}^{\lambda} \sum_{s\ 0}^{\lambda+\mu-1} A_r B_s \cos \tfrac{1}{2}(\phi_r - \psi_s)\,\pi,$$

and therefore, by §§ 1 . 2 and 1 . 4,

$$D_1(z) = D_2(z) = -\tfrac{1}{2} \sum_{r=0}^{\lambda} \sum_{s=0}^{\lambda+\mu-1} A_r B_s \sin (\phi_0 + \rho_r + \rho_s)\,\pi.$$

This completes the proof of Theorem 2.

5. *Special cases.*

5 . 1. The following theorem is due to Hardy* :—

$$\textit{If}\quad f(y) = \int_0^\infty F(x)\, F_\nu(xy)\, y\, dy, \quad \textit{then}\quad F(z) = \int_0^\infty f(y)\, C_\nu(yz)\, y\, dy,$$

where

$$F_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{1}{2}x)^{\nu+2a+2n}}{\Gamma(a+1+n)\,\Gamma(\nu+a+1+n)},$$

and

$$C_\nu(x) = \cos a\pi J_\nu(x) + \sin a\pi Y_\nu(x) = \frac{\sin (a+\nu)\,\pi J_\nu(x) - \sin a\pi J_{-\nu}(x)}{\sin \nu\pi}.$$

To prove that this theorem is a special case of Theorem 1, we observe that, in the notation of § 1,

$$\Phi_0(x) = F_\nu(x) = (\tfrac{1}{2}x)^{(2a+\nu+\frac{1}{2})-\frac{1}{2}}\, {}_1f_2(1;\ a+1,\ \nu+a+1;\ -\tfrac{1}{4}x^2),$$

and we accordingly take

$$\rho_1 = a+1, \quad \rho_2 = \nu+a+1; \quad \alpha_1 = 1; \quad \phi_0 = -\nu-2a-\tfrac{1}{2};$$
$${}_1\rho_1 = a+1, \quad {}_1\rho_2 = 1-\nu; \quad {}_1\alpha_1 = a+1; \quad \psi_1 = \nu-\tfrac{1}{2};$$
$${}_2\rho_1 = \nu+1, \quad {}_2\rho_2 = \nu+a+1; \quad {}_2\alpha_1 = \nu+a+1; \quad \psi_2 = -\nu-\tfrac{1}{2}.$$

Hence $\Psi_1(x) = (\tfrac{1}{2}x)^{-\nu}\, {}_1f_2(a+1;\ a+1,\ 1-\nu;\ -\tfrac{1}{4}x^2) = J_{-\nu}(x),$

and similarly $\Psi_2(x) = J_\nu(x).$

In accordance with (1 . 22) we have

$$\frac{B_1}{\sin a\pi} + \frac{B_2}{\sin (\nu+a)\,\pi} = 0;$$

---

* See § 1.4, foot-note *, *supra*, 406.

and values of B_1 and B_2 which satisfy this equation are

$$B_1 = -\frac{\sin a\pi}{\sin \nu\pi}, \quad B_2 = \frac{\sin (\nu+a)\,\pi}{\sin \nu\pi}.$$

Adopting these values we get

$$B_1\Psi_1(x)+B_2\Psi_2(x) = C_\nu(x)$$

and

$$B_1 \sin (\phi_0+\rho_1)\,\pi+B_2 \sin (\phi_0+\rho_2)\,\pi = 1.$$

It follows that Hardy's theorem, with the conditions $-\frac{3}{2} < \nu < \frac{3}{2}$ and $-\frac{1}{2} \leqslant 2a+\nu < \frac{5}{2}$, is a special case of Theorem 1.

5.2. The following integral transform is due to Titchmarsh* :—

If

$$h(y) = \int_0^\infty H(x)\,\frac{d}{dy}\{yG_\nu(xy)\}\,x\,dx,$$

then

$$H(z) = \int_0^\infty h(y)\,\Xi_\nu(yz)\,y\,dy,$$

where

$$G_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(\nu+a+n+\frac{1}{2})\,x^{2\nu+2a+2n}}{\Gamma(\frac{1}{2})\,\Gamma(a+n+1)\,\Gamma(\nu+a+n+1)\,\Gamma(2\nu+a+n+1)},$$

and

$$\Xi(x) = \sin a\pi\{J_\nu^2(x)-Y_\nu^2(x)\} - 2\cos a\pi J_\nu(x)\,Y_\nu(x)$$

$$= -\{\sin a\pi J_{-\nu}^2(x) - 2\sin(\nu+a)\pi\,J_{-\nu}(x)\,J_\nu(x) + \sin(2\nu+a)\pi\,J_\nu^2(x)\}\operatorname{cosec}^2\nu\pi.$$

We have

$$\frac{d}{dx}\{xG_\nu(\tfrac{1}{2}x)\} = \frac{2}{\sqrt{\pi}}(\tfrac{1}{2}x)^{2\nu+2a}\,{}_1f_2(1,\ \nu+a+\tfrac{3}{2}\,;\ a+1,\ \nu+a+1,\ 2\nu+a+1\,;\ -\tfrac{1}{4}x^2),$$

and so

$$\rho_1 = a+1,\ \rho_2 = \nu+a+1,\ \rho_3 = 2\nu+a+1\,;\quad a_1 = 1,\ a_2 = \nu+a+\tfrac{3}{2}\,;\ \phi_0 = -2\nu-2a,$$

$$\Phi_0(x) = \left(\frac{2x}{\pi}\right)^{\frac{1}{2}}\frac{d}{dx}\{xG_\nu(\tfrac{1}{2}x)\}.$$

Also

$${}_1\rho_1 = a+1,\ {}_1\rho_2 = \nu+1,\ {}_1\rho_3 = 1-2\nu\,;$$

$${}_1a_1 = a+1,\ {}_1a_2 = \tfrac{1}{2}-\nu,\ \psi_1 = 2\nu-1,$$

whence it follows that

$$\Psi_1(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-\nu}^2(\tfrac{1}{2}x).$$

Similarly $\Psi_2(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-\nu}(\tfrac{1}{2}x)\,J_\nu(\tfrac{1}{2}x)$, $\Psi_3(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_\nu^2(\tfrac{1}{2}x)$.

* *Proc. London Math. Soc.* (2), 24 (1925), vi (*Records* for 14 November, 1924).

In accordance with (1 . 22) we have

$$\frac{B_1}{\sin a\pi}+\frac{B_2}{\sin(\nu+a)\pi}+\frac{B_3}{\sin(2\nu+a)\pi}=0,$$

$$-\frac{B_2}{\cos\nu\pi}-B_2-\frac{B_3}{\cos\nu\pi}=0,$$

and values of B_1, B_2, B_3 which satisfy these equations are

$$B_1=\sin a\pi,\quad B_2=-2\sin(\nu+a)\pi,\quad B_3=\sin(2\nu+a)\pi.$$

Adopting these values we get

$$\sum_{r=1}^{3} B_r\sin(\phi_0+\rho_r)\pi=-2\sin^2\nu\pi.$$

Theorem 1 now assumes the special form :—

If
$$f(y)=\int_0^\infty F(x)\left(\frac{2xy}{\pi}\right)^{-\frac{1}{2}}\frac{d}{dy}\{yG_\nu(\tfrac{1}{2}xy)\}\,x\,dx,$$

then

$$-2\sin^2\nu\pi F(z)=\int_0^\infty f(y)(\tfrac{1}{2}\pi yz)^{\frac{1}{2}}\{\sin a\pi J_{-\nu}^2(\tfrac{1}{2}yz)$$
$$-2\sin(\nu+a)\pi J_{-\nu}(\tfrac{1}{2}yz)J_\nu(\tfrac{1}{2}yz)+\sin(2\nu+a)\pi J_\nu^2(\tfrac{1}{2}yz)\}\,y\,dy.$$

When we write $x=2u$, $z=2v$, $y^{\frac{1}{2}}f(y)=2\pi^{\frac{1}{2}}h(y)$, $v^{-\frac{1}{2}}F(2v)=H(v)$, this transform becomes the transform given at the beginning of this section, with the conditions $-1<\nu<1$ and $0\leqslant\nu+a<1$.

5 . 3. *Integral transforms involving products of Bessel functions.*

Since

$$\pi^{\frac{1}{2}}J_\mu(x)J_\nu(x)=x^{\mu+\nu}\,{}_2f_3\{\tfrac{1}{2}(\mu+\nu+1),\ \tfrac{1}{2}(\mu+\nu+2);$$
$$\mu+\nu+1,\ \mu+1,\ \nu+1,\ -x^2\},$$

we have $\rho_1=\mu+\nu+1$, $\rho_2=\mu+1$, $\rho_3=\nu+1$;

$$a_1=\tfrac{1}{2}(\mu+\nu+1),\quad a_2=\tfrac{1}{2}(\mu+\nu+2);\quad \phi_0=-1-\mu-\nu,$$

from which it is easy to construct the set of integral associates and conjugates given in § 1 . 4.

In consequence of the conditions to which Theorem 1 is subject, it is not possible to obtain an integral transform for

$$f(y)=\int_0^\infty F(x)\,J_\mu(xy)\,J_\nu(xy)\,x\,dx,$$

because $\phi_0+2a_1=0$, whereas condition (vi) requires that*

$$\phi_0+2a_r>\lambda \quad (r=1, 2, 3, \ldots, p),$$

where $0\leqslant\lambda\leqslant 1$.

In accordance with the remark at the end of § 1 . 52, we endeavour to find a linear combination

$$A_0\Phi_0(x)+A_1\Phi_1(x),$$

in whose asymptotic expansion the coefficient of $x^{-\phi_0-2a_r-\frac{1}{2}}$ vanishes when $r=1$, for then the condition $\phi_0+2a_1>\lambda$ may be replaced by the condition $\phi_0+2a_1>\lambda-2$.

For the functions under consideration, the required combination is obtained by taking $A_1=-A_0$, for then the coefficient of $x^{-\frac{1}{2}}$ in the asymptotic expansion vanishes. We then derive the following theorem from Theorem 2 :—

If $$f(y)=\int_0^\infty F(x)\{\Phi_0(xy)-\Phi_1(xy)\}\,x\,dx,$$

then $$-2(\sin\mu\pi+\sin\nu\pi)\,F(z)=\int_0^\infty f(y)\{\Psi_2(yz)+\Psi_3(yz)\}\,y\,dy,$$

provided that

$$-1\leqslant\mu+\nu\leqslant 1, \quad -1<\mu<1, \quad -1<\nu<1.$$

When we simplify the functions $\Phi_1(xy)$, $\Psi_2(yz)$, $\Psi_3(yz)$ by means of substitutions closely resembling the substitutions of § 5 . 2, the theorem assumes the following form :—

If $$f(y)=\pi\int_0^\infty F(x)\{J_{-\mu}(xy)\,J_{-\nu}(xy)-J_\mu(xy)\,J_\nu(xy)\}\,x\,dx,$$

then

$$(\sin\mu\pi+\sin\nu\pi)\,F(z)=\int_0^\infty f(y)\left[\frac{d}{dy}\{yJ_{-\mu}(yz)\,J_\nu(yz)+yJ_\mu(yz)\,J_{-\nu}(yz)\}\right]y\,dy.$$

Similarly, if we apply Theorem 2 to the integral

$$f(y)=\int_0^\infty F(x)\{A_0\Phi_0(xy)+A_2\Phi_2(xy)\}\,x\,dx$$

with $$A_0=\cos\tfrac{1}{2}(\mu+\nu)\,\pi \quad \text{and} \quad A_2=-\cos\tfrac{1}{2}(\mu-\nu)\,\pi,$$

we obtain the following transform :—

If $$\sin\mu\pi\,f(y)=2\pi\int_0^\infty F(x)[\cos\tfrac{1}{2}(\mu+\nu)\,\pi J_\mu(xy) - \cos\tfrac{1}{2}(\mu-\nu)\,\pi\,J_{-\mu}(xy)]J_\nu(xy)\,x\,dx,$$

* For obvious reasons I replace the ν of condition (vi) by λ.

then
$$-\sin\mu\pi F(z)=\int_0^\infty f(y)\frac{d}{dy}\left[\left\{\sin\tfrac{1}{2}(\mu+\nu)\pi J_\mu(yz)+\sin\tfrac{1}{2}(\mu-\nu)\pi J_{-\mu}(yz)\right\}yJ_\nu(yz)\right]y\,dy,$$

provided that $\mu+\nu>-1$, $\mu-\nu<1$, *and* $\nu\geq-1$, *the last condition being a consequence of the other two.*

Another integral transform of the same class is as follows :—

If
$$f(y)=\int_0^\infty F(x)\frac{d}{dx}\left\{xJ_\mu(xy)\,J_\nu(xy)\right\}x\,dx,$$

then
$$\sin\mu\pi\sin\nu\pi F(z)=\pi\int_0^\infty f(y)\left[\sin\mu\pi J_\mu(yz)\,J_{-\nu}(yz)+\sin\nu\pi J_{-\mu}(yz)\,J_\nu(yz)-\sin(\mu+\nu)\pi J_\mu(yz)\,J_\nu(yz)\right]y\,dy,$$

provided that $\mu+\nu>0$, $-2<\mu-\nu<2$.

By taking $\mu=\nu$, some of these results are reduced to known theorems.

ON THE REPRESENTATIONS OF A NUMBER AS THE SUM OF THREE PRODUCTS

By T. ESTERMANN.

[Received 25 April, 1928.—Read 26 April, 1928.]

1. *The problem.*

This paper is a new application of a method invented by Hardy and Littlewood* and simplified by Landau†. It deals with the number of solutions of the equation $xy+zu+vw=n$ in positive integers x, y, z, u, v, w, the integer n being given. Denoting this number of solutions by $\nu(n)$ and the number of positive divisors of n by $d(n)$, we have

$$\nu(n) = \sum_{k+l+m=n} d(k)\,d(l)\,d(m). \tag{1}$$

We put

$$f(x) = \sum_{n=1}^{\infty} d(n)\,x^n \quad (|x|<1). \tag{2}$$

Then, by (1)‡,

$$f^3(x) = \sum_{n=3} \nu(n)\,x^n \quad (|x|<1);$$

hence, by Cauchy's formula,

$$\nu(n) = \frac{1}{2\pi i}\int_C \frac{f^3(x)}{x^{n+1}}\,dx, \tag{3}$$

where C denotes the circle $|x| = e^{-1/n}$.

It lies in the nature of the method that the function $f(x)$ has to be investigated in the neighbourhood of an arbitrary root of unity. This will be done in §§ 3–6.

---

* *Acta Math.*, 44 (1922), 1–70.

† *Vorlesungen über Zahlentheorie*, 1 (1927), Part V.

‡ $f^3(x)$, $\log^m n$, etc., are abbreviations for $[f(x)]^3$, $(\log n)^m$, etc.

§ 7 is of an elementary character. It deals with the coefficients of the four power series

$$\sum_{n=0}^{\infty} a_{m,n} x^n = (1-x)^{-3}[\gamma - \log(1-x)]^m \quad (m = 0, 1, 2, 3),$$

where γ is Euler's constant.

In § 8 I prove the main theorem. The expression which I obtain for $v(n)$ is fairly complicated. The dominant term is $[2\zeta(3)]^{-1}\sigma_2(n)\log^3 n$, where $\sigma_2(n)$ is the sum of the squares of the positive divisors of n. Then follow terms of the orders $n^2\log^2 n$, $n^2\log n$, and n^2; then an error term of order $n^{\frac{3}{2}}\log^3 n$.

2. *Notation.*

1. The letters l, m, n, r denote integers, the letters $h, k, \alpha, \beta, \delta, \kappa, \lambda, \omega$ positive integers.

2. $m \mid n$ means "m is a divisor of n". $m \nmid n$ means "m is not a divisor of n". In $\sum_{m\ n}$, m runs through the positive divisors of n.

3. Always $\delta \mid k$, $\alpha = k/\delta$, $\beta \mid \alpha$, $\lambda = \alpha/\beta$, $\omega \mid \delta$, and $\kappa \mid \lambda^2$.

4. p always denotes a prime.

5. $v(p, m)$ denotes the multiplicity of p in the decomposition of m into prime factors, that is the number defined by

$$p^{v(p,m)} \mid m, \quad p^{v(p,m)+1} \nmid m.$$

6. (m, n) denotes the greatest common divisor of m and n.

7. In the case $n > 0$, $(m, n) = 1$, we define $[m, n]$ by

$$m[m, n] \equiv 1 \pmod{n}, \quad 0 < [m, n] \leqslant n.$$

8. $\chi_m(n)$ denotes a character of residues mod m, $\bar{\chi}_m$ the character conjugate to χ_m, and χ_m^0 the principal character mod m.

9. Two characters χ_m and χ_r will be called equivalent ($\chi_m \sim \chi_r$) if $\chi_m(n) = \chi_r(n)$ for every n prime to mr.

10. In $\sum_{\chi_m}$, χ_m runs through all the characters mod m, in $\sum_{\chi_m}^{(r)}$ through those that are equivalent to characters mod r, and in $\sum_{\chi_m}^{r}$ through those that are equivalent to primitive characters mod r.

11. We put $B(x, \chi_m) = \sum_{r=1}^{m} \chi_m(r) x^r$ and $\xi_m = e^{2\pi i/m}$.

12. $\mu(m)$ denotes Moebius's function* :—

$$\mu(m) = \begin{cases} \prod_{p \mid m} (-1) & (m = \prod_{p \mid m} p) \\ 0 & (m > \prod_{p \mid m} p). \end{cases}$$

13. $\phi(m)$ denotes Euler's function (the number of positive integers prime to m and less than or equal to m).

14. $\mathfrak{R}x$ and $\mathfrak{I}x$ denote the real and imaginary parts of x respectively. We write σ for $\mathfrak{R}s$ and t for $\mathfrak{I}s$.

15. $\log x$ denotes that value of the logarithm for which

$$-\pi < \mathfrak{I} \log x \leqslant \pi, \quad \text{and} \quad x^s = e^{s \log x}.$$

3. *The connection between $f(x)$ and the L functions.*

1. Let

$$x = \xi_k^h e^{-y}, \quad (h, k) = 1, \ \mathfrak{R}y > 0. \tag{4}$$

Then, by (2),

$$f(x) = \sum_{n=1}^{\infty} d(n) \xi_k^{hn} e^{-ny}. \tag{5}$$

Putting

$$D_1(s) = \sum_{n=1}^{\infty} d(n) \xi_k^{hn} n^{-s} \quad (\sigma > 1), \tag{6}$$

we have, by (5) and Mellin's formula,

$$f(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) y^{-s} D_1(s) \, ds. \tag{7}$$

Let

$$D_2(s, \delta) = \sum_{\substack{n=1 \\ (n, k)=\delta}}^{\infty} d(n) \xi_k^{hn} n^{-s} = \sum_{\substack{m=1 \\ (m, \alpha)=1}}^{\infty} d(\delta m) \xi_\alpha^{hm} (\delta m)^{-s} \quad (\sigma > 1). \tag{8}$$

Then, by (6),

$$D_1(s) = \sum_{\delta \mid k} D_2(s, \delta). \tag{9}$$

2. LEMMA 1. *If ρ be an m-th root of unity, $m > 0$, and $(n, m) = 1$, then*

$$B(\rho^n, \chi_m) = \bar{\chi}_m(n) B(\rho, \chi_m).$$

* An empty product means 1, an empty sum 0.

Proof. We have $\bar{\chi}_m(n)\chi_m(n) = 1$; hence, by § 2, 11,

$$B(\rho^n, \chi_m) = \sum_{r=1}^{m} \chi_m(r)\rho^{nr} = \bar{\chi}_m(n) \sum_{r=1}^{m} \chi_m(nr)\rho^{nr}.$$

Now $\chi_m(nr)\rho^{nr}$ depends only on the class of residues mod m to which nr belongs, and nr runs with r through a complete set of residues; hence

$$\sum_{r=1}^{m} \chi_m(nr)\rho^{nr} = \sum_{r=1}^{m} \chi_m(r)\rho^{r} = B(\rho, \chi_m),$$

which proves the lemma

LEMMA 2. *With the above conditions*

$$\phi(m)\rho^n = \sum_{\chi_m} \bar{\chi}_m(n) B(\rho, \chi_m).$$

Proof. By Lemma 1 and § 2, 11,

$$\sum_{\chi_m} \bar{\chi}_m(n) B(\rho, \chi_m) = \sum_{\chi_m} B(\rho^n, \chi_m) = \sum_{r=1}^{m} \rho^{nr} \sum_{\chi_m} \chi_m(r) = \phi(m)\rho^n,$$

since $\sum_{\chi_m} \chi_m(1) = \phi(m)$ and $\sum_{\chi_m} \chi_m(r) = 0$ for $r = 2, 3, \ldots, m$.

3. By Lemma 2 and § 2, 11,

$$\xi_a^{hm} = \frac{1}{\phi(a)} \sum_{\chi_a} \bar{\chi}_a(m) B(\xi_a^h, \chi_a) \quad [(m, a) = 1];$$

hence, by (8),

10) $$D_2(s, \delta) = \frac{\delta^{-s}}{\phi(a)} \sum_{\chi_a} B(\xi_a^h, \chi_a) \sum_{m=1}^{\infty} d(\delta m)\bar{\chi}_a(m) m^{-s} \quad (\sigma > 1).$$

By § 2, 5

$$d(\delta m)\bar{\chi}_a(m) m^{-s} = \prod_p \{[v(p, \delta) + v(p, m) + 1]\bar{\chi}_a(p^{v(p, m)}) p^{-v(p, m)s}\},$$

and

$$\prod_p \sum_{v=0}^{\infty} |[v(p, \delta) + v + 1]\bar{\chi}_a(p^v) p^{-vs}|$$

converges for $\sigma > 1$; hence

11) $$\prod_p \sum_{v=0}^{\infty} \{[v(p, \delta) + v + 1]\bar{\chi}_a(p^v) p^{-vs}\} = \sum_{m=1}^{\infty} d(\delta m)\bar{\chi}_a(m) m^{-s} \quad (\sigma > 1),$$

as can be shown by formal multiplication. Now

$$[1 - \bar{\chi}_a(p) p^{-s}] \sum_{v=0}^{\infty} [v(p, \delta) + v + 1]\bar{\chi}_a(p^v) p^{-vs}$$

$$= \sum_{v=0}^{\infty} [v(p, \delta) + v + 1]\bar{\chi}(p^v) p^{-vs} - \sum_{v=1}^{\infty} [v(p, \delta) + v]\bar{\chi}_a(p^v) p^{-vs}$$

$$= \sum_{v=0}^{\infty} \bar{\chi}_a(p^v) p^{-vs} + v(p, \delta);$$

hence

$$[1-\bar{\chi}_a(p)\,p^{-s}]^2 \sum_{v=0}^{\infty} [v(p,\,\delta)+v+1]\,\bar{\chi}_a(p^v)\,p^{-vs} = 1+v(p,\,\delta)[1-\bar{\chi}_a(p)\,p^{-s}].$$

Hence, since $\prod_p [1-\bar{\chi}_a(p)\,p^{-s}]^{-1} = L(s,\,\bar{\chi}_a)$, by (11)

$$\sum_{m=1}^{\infty} d(\delta m)\,\bar{\chi}_a(m)\,m^{-s} = L^2(s,\,\bar{\chi}_a)\,F(s,\,\delta,\,\chi_a) \quad (\sigma > 1), \tag{12}$$

where

$$F(s,\,\delta,\,\chi_a) = \prod_{p\,|\,\delta} [1+v(p,\,\delta)-v(p,\,\delta)\,\bar{\chi}_a(p)\,p^{-s}]. \tag{13}$$

4. We put

$$D_3(s,\,\chi_a) = \delta^{-s}\,B(\xi_a^h,\,\chi_a)\,L^2(s,\,\bar{\chi}_a)\,F(s,\,\delta,\,\chi_a). \tag{14}$$

Then, by (10) and (12),

$$D_2(s,\,\delta) = \frac{1}{\phi(a)} \sum_{\chi_a} D_3(s,\,\chi_a). \tag{15}$$

The formulae (14), (15), and (9) show that the analytic function $D_1(s)$ exists in the whole plane and cannot have any singularities apart from the point $s=1$. We shall see that it actually has a pole of the second order at that point. We now consider the function $\Gamma(s)\,y^{-s}\,D_1(s)$ [the integrand in (7)] in the strip $-\frac{1}{2} \leqslant \sigma \leqslant 2$. This function is certainly regular in the whole strip apart from the points $s=1$ and $s=0$. At the latter it has a single pole of residue $D_1(0)$ unless $D_1(0)=0$, in which case it would be regular. Now let $t \to \pm\infty$. Then, again by (14), (15), and (9), $\Gamma(s)\,y^{-s}\,D_1(s)$ tends to 0 uniformly in the strip; for so does $\Gamma(s)\,y^{-s}\,L^2(s,\,\bar{\chi}_a)$, and δ^{-s} and $F(s,\,\delta,\,\chi_a)$ are bounded for $\sigma \geqslant -\frac{1}{2}$. Hence, by (7),

$$f(x) = Z+D_1(0)+\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\,y^{-s}\,D_1(s)\,ds, \tag{16}$$

where Z is the residue of the integrand at the point $s=1$.

4. *Calculation of Z.*

1. We put

$$D_0(s) = \sum_{\delta\,|\,k} \frac{1}{\phi(a)}\,D_3(s,\,\chi_a^0). \tag{17}$$

Then, since the L functions to non-principal characters are regular at

the point $s=1$, it follows from (14), (15), and (9) that $D_1(s)-D_0(s)$ is also regular at that point. Hence Z is also the residue of $\Gamma(s)\,y^{-s}D_0(s)$ at the point $s=1$.

2. We have*

$$B(\xi_\alpha^h, \chi_a^0) = \mu(\alpha), \tag{18}$$

$$L(s, \chi_a^0) = \zeta(s) \prod_{p\mid a}(1-p^{-s}), \tag{19}$$

and, by (13),

$$F(s, \delta, \chi_a^0) = \prod_{p\mid a}[1+v(p,\delta)] \prod_{p\nmid a}[1+v(p,\delta)-v(p,\delta)p^{-s}]; \tag{20}$$

hence, by (14) and § 2, 3,

$$D_3(s, \chi_a^0) = k^{-s}\alpha^s\mu(\alpha)\zeta^2(s)\prod_{p\mid a}\{(1-p^{-s})^2[1+v(p,\delta)]\} \times \prod_{p\nmid a}[1+v(p,\delta)-v(p,\delta)p^{-s}]. \tag{21}$$

Let $k' = \prod_{p\mid k} p$. Then, by (21),

$$D_3(s, \chi_a^0) = 0 \quad (\alpha\nmid k'), \tag{22}$$

and in the case $\alpha\mid k'$ we obviously have

$$\alpha^s\mu(\alpha) = \prod_{p\mid a}(-p^s), \quad \phi(\alpha) = \prod_{p\mid a}(p-1), \quad v(p,\delta) = \begin{cases} v(p,k)-1 & (p\mid a), \\ v(p,k) & (p\nmid a). \end{cases} \tag{23}$$

Hence, by (21),

$$\frac{1}{\phi(\alpha)}D_3(s, \chi_a^0) = k^{-s}\zeta^2(s)\prod_{p\mid a}\frac{-p^s(1-p^{-s})^2\,v(p,k)}{p-1} \times \prod_{p\nmid a}[1+v(p,k)(1-p^{-s})] \quad (\alpha\mid k'). \tag{24}$$

By (17), (22), and (24),

$$D_0(s) = k^{-s}\zeta^2(s)\,G(s), \tag{25}$$

where

$$G(s) = \sum_{\alpha\mid k'}\left\{\prod_{p\mid a}\frac{-p^s(1-p^{-s})^2\,v(p,k)}{p-1}\cdot\prod_{p\mid(k'/\alpha)}[1+v(p,k)(1-p^{-s})]\right\} = \prod_{p\mid k}\left[\frac{-p^s(1-p^{-s})^2\,v(p,k)}{p-1}+1+v(p,k)(1-p^{-s})\right].$$

Putting

$$G(s,p) = 1+v(p,k)(1-p^{-s})\left(1-\frac{p^s-1}{p-1}\right), \tag{26}$$

* Landau, *loc. cit.*, Theorem 222.

we have

$$G(s) = \prod_{p \mid k} G(s, p), \tag{27}$$

which implies

$$\frac{G'(s)}{G(s)} = \sum_{p \mid k} \frac{G'(s, p)}{G(s, p)}, \tag{28}$$

the dash indicating differentiation with respect to s. By (26)

$$G(1, p) = 1, \quad G'(1, p) = v(p, k)(1-p^{-1})\frac{-p \log p}{p-1} = -v(p, k) \log p;$$

hence, by (27) and (28),

$$G(1) = 1, \quad G'(1) = -\sum_{p \mid k} v(p, k) \log p = -\log k. \tag{29}$$

3. We are now in a position to find the first two terms of the Laurent expansion of $\Gamma(s) y^{-s} D_0(s)$ for the neighbourhood of the point $s = 1$. Let γ denote Euler's constant. Then

$$\Gamma(s) = 1-\gamma(s-1)+\ldots,$$

$$\zeta^2(s) = (s-1)^{-2}+2\gamma(s-1)^{-1}+\ldots,$$

$$(ky)^{-s} = (ky)^{-1}[1-\log(ky).(s-1)+\ldots],$$

and, by (29),

$$G(s) = 1-\log k\,.(s-1)+\ldots;$$

hence, by (25),

$$\Gamma'(s)\, y^{-s} D_0(s) = \frac{ky^{-1}}{(s-1)^2} + \frac{(ky)^{-1}(\gamma-\log y-2\log k)}{(s-1)} + \ldots.$$

Hence, by § 4, 1,

$$Z = (ky)^{-1}(\gamma-\log y-2\log k). \tag{30}$$

5. *The inequality for* $\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\, y^{-s} D_1(s)\, ds$.

1. Let

$$u(\delta, \omega) = \mu(\omega) \prod_{p \nmid \omega} [1+v(p, \delta)] . \prod_{p \mid \omega} v(p, \delta). \tag{31}$$

Then

$$\sum_{\omega \mid \delta} |u(\delta, \omega)| = \prod_{p \mid \delta} [1+v(p, \delta)+v(p, \delta)] = d(\delta^2), \tag{32}$$

and, by (13),

$$F(s, \delta, \chi_a) = \sum_{\omega \mid \delta} u(\delta, \omega)\, \bar{\chi}_a(\omega)\, \omega^{-s}. \tag{33}$$

Putting

$$D_4(s, \chi_a, \omega) = (\delta\omega)^{-s} \bar{\chi}_a(\omega)\, B(\xi_a^h, \chi_a)\, L^2(s, \bar{\chi}_a), \tag{34}$$

we have, by (14) and (33),

$$D_3(s, \chi_a) = \sum_{\omega \mid \delta} u(\delta, \omega)\, D_4(s, \chi_a, \omega). \tag{35}$$

We now put (in the notation of § 2, 10)

$$D_5(s, \beta) = \sum_{\chi_a}^{\beta} D_3(s, \chi_a), \quad D_6(s, \beta, \omega) = \sum_{\chi_a}^{\beta} D_4(s, \chi_a, \omega). \tag{36}$$

Then, by (35),

$$D_5(s, \beta) = \sum_{\omega \mid \delta} u(\delta, \omega)\, D_6(s, \beta, \omega), \tag{37}$$

and by (15) and § 2, 10,

$$D_2(s, \delta) = \frac{1}{\phi(a)} \sum_{\beta} \sum_{a} D_5(s, \beta). \tag{38}$$

2. The following lemma has been proved by H. Rademacher* :—

LEMMA 3. *Let* $(h, a) = 1$ *and* $(n, \beta) = 1$. *Then (in the notation of* § 2, *sections* 3, 7, *and* 11)

$$\sum_{\substack{1 \le l \le a \\ (l, a) = 1 \\ l \equiv n \pmod{\beta}}} \xi_a^{hl} = \begin{cases} 0 & [(\lambda, \beta) > 1] \\ \mu(\lambda)\, \xi_\beta^{h\,[\lambda, \beta]\, n} & [(\lambda, \beta) = 1]. \end{cases}$$

3. LEMMA 4. *Let* $(h, a) = 1$ *and* $\chi_a \sim \chi_\beta$. *Then*

$$B(\xi_a^h, \chi_a) = \begin{cases} 0 & [(\lambda, \beta) > 1] \\ \mu(\lambda)\, B(\xi_\beta^{h\,[\lambda, \beta]}, \chi_\beta) & [(\lambda, \beta) = 1]. \end{cases}$$

Proof. By § 2, 11 and § 2, 9,

$$B(\xi_a^h, \chi_a) = \sum_{\substack{1 \le r \le a \\ (r, a) = 1}} \chi_\beta(r)\, \xi_a^{hr} = \sum_{n=1}^{\beta} \sum_{\substack{1 \le r \le a \\ (r, a) = 1 \\ r \equiv n \pmod{\beta}}} \chi_\beta(r)\, \xi_a^{hr} = \sum_{\substack{1 \le n \le \beta \\ (n, \beta) = 1}} \chi_\beta(n) \sum_{\substack{1 \le r \le a \\ (r, a) = 1 \\ r \equiv n \pmod{\beta}}} \xi_a^{hr}.$$

Applying Lemma 3 to the last sum, we obtain the above result.

4. Let $\chi_a \sim \chi_\beta$. Then

$$L(s, \bar{\chi}_a) = L(s, \bar{\chi}_\beta) \prod_{p \mid \lambda} [1 - \bar{\chi}_\beta(p)\, p^{-s}];$$

hence, since $(h, a) = 1$ by (4) and § 2, 3, it follows, by (34) and Lemma 4 that

$$D_4(s, \chi_a, \omega) = \begin{cases} 0 & [(a, \omega).(\lambda, \beta) > 1], \\ \mu(\lambda)(\delta\omega)^{-s}\, \bar{\chi}_\beta(\omega)\, B(\xi_\beta^{h\,[\lambda, \beta]}, \chi_\beta)\, L^2(s, \bar{\chi}_\beta) \prod_{p \mid \lambda} [1 - \bar{\chi}_\beta(p)\, p^{-s}]^2 \\ \qquad [(a, \omega) = (\lambda, \beta) = 1]. \end{cases} \tag{39}$$

* *Math. Zeitschrift*, 25 (1926), 629–631.

We now restrict ourselves to the last case, and assume, moreover, that λ is not divisible by squares of primes. So we exclude only the trivial cases in which $D_4(s, \chi_\alpha, \omega) = 0$.

For all numbers κ (which, by § 2, 3, are not divisible by cubes of primes) we define $w(\kappa)$ by

$$w(p) = -2, \quad w(p^2) = 1, \quad w(\kappa) = \prod_{p|\kappa} w(p^{\nu(p,\kappa)}). \tag{40}$$

Then, obviously,

$$\prod_{p|\lambda} [1-\bar{\chi}_\beta(p)\, p^{-s}]^2 = \sum_{\kappa|\lambda^2} w(\kappa)\, \bar{\chi}_\beta(\kappa)\, \kappa^{-s} \tag{41}$$

and

$$\sum_{\kappa|\lambda^3} |w(\kappa)| = \prod_{p|\lambda} [1+|w(p)|+w(p^2)] = \prod_{p|\lambda} 4 = d^2(\lambda). \tag{42}$$

Putting

$$D_7(s, \chi_\beta, \kappa) = \mu(\lambda)(\delta\omega\kappa)^{-s}\, \bar{\chi}_\beta(\omega\kappa)\, B(\xi_\beta^{h[\lambda,\beta]}, \chi_\beta)\, L^2(s, \bar{\chi}_\beta) \tag{43}$$

and

$$D_8(s, \beta, \kappa) = \sum_{\chi_\beta} \beta D_7(s, \chi_\beta, \kappa), \tag{44}$$

we have, by (39) and (41),

$$D_4(s, \chi_\alpha, \omega) = \sum_{\kappa|\lambda^2} w(\kappa)\, D_7(s, \chi_\beta, \kappa) \quad (\chi_\alpha \sim \chi_\beta);$$

hence, by (36) and (44),

$$D_6(s, \beta, \omega) = \sum_{\kappa|\lambda^2} w(\kappa)\, D_8(s, \beta, \kappa). \tag{45}$$

5. The following two lemmas are well known :—

LEMMA 5. *Let χ_β be a primitive character. Then*

$$B(\xi_\beta^{-1}, \chi_\beta)\, B(\xi_\beta, \bar{\chi}_\beta) = |B(\xi_\beta, \bar{\chi}_\beta)|^2 = \beta.$$

LEMMA 6. *Let χ_β be a primitive character. Then*

$$L(s, \bar{\chi}_\beta) = B(\xi_\beta, \bar{\chi}_\beta)\, \frac{(2\pi)^s \beta^{-s}}{2\cos(\frac{1}{2}\pi s)\,\Gamma(s)}\, L(1-s, \chi_\beta)$$

if $\chi_\beta(-1) = 1$, *and*

$$L(s, \bar{\chi}_\beta) = B(\xi_\beta, \bar{\chi}_\beta)\, \frac{(2\pi)^s \beta^{-s}}{2i\sin(\frac{1}{2}\pi s)\,\Gamma(s)}\, L(1-s, \chi_\beta)$$

if $\chi_\beta(-1) = -1$.

6. Let χ_β be a primitive character. Then, by Lemma 1 and Lemma 5,

$$B(\xi_\beta^{h[\lambda,\beta]}, \chi_\beta) B^2(\xi_\beta, \bar{\chi}_\beta) = \bar{\chi}_\beta(-h[\lambda, \beta])\, \beta\, B(\xi_\beta, \bar{\chi}_\beta); \tag{46}$$

hence, by (43) and Lemma 6,

$$D_7(s, \chi_\beta, \kappa) = \mu(\lambda)\, \bar{\chi}_\beta(\omega\kappa h[\lambda, \beta])\, \beta\, B(\xi_\beta, \bar{\chi}_\beta)\, \frac{(2\pi)^{2s}(\delta\omega\kappa\beta^2)^{-s}}{4\cos^2(\frac{1}{2}\pi s)\,\Gamma^2(s)}\, L^2(1-s, \chi_\beta) \tag{47}$$

$$[\chi_\beta(-1) = 1]$$

and

$$D_7(s, \chi_\beta, \kappa) = \mu(\lambda)\, \bar{\chi}_\beta(\omega\kappa h[\lambda, \beta])\, \beta\, B(\xi_\beta, \bar{\chi}_\beta)\, \frac{(2\pi)^{2s}(\delta\omega\kappa\beta^2)^{-s}}{4\sin^2(\frac{1}{2}\pi s)\,\Gamma^2(s)}\, L^2(1-s, \chi_\beta) \tag{48}$$

$$[\chi_\beta(-1) = -1],$$

and we have

$$L^2(1-s, \chi_\beta) = \sum_{n=1}^{\infty} d(n)\, \chi_\beta(n)\, n^{s-1} \quad (\sigma < 0). \tag{49}$$

We still assume that $(\alpha, \omega) = (\lambda, \beta) = 1$, so that, by (4) and § 2, 3, the numbers ω, κ, h, and λ are prime to β. Hence, by § 2, 7,

$$\bar{\chi}_\beta(\omega\kappa h[\lambda, \beta]) = \chi_\beta([\omega\kappa h, \beta]\lambda);$$

hence, if we put

$$\rho_0 = \xi_\beta^{[\omega\kappa h, \beta]\lambda}, \tag{50}$$

by Lemma 1,

$$\chi_\beta(n)\, \bar{\chi}_\beta(\omega\kappa h[\lambda, \beta])\, B(\xi_\beta, \bar{\chi}_\beta) = B(\rho_0^n, \bar{\chi}_\beta) \quad [(n, \beta) = 1]. \tag{51}$$

We now put

$$D_9(s, \chi_\beta, n) = \begin{cases} \mu(\lambda)\, \dfrac{\beta}{n} \left(\dfrac{4\pi^2 n}{\delta\omega\kappa\beta^2}\right)^s \dfrac{B(\rho_0^n, \bar{\chi}_\beta)}{4\cos^2(\frac{1}{2}\pi s)\,\Gamma^2(s)} & [\chi_\beta(-1) = 1], \\ \mu(\lambda)\, \dfrac{\beta}{n} \left(\dfrac{4\pi^2 n}{\delta\omega\kappa\beta^2}\right)^s \dfrac{B(\rho_0^n, \bar{\chi}_\beta)}{4\sin^2(\frac{1}{2}\pi s)\,\Gamma^2(s)} & [\chi_\beta(-1) = -1]. \end{cases} \tag{52}$$

Then, by (47), (48), (49), and (51),

$$D_7(s, \chi_\beta, \kappa) = \sum_{\substack{n=1 \\ (n, \beta)=1}}^{\infty} d(n)\, D_9(s, \chi_\beta, n) \quad (\sigma < 0). \tag{53}$$

Putting

$$D_{10}(s, \beta, n) = \sum_{\chi_\beta}^{\beta} D_9(s, \chi_\beta, n), \tag{54}$$

we have, by (44) and (53),

$$D_8(s, \beta, \kappa) = \sum_{\substack{n=1 \\ (n, \beta)=1}}^{\infty} d(n)\, D_{10}(s, \beta, n) \quad (\sigma < 0). \tag{55}$$

7. We put

$$B_a(\rho, \beta) = \sum_{\chi_\beta(-1)=a}^{\beta} B(\rho, \chi_\beta) \quad (a = \pm 1). \tag{56}$$

Then, by (52) and (54),

$$D_{10}(s, \beta, n) = \mu(\lambda)\, \frac{\beta}{n} \left(\frac{4\pi^2 n}{\delta\omega\kappa\beta^2}\right)^s \Gamma^{-2}(s) \left(\frac{B_1(\rho_0^n, \beta)}{4\cos^2(\frac{1}{2}\pi s)} + \frac{B_{-1}(\rho_0^n, \beta)}{4\sin^2(\frac{1}{2}\pi s)}\right). \tag{57}$$

LEMMA 7. *Let ρ be a primitive β-th root of unity. Then*

$$|B_a(\rho, \beta)| \leqslant \beta \quad (a = \pm 1).$$

Proof. For every positive divisor m of β let

$$g_1(m) = \sum_{\chi_\beta(-1)=a}^{m} B(\rho, \chi_\beta), \qquad g_2(m) = \sum_{\chi_\beta(-1)=a}^{(m)} B(\rho, \chi_\beta). \tag{58}$$

Then, by (56),

$$B_a(\rho, \beta) = g_1(\beta), \tag{59}$$

by § 2, 10 $g_2(m) = \sum_{d \mid m} g_1(d)$, and therefore, by Moebius's inversion formula,

$$g_1(m) = \sum_{d \mid m} \mu\left(\frac{m}{d}\right) g_2(d).$$

Hence, by (59),

$$B_a(\rho, \beta) = \sum_{m \mid \beta} \mu\left(\frac{\beta}{m}\right) g_2(m). \tag{60}$$

By (58) and § 2, 11,

$$g_2(m) = \sum_{r=1}^{\beta} \rho^r \sum_{\chi_\beta(-1)=a}^{(m)} \chi_\beta(r). \tag{61}$$

Now obviously

$$\sum_{\chi_\beta(-1)=a}^{(m)} \chi_\beta(r) = 0 \quad [(r, \beta) > 1],$$

and in the case $(r, \beta) = 1$, $m > 2$,

$$\sum_{\chi_\beta(-1)=a}^{(m)} \chi_\beta(r) = \sum_{\chi_m(-1)=a} \chi_m(r) = \begin{cases} \frac{1}{2}\phi(m) & [r \equiv 1 \pmod m], \\ \frac{1}{2}a\phi(m) & [r \equiv -1 \pmod m], \\ 0 & \text{(otherwise)}; \end{cases}$$

hence, by (61),

$$g_2(m) = \tfrac{1}{2}\phi(m) \sum_{\substack{1\leqq r\leqq\beta\\(r,\beta)=1\\ r\equiv 1 \pmod m}} \rho^r + \tfrac{1}{2}a\phi(m) \sum_{\substack{1\leqq r\leqq\beta\\(r,\beta)=1\\ r\equiv -1 \pmod m}} \rho^r. \tag{62}$$

This is also true (even trivial) if $m \leqq 2$. By (62) and Lemma 3, $|g_2(m)| \leqq \phi(m)$; hence, by (60), $|B_a(\rho, \beta)| \leqq \sum_{m|\beta} \phi(m) = \beta$, which was to be proved.

8. We put

$$z_0 = \frac{4\pi^2}{\delta\omega\kappa\beta^2 y}, \tag{63}$$

and, if z be any number whose real part is positive,

$$H_1(z) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{z^s\,ds}{4\cos^2(\frac{1}{2}\pi s)\Gamma(s)}, \quad H_{-1}(z) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{z^s\,ds}{4\sin^2(\frac{1}{2}\pi s)\Gamma(s)}. \tag{64}$$

Then, by (57),

$$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\,y^{-s} D_{10}(s, \beta, n)\,ds = \mu(\lambda)\frac{\beta}{n} \sum_{a=\pm 1} B_a(\rho_0^n, \beta)\,H_a(nz_0).$$

Hence, if we multiply (55) by $\Gamma(s)y^{-s}$ and integrate term by term (which is permissible) we obtain

$$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\,y^{-s} D_8(s, \beta, \kappa)\,ds = \mu(\lambda)\beta \sum_{\substack{n=1\\(n,\beta)=1}}^{\infty} \frac{d(n)}{n} \sum_{a=\pm 1} B_a(\rho_0^n, \beta)\,H_a(nz_0). \tag{65}$$

Henceforth A_1, A_2, ... denote absolute positive constants.

LEMMA 8. *Let* $\Re z > 0$. *Then*

$$|H_a(z)| < A_1|z|^{-\frac{1}{2}} + |z|e^{-\Re z} \quad (a = \pm 1).$$

Proof. We may suppose $\Im z \leqq 0$, for $|H_a(z)|$ remains unchanged if we replace z by its conjugate. Then $0 \geqq \Im \log z > -\frac{1}{2}\pi$, and

$$|z^s e^{\frac{1}{2}\pi i s}| = |z|^\sigma \exp[-(\tfrac{1}{2}\pi + \Im \log z)\,t] \leqq |z|^\sigma e^{\frac{1}{2}\pi |t|}.$$

Hence

$$\left|\frac{z^s e^{\frac{1}{2}\pi i s}}{\cos(\frac{1}{2}\pi s)}\right| < A_2|z|^{-\frac{1}{2}} \quad (\sigma = -\tfrac{1}{2})$$

and

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(1-s)\,z^s \frac{e^{\frac{1}{2}\pi i s}}{\cos(\frac{1}{2}\pi s)}\,ds\right| < A_2|z|^{-\frac{1}{2}} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} |\Gamma(1-s)\,ds| = A_3|z|^{-\frac{1}{2}}. \tag{66}$$

Now

$$\frac{1}{4\cos^2(\frac{1}{2}\pi s)\,\Gamma(s)} = \frac{\Gamma(1-s)\sin(\pi s)}{4\pi\cos^2(\frac{1}{2}\pi s)} = \frac{\Gamma(1-s)}{2\pi i}\,\frac{i\sin(\frac{1}{2}\pi s)}{\cos(\frac{1}{2}\pi s)}$$

$$= \frac{\Gamma(1-s)}{2\pi i}\left(\frac{e^{\frac{1}{2}\pi i s}}{\cos(\frac{1}{2}\pi s)} - 1\right);$$

hence, by (64),

$$(67)\quad H_1(z) = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(1-s)\,z^s\,\frac{e^{\frac{1}{2}\pi i s}}{\cos(\frac{1}{2}\pi s)}\,ds - \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(1-s)\,z^s\,ds.$$

Now $$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(1-s)\,z^s\,ds = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\Gamma(u)\,z^{1-u}\,du = ze^{-z},$$

and $|ze^{-z}| = |z|e^{-\mathfrak{R}z}$; hence, by (67) and (66),

$$(68)\qquad |H_1(z)| < A_3|z|^{-\frac{1}{2}} + |z|e^{-\mathfrak{R}z}.$$

By (64)

$$H_1(z)+H_{-1}(z) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{z^s\,ds}{\sin^2(\pi s)\,\Gamma(s)} = \frac{1}{\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{\Gamma(1-s)\,z}{\sin(\pi s)}\,ds.$$

Now $|z^s| \leqslant |z|^\sigma e^{\frac{1}{2}\pi|t|}$; hence $|z^s\sin^{-1}(\pi s)| < 2|z|^{-\frac{1}{2}}$ $(\sigma = -\frac{1}{2})$,

and $$|H_1(z)+H_{-1}(z)| < |z|^{-\frac{1}{2}}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}|\Gamma(1-s)\,ds| = A_4|z|^{-\frac{1}{2}}.$$

Hence, by (68), $$|H_{-1}(z)| < (A_3+A_4)|z|^{-\frac{1}{2}} + |z|e^{-\mathfrak{R}z},$$

which completes the proof.

10. Now let

$$(69)\qquad k|y| < 8\sqrt{(\mathfrak{R}y)}.$$

Then, by (63) and § 2, 3,

$$(70)\qquad |z_0 \geqslant \mathfrak{R}z_0 = \frac{4\pi^2\mathfrak{R}y}{\delta\omega\kappa\beta^2|y|^2} \geqslant \frac{4\pi^2\mathfrak{R}y}{k^2|y|^2} > \frac{\pi^2}{16} > \tfrac{1}{2}.$$

Hence $$ne^{(1-n)\mathfrak{R}z_0} \leqslant ne^{\frac{1}{2}(1-n)} < A_5 n^{-\frac{1}{2}} \quad (n \geqslant 1),$$

and, since $|z_0|\mathfrak{K}^{-1}z_0 = |y|\mathfrak{K}^{-1}y$, by Lemma 8,

$$|H_a(nz_0)| < A_1(\tfrac{1}{2}n)^{-\frac{1}{2}} + |z_0|e^{-\mathfrak{R}z_0}A_5 n^{-\frac{1}{2}} < A_6 n^{-\frac{1}{2}}|y|\mathfrak{K}^{-1}y \quad (a = \pm 1).$$

Hence, by (65) and Lemma 7,

$$(71)\qquad \left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(s)\,y^{-s}D_8(s,\beta,\kappa)\,ds\right| < 2A_6\beta^2|y|\mathfrak{K}^{-1}y\sum_{n=1}^{\infty}d(n)\,n^{-\frac{3}{2}}$$

$$= A_7\beta^2|y|\mathfrak{K}^{-1}y.$$

11. By (45), (71), and (42),

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\, y^{-s} D_6(s, \beta, \omega)\, ds\right| < A_7 d^2(\lambda)\, \beta^2 |y|\, \mathfrak{K}^{-1} y\,;$$

hence, by (37) and (32),

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\, y^{-s} D_5(s, \beta)\, ds\right| < A_7 d(\delta^2)\, d^2(\lambda)\, \beta^2 |y|\, \mathfrak{K}^{-1} y\,;$$

hence, by (38) and § 2, 3,

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\, y^{-s} D_2(s, \delta)\, ds\right| < A_7 \phi^{-1}(a)\, d(\delta^2)\, |y|\, \mathfrak{K}^{-1} y \sum_{\beta \mid a} d^2(\lambda)\, \beta^2$$

$$= A_7 a^2 \phi^{-1}(a)\, d(\delta^2) |y|\, \mathfrak{K}^{-1} y \sum_{\lambda \mid a} d^2(\lambda)\, \lambda^{-2} < A_8 a^2 \phi^{-1}(a)\, d(\delta^2)\, |y|\, \mathfrak{K}^{-1} y\,;$$

hence, by (9)

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s)\, y^{-s} D_1(s)\, ds\right| < A_8 |y|\, \mathfrak{K}^{-1} y \sum_{\delta \mid k} a^2 \phi^{-1}(a)\, d(\delta^2). \tag{72}$$

Now $a\phi^{-1}(a) \leqslant k\phi^{-1}(k)$, since $a \mid k$, and

$$\sum_{\delta \mid k} d(\delta^2)\, \delta^{-1} = \prod_{p \mid k} \sum_{m=0}^{v(p,k)} (2m+1)\, p^{-m} \leqslant \prod_{p \mid k} (1-p^{-1})^{-3} = [k\phi^{-1}(k)]^3\,;$$

hence, by § 2, 3,

$$\sum_{\delta \mid k} a^2 \phi^{-1}(a)\, d(\delta^2) \leqslant k^2 \phi^{-1}(k) \sum_{\delta \mid k} d(\delta^2)\, \delta^{-1} \leqslant k\,[k\phi^{-1}(k)]^4. \tag{73}$$

12. LEMMA 9. *Let $k > e^e$. Then $k\phi^{-1}(k) < \log^{A_9} \log k$.*

This lemma is implied in a theorem of Landau*, but the proof of that theorem is complicated, whereas that of Lemma 9 can be given in a few lines.

Proof. Let p_r denote the r-th prime. By an elementary theorem

$$\sum_{r=1}^{u} p_r^{-1} < A_{10} \log \log u \quad (u \geqslant 3).$$

Now $\quad k\phi^{-1}(k) = \prod_{p \mid k} (1-p^{-1})^{-1}$ and $\log[(1-p^{-1})^{-1}] < 2p^{-1}$,

and k is divisible by no more than $2 \log k$ different primes. Hence

$$\log[k\phi^{-1}(k)] < 2 \sum_{p \mid k} p^{-1} \leqslant 2 \sum_{r=1}^{2 \log k} p_r^{-1} < 2A_{10} \log \log (2 \log k)$$

$$< A_9 \log \log \log k \quad (k > e^e),$$

which proves the lemma.

---

* *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (1909), 216–219.

By (72), (73), and Lemma 9,

$$\left|\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\Gamma(s)\,y^{-s}D_1(s)\,ds\right| < A_{11}|y|^{\mathfrak{R}-1}yk\log(2k) \tag{74}$$

for all values of k and y that satisfy (69).

6. *The inequality for* $D_1(0)$.

1. LEMMA 10. *We have*

$$|f(\xi_k^h e^{-k^{-2}})| < A_{12}k\log(2k) \quad [(h,k)=1].$$

Proof. By (2)

$$f(\xi_k^h e^{-k^{-2}}) = \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}(\xi_k^h e^{-k^{-2}})^{mr} = f_1+f_2, \tag{75}$$

where

$$f_1 = \sum_{m=1}^{k^2}\sum_{r=1}^{\infty}(\xi_k^{hm}e^{-k^{-2}m})^r = \sum_{m=1}^{k^2}(\xi_k^{-hm}e^{k^{-2}m}-1)^{-1} \tag{76}$$

and

$$f_2 = \sum_{r=1}^{\infty}\sum_{m=k^2+1}^{\infty}(\xi_k^{hr}e^{-k^{-2}r})^m = \sum_{r=1}^{\infty}e^{-r}(\xi_k^{-hr}e^{k^{-2}r}-1)^{-1}. \tag{77}$$

Now let $k\geqslant 4$. Then $\sin(2\pi/k)\geqslant 4/k$, and it is geometrically obvious that $|\xi_k^n u-1|\geqslant\sin(2\pi/k)$ for any positive u and any n that is not a multiple of k. Hence, in particular, $|\xi_k^{-hr}e^{k^{-2}r}-1|>k^{-1}$ for $1\leqslant r\leqslant k-1$. In the case $r\geqslant k$, on the other hand,

$$|\xi_k^{-hr}e^{k^{-2}r}-1|\geqslant e^{k^{-2}r}-1\geqslant e^{k^{-1}}-1>k^{-1}.$$

Hence, by (77),

$$|f_2|<k\sum_{r=1}^{\infty}e^{-r}<k. \tag{78}$$

By (76)

$$f_1 = \sum_{r=1}^{k}g_r, \tag{79}$$

where

$$g_r = \sum_{m=kr-k+1}^{kr}(\xi_k^{-hm}e^{k^{-2}m}-1)^{-1} = \sum_{n=0}^{k-1}(\xi_k^{hn}e^{k^{-2}(kr-n)}-1)^{-1}. \tag{80}$$

We put

$$a_n = (\xi_k^{hn}e^{k^{-2}(kr-n)}-1)^{-1}, \qquad b_n = (\xi_k^{hn}e^{k^{-1}r}-1)^{-1}. \tag{81}$$

Then, by (80),

$$g_r = \sum_{n=0}^{k-1}a_n = \sum_{n=0}^{k-1}b_n+\sum_{n=1}^{k-1}(a_n-b_n). \tag{82}$$

By (81)

$$\sum_{n=0}^{k-1} b_n = k(e^r-1)^{-1} < 2ke^{-r} \tag{83}$$

and $a_n-b_n = a_n b_n \xi_k^{hn} e^{k^{-1}r}(1-e^{-k^{-2}n})$, so that

$$\sum_{n=1}^{k-1} |a_n-b_n| < ek^{-1} \sum_{n=1}^{k-1} |a_n b_n| \quad (r \leqslant k). \tag{84}$$

Let l be defined by the conditions

$$l \equiv hn \pmod{k}, \quad -\tfrac{1}{2}k < l \leqslant \tfrac{1}{2}k.$$

Then it is obvious that $|\xi_k^{hn} u-1| \geqslant \frac{1}{2}|\xi_k^{hn}-1| = \sin|\pi l/k| \geqslant 2|l|k^{-1}$ for any positive u. Hence, by (81),

$$|a_n| \leqslant \tfrac{1}{2}k\,|l|^{-1}, \quad |b_n| \leqslant \tfrac{1}{2}k|l|^{-1} \quad (l \neq 0). \tag{85}$$

On the other hand,

$$|a_n^{-1}| \geqslant e^{k^{-2}(kr-n)}-1 > k^{-2}(kr-n) > \tfrac{1}{2}k^{-1}r \quad (r \geqslant 2,\ n < k)$$

and $|b_n^{-1}| \geqslant e^{k^{-1}r}-1 > k^{-1}r$; hence

$$|a_n b_n| < 2k^2 r^{-2} \quad (r \geqslant 2,\ n < k). \tag{86}$$

Now $|l|$ is never zero and it does not assume any value more than twice, while n runs from 1 to $k-1$. Hence, applying (86), if $|l| < r$ and (85), if $|l| \geqslant r$ [so that (86) is not applied in the case $r = 1$], we obtain

$$\sum_{n=1}^{k-1} |a_n b_n| < 2 \sum_{l=1}^{r-1} 2k^2 r^{-2} + 2 \sum_{l=r}^{\infty} \tfrac{1}{4}k^2 l^{-2} < 5k^2 r^{-1}.$$

Hence, by (82), (83), and (84), $|g_r| < (2+5e)kr^{-1}$; hence, by (75), (78), and (79),

$$|f(\xi_k^h e^{-k^{-2}})| < A_{13} k \log k \quad (k \geqslant 4).$$

In the case $k < 4$, however,

$$|f(\xi_k^h e^{-k^{-2}})| \leqslant \sum_{n=1}^{\infty} d(n)\, e^{-\frac{1}{2}n} = A_{14} < 2A_{14} k \log(2k).$$

The last two inequalities prove the lemma.

2. $D_1(0)$ is, of course, independent of y. Taking $y = k^{-2}$, which is consistent with (69), we obtain from (4), (16), (30), (74), and Lemma 10,

$$|D_1(0)| < A_{12} k \log(2k) + \gamma k + A_{11} k \log(2k) < A_{15} k \log(2k). \tag{87}$$

The results hitherto attained are expressed in the following lemma :—

LEMMA 11. *Let* $(h, k) = 1$, $k|y| < 8\sqrt{\Re y}$, *and* $x = \xi_k^h e^{-y}$. *We put*

$$Z(k, y) = (ky)^{-1}(\gamma - \log y - 2\log k), \qquad U = f(x) - Z(k, y). \tag{88}$$

Then

$$|U| < A_{16}|y|\Re^{-1} yk \log(2k).$$

This follows from (16), (30), (87), and (74).

7. *The coefficients of certain power series.*

1. In this paragraph I calculate the approximate values of the coefficients $a_{m,n}$ of the four power series

$$\sum_{n=0}^{\infty} a_{m,n} x^n = (1-x)^{-3}[\gamma - \log(1-x)]^m \quad (m = 0, 1, 2, 3). \tag{89}$$

They will be used in the proof of the main theorem.

Obviously

$$a_{m+1,n} = \gamma a_{m,n} + \sum_{r=1}^{n} r^{-1} a_{m,n-r} \tag{90}$$

and*

$$a_{0,n} = \tfrac{1}{2}(n+1)(n+2) = \tfrac{1}{2}n^2 + O(n). \tag{91}$$

Hence

$$a_{1,n} = \tfrac{1}{2}\gamma n^2 + \tfrac{1}{2}\sum_{r=1}^{n} r^{-1}(n-r)^2 + O(n\log n),$$

and, since

$$\begin{aligned}\sum_{r=1}^{n} r^{-1}(n-r)^2 &= n^2 \sum_{r=1}^{n} r^{-1} - 2n\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} r \\ &= n^2\log n + (\gamma - \tfrac{3}{2})n^2 + O(n),\end{aligned} \tag{92}$$

we have

$$a_{1,n} = \tfrac{1}{2}n^2\log n + (\gamma - \tfrac{3}{4})n^2 + O(n\log n). \tag{93}$$

2. By (90) and (93)

$$\begin{aligned}a_{2,n} = \tfrac{1}{2}\gamma n^2\log n + \gamma(\gamma - \tfrac{3}{4})n^2 + \tfrac{1}{2}\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log(n-r) \\ + (\gamma - \tfrac{3}{4})\sum_{r=1}^{n} r^{-1}(n-r)^2 + O(n\log^2 n),\end{aligned} \tag{94}$$

* $f_1(n) = f_2(n) + O[f_3(n)]$ means here that $[f_1(n) - f_2(n)]f_3^{-1}(n)$ is bounded for $n \geqslant 2$.

and we have

$$(95)\quad \sum_{r=1}^{n-1} r^{-1}(n-r)^2 \log(n-r) = \log n \sum_{r=1}^{n-1} r^{-1}(n-r)^2 + \sum_{r=1}^{n-1} r^{-1}(n-r)^2 \log(1-r/n).$$

LEMMA 12. *Let the function $g(x)$ be continuous at the point $x=n$ and continuously differentiable throughout the open interval $0<x<n$, and let $\int_0^n |g'(x)|\,dx$ exist. Then*

$$\left| \sum_{r=1}^{n} g(r) - \int_0^n g(x)\,dx \right| \leqslant \int_0^n |g'(x)|\,dx.$$

Proof. We obtain by partial integration

$$\int_{r-1}^{r} g(x)\,dx = g(x)(x-r+1)\Big|_{r-1}^{r} - \int_{r-1}^{r} g'(x)(x-r+1)\,dx$$

$$= g(r) - \int_{r-1}^{r} g'(x)(x-[x])\,dx.$$

Hence $$\sum_{r=1}^{n} g(r) - \int_0^n g(x)\,dx = \int_0^n g'(x)(x-[x])\,dx,$$

which proves the lemma.

Putting $g_1(n)=0$ and

$$g_1(x) = x^{-1}(n-x)^2 \log(1-x/n) \quad (0<x<n),$$

we have

$$(96)\quad \sum_{r=1}^{n-1} r^{-1}(n-r)^2 \log(1-r/n) = \sum_{r=1}^{n} g_1(r),$$

and $g_1(x)$ satisfies the conditions of Lemma 12.

Let $$g_2(x) = x^{-1}(1-x)^2 \log(1-x).$$

Then $$g_1(x) = n g_2(x/n), \quad \text{and} \quad g_1'(x) = g_2'(x/n);$$

hence

$$(97)\quad \int_0^n g_1(x)\,dx = n\int_0^1 g_1(ny)\,dy = n^2 \int_0^1 g_2(y)\,dy$$

and

$$(98)\quad \int_0^n |g_1'(x)|\,dx = n\int_0^1 |g_2'(y)|\,dy = A_{17} n.$$

We obtain by partial integration

$$\int_0^1 x^{m-1}\log x\,dx = -m^{-1}\int_0^1 x^{m-1}\,dx = -m^{-2} \quad (m>0); \tag{99}$$

hence $$\int_0^1 g_2(y)\,dy = \int_0^1 g_2(1-x)\,dx = \int_0^1 \sum_{m=3}^{\infty} x^{m-1}\log x\,dx = -\sum_{m=3}^{\infty} m^{-2}$$
$$= -(\tfrac{1}{6}\pi^2-\tfrac{5}{4});$$

hence, by (96), Lemma 12, (97), and (98),

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log(1-r/n) = -(\tfrac{1}{6}\pi^2-\tfrac{5}{4})n^2+O(n). \tag{100}$$

By (95), (92), and (100),

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log(n-r) = n^2\log^2 n+(\gamma-\tfrac{3}{2})n^2\log n \tag{101}$$
$$-(\tfrac{1}{6}\pi^2-\tfrac{5}{4})n^2+O(n\log n);$$

hence, by (94) and (92),

$$a_{2,n} = \tfrac{1}{2}n^2\log^2 n+2(\gamma-\tfrac{3}{4})n^2\log n+[2(\gamma-\tfrac{3}{4})^2 \tag{102}$$
$$-\tfrac{1}{2}(\tfrac{1}{6}\pi^2-\tfrac{5}{4})]n^2+O(n\log^2 n).$$

3. By (90) and (102)

$$a_{3,n} = \tfrac{1}{2}\gamma n^2\log^2 n+2\gamma(\gamma-\tfrac{3}{4})n^2\log n+\gamma[2(\gamma-\tfrac{3}{4})^2-\tfrac{1}{2}(\tfrac{1}{6}\pi^2-\tfrac{5}{4})]n^2 \tag{103}$$
$$+\tfrac{1}{2}\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log^2(n-r)+2(\gamma-\tfrac{3}{4})\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log(n-r)$$
$$+[2(\gamma-\tfrac{3}{4})^2-\tfrac{1}{2}(\tfrac{1}{6}\pi^2-\tfrac{5}{4})]\sum_{r=1}^{n} r^{-1}(n-r)^2+O(n\log^3 n),$$

and we have

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log^2(n-r) = \log^2 n\sum_{r=1}^{n} r^{-1}(n-r)^2 \tag{104}$$
$$+2\log n\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log(1-r/n)+\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log^2(1-r/n).$$

Putting $g_3(n)=0$ and

$$g_3(x) = x^{-1}(n-x)^2\log^2(1-x/n) \quad (0<x<n),$$

we have

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2\log^2(1-r/n) = \sum_{r=1}^{n} g_3(r), \tag{105}$$

and $g_3(x)$ satisfies the conditions of Lemma 12.

Let $$g_4(x) = x^{-1}(1-x)^2 \log^2(1-x).$$

Then $$g_3(x) = n g_4(x/n) \quad \text{and} \quad g_3'(x) = g_4'(x/n);$$

hence

$$\int_0^n g_3(x)\,dx = n^2 \int_0^1 g_4(y)\,dy \tag{106}$$

and

$$\int_0^n |g_3'(x)|\,dx = n \int_0^1 |g_4'(y)|\,dy = A_{18} n. \tag{107}$$

We obtain by partial integration and by (99)

$$\int_0^1 x^{m-1} \log^2 x\,dx = -2m^{-1} \int_0^1 x^{m-1} \log x\,dx = 2m^{-3} \quad (m > 0);$$

hence

$$\int_0^1 g_4(y)\,dy = \int_0^1 g_4(1-x)\,dx = \int_0^1 \sum_{m=3}^{\infty} x^{m-1} \log^2 x\,dx = 2 \sum_{m=3}^{\infty} m^{-3}$$
$$= 2\left[\zeta(3) - \tfrac{9}{8}\right];$$

hence, by (105), Lemma 12, (106), and (107),

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2 \log^2(1-r/n) = 2\left[\zeta(3) - \tfrac{9}{8}\right] n^2 + O(n). \tag{108}$$

By (104), (92), (100), and (108),

$$\sum_{r=1}^{n-1} r^{-1}(n-r)^2 \log^2(n-r) = n^2 \log^3 n + (\gamma - \tfrac{3}{2})\, n^2 \log^2 n$$
$$- 2(\tfrac{1}{6}\pi^2 - \tfrac{5}{4})\, n^2 \log n + 2\left[\zeta(3) - \tfrac{9}{8}\right] n^2 + O(n \log^2 n). \tag{109}$$

By (103), (109), (101), and (92),

$$a_{3,n} = \tfrac{1}{2} n^2 \log^3 n + 3(\gamma - \tfrac{3}{4})\, n^2 \log^2 n + \left[6(\gamma - \tfrac{3}{4})^2 - \tfrac{3}{2}(\tfrac{1}{6}\pi^2 - \tfrac{5}{4})\right] n^2 \log n$$
$$+ \left[4(\gamma - \tfrac{3}{4})^3 - 3(\gamma - \tfrac{3}{4})(\tfrac{1}{6}\pi^2 - \tfrac{5}{4}) + \zeta(3) - \tfrac{9}{8}\right] n^2 + O(n \log^3 n). \tag{110}$$

4. We put

$$b_{0,n} = \tfrac{1}{2} n^2, \quad b_{1,n} = \tfrac{1}{2} n^2 \log n + (\gamma - \tfrac{3}{4})\, n^2, \tag{111}$$

$$b_{2,n} = \tfrac{1}{2} n^2 \log^2 n + 3(\gamma - \tfrac{3}{4})\, n^2 \log n + \left[2(\gamma - \tfrac{3}{4})^2 - \tfrac{1}{2}(\tfrac{1}{6}\pi^2 - \tfrac{5}{4})\right] n^2,$$

$$b_{3,n} = \tfrac{1}{2} n^2 \log^3 n + 3(\gamma - \tfrac{3}{4})\, n^2 \log^2 n + \left[6(\gamma - \tfrac{3}{4})^2 - \tfrac{3}{2}(\tfrac{1}{6}\pi^2 - \tfrac{5}{4})\right] n^2 \log n$$
$$+ \left[4(\gamma - \tfrac{3}{4})^3 - 3(\gamma - \tfrac{3}{4})(\tfrac{1}{6}\pi^2 - \tfrac{5}{4}) + \zeta(3) - \tfrac{9}{8}\right] n^2.$$

Then, by (91), (93), (102), and (110),

$$|a_{m,n}-b_{m,n}|<A_{19}n\log^m n \quad (n\geqslant 2;\ m=0,1,2,3). \tag{112}$$

8. *The main theorem.*

1. Henceforth let n be a fixed integer greater than or equal to 4. We return to formula (3), and apply to the circle C the Farey dissection of order $[\sqrt{n}]$, particulars of which are given in Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 217–218. We need only the following facts :—

The circle C is divided into arcs $A(h,k)$, where

$$h\leqslant k\leqslant\sqrt{n} \quad (h,k)=1; \tag{113}$$

and there is exactly one arc $A(h,k)$ for each pair of numbers h, k which satisfy the above conditions. That arc is given by

$$x=\xi_k^h e^{-y}, \quad y=\frac{1}{n}-\vartheta i, \quad -\vartheta_1\leqslant\vartheta\leqslant\vartheta_2, \tag{114}$$

where ϑ_1 and ϑ_2 are certain numbers (depending on n, h, and k) such that

$$\pi k^{-1}n^{-\frac{1}{2}}\leqslant\vartheta_m<2\pi k^{-1}n^{-\frac{1}{2}} \quad (m=1,2). \tag{115}$$

We have, by (3),

$$\nu(n)=\sum_{k=1}^{\sqrt{n}}\sum_{\substack{h\leqslant k\\(h,k)=1}}\frac{1}{2\pi i}\int_{A(h,k)}\frac{f^3(x)}{x^{n+1}}\,dx. \tag{116}$$

2. We now consider the last integral. By (114)

$$-i\int_{A(h,k)}\frac{f^3(x)}{x^{n+1}}\,dx=\int_{-\vartheta_1}^{\vartheta_2}f^3(x)\,x^{-n}\,d\vartheta=\xi_k^{-hn}e\int_{-\vartheta_1}^{\vartheta_2}f^3(x)\,e^{-n\vartheta i}\,d\vartheta. \tag{117}$$

In the notation of Lemma 11,

$$f^3(x)=Z^3(k,y)+3Z^2(k,y)\,U+3Z(k,y)\,U^2+U^3,$$

and obviously $$|Z(k,y)\,U^2|\leqslant|Z^2(k,y)\,U|+|U^3|;$$

hence

$$\left|\int_{-\vartheta_1}^{\vartheta_2}f^3(x)\,e^{-n\vartheta i}\,d\vartheta-\int_{-\vartheta_1}^{\vartheta_2}Z^3(k,y)\,e^{-n\vartheta i}\,d\vartheta\right| \leqslant 6\int_{-\vartheta_1}^{\vartheta_2}|Z^2(k,y)\,U|\,d\vartheta+4\int_{-\vartheta_1}^{\vartheta_2}|U^3|\,d\vartheta. \tag{118}$$

(113), (114), and (115) imply the conditions of Lemma 11. Moreover,

$$\mathfrak{R}^{-1}y = n \text{ by (114)}, \quad \log(2k) \leqslant \log n \text{ by (113)},$$

and $$|\log y| < \big|\log|y|\big| + \tfrac{1}{2}\pi \leqslant \log n + \tfrac{1}{2}\pi \text{ by (114) and (115)}.$$

Hence, by Lemma 11,

$$|U| < A_{16}k|y|n\log n < 8A_{16}\sqrt{n}\log n,$$

and, by (88),

$$|Z(k, y)| < A_{20}|ky|^{-1}\log n; \tag{119}$$

hence, by (115),

$$\int_{-\vartheta_1}^{\vartheta_2} |U^3|\,d\vartheta < 4\pi k^{-1}n^{-\frac{1}{2}}8^3A_{16}^3n^{\frac{3}{2}}\log^3 n = A_{21}k^{-1}n\log^3 n \tag{120}$$

and

$$\int_{-\vartheta_1}^{\vartheta_2} |Z^2(k, y)\,U|\,d\vartheta < 2A_{16}A_{20}^2k^{-1}n\log^3 n\int_0^{2\pi k^{-1}n^{-\frac{1}{2}}} |y|^{-1}d\vartheta. \tag{121}$$

Now, by (114), $|y| > \frac{1}{2}(n^{-1}+\vartheta)$, and, by (113), $n^{-1}+2\pi k^{-1}n^{-\frac{1}{2}} < 8k^{-1}n^{-\frac{1}{2}}$, so that

$$\int_0^{2\pi k^{-1}n^{-\frac{1}{2}}} |y|^{-1}d\vartheta < 2[\log(8k^{-1}n^{-\frac{1}{2}}) - \log(n^{-1})] = 2\log 8 + \log n - 2\log k;$$

hence, by (121),

$$\int_{-\vartheta_1}^{\vartheta_2} |Z^2(k, y)\,U|\,d\vartheta < A_{22}k^{-1}n\log^3 n(6+\log n-2\log k). \tag{122}$$

By (117), (118), (120), and (122),

$$\left|\frac{1}{2\pi i}\int_{A(h,k)}\frac{f^3(x)}{x^{n+1}}\,dx - \xi_h^{-hn}\frac{e}{2\pi}\int_{-\vartheta_1}^{\vartheta_2} Z^3(k, y)\,e^{-n\vartheta i}\,d\vartheta\right| < A_{23}k^{-1}n\log^3 n(1+\log n-2\log k). \tag{123}$$

3. LEMMA 13. *Let* $0 < |y| \leqslant 4$. *Then*

$$|y^{-1} - (1-e^{-y})^{-1}| < A_{24} \tag{124}$$

and

$$|-y^{-1}\log y + (1-e^{-y})^{-1}\log(1-e^{-y})| < A_{24}|\log y| + A_{25}. \tag{125}$$

Proof. (124) is trivial, for the function $y^{-1}-(1-e^{-y})^{-1}$ is regular in the circle $|y| \leqslant 4$. To prove (125), we put

$$g(y) = (1-e^{-y})^{-1}[\log(1-e^{-y}) - \log y]. \tag{126}$$

Then $g(y)$ is regular in the circle $|y| \leqslant 4$ (by § 2, 15 also at the point $y = 0$), so that

$$|g(y)| < A_{25}. \tag{127}$$

By (126)

$$-y^{-1}\log y+(1-e^{-y})^{-1}\log(1-e^{-y}) = [(1-e^{-y})^{-1}-y^{-1}]\log y+g(y),$$

which, together with (124) and (127), proves (125).

4. Let

$$y = n^{-1}-\vartheta i, \quad z = e^{-y}, \tag{128}$$

$$V(k, z) = k^{-1}(1-z)^{-1}[\gamma-\log(1-z)-2\log k], \quad W = Z(k, y)-V(k, z).$$

Then, by (88) and Lemma 13,

$$|W| < A_{24}k^{-1}(\gamma+2\log k+|\log y|)+A_{25}k^{-1} < A_{26}k^{-1}\log n \tag{129}$$

$$(-\pi \leqslant \vartheta \leqslant \pi).$$

Now $Z^3(k, y)-V^3(k, z) = 3Z^2(k, y)\,W-3Z(k, y)\,W^2+W^3$; hence, by (119) and (129), $|Z^3(k, y)-V^3(k, z)| < A_{27}k^{-3}|y|^{-2}\log^3 n$; hence

$$\left|\int_{-\vartheta_1}^{\vartheta_2} Z^3(k, y)\,e^{-n\vartheta i}\,d\vartheta-\int_{-\vartheta_1}^{\vartheta_2} V^3(k, z)\,e^{-n\vartheta i}\,d\vartheta\right| \tag{130}$$

$$< A_{27}k^{-3}\log^3 n\int_{-\infty}^{\infty}(n^{-2}+\vartheta^2)^{-1}\,d\vartheta$$

$$= A_{28}k^{-3}n\log^3 n.$$

By (115), (128), and (113),

$$\left|\int_{-\pi}^{\pi} V^3(k, z)\,e^{-n\vartheta i}\,d\vartheta-\int_{-\vartheta_1}^{\vartheta_2} V^3(k, z)\,e^{-n\vartheta i}\,d\vartheta\right| \leqslant 2\int_{\pi k^{-1}n^{-\frac{1}{2}}}^{\pi} |V^3(k, z)|\,d\vartheta \tag{131}$$

$$< A_{29}k^{-3}\log^3 n\int_{\pi k^{-1}n^{-\frac{1}{2}}}^{\pi}\vartheta^{-3}\,d\vartheta < A_{29}k^{-1}n\log^3 n.$$

Now, by (128),

$$\frac{e}{2\pi}\int_{-\pi}^{\pi} V^3(k, z)\,e^{-n\vartheta i}\,d\vartheta = \frac{1}{2\pi i}\int_C V^3(k, z)\,z^{-n-1}\,dz, \tag{132}$$

and the right-hand side is the coefficient of z^n in the power series for

$V^3(k, z)$. By (128)*

$$V^3(k, z) = k^{-3}(1-z)^{-3} \sum_{m=0}^{3} \binom{3}{m} [\gamma - \log(1-z)]^{3-m}(-2\log k)^m;$$

hence, by (132) and (89),

$$\frac{e}{2\pi}\int_{-\pi}^{\pi} V^3(k, z)\, e^{-n\vartheta i}\, d\vartheta = k^{-3} \sum_{m=0}^{3} \binom{3}{m} a_{3-m,n}(-2\log k)^m;$$

hence, by (112) and (113),

$$\left| \frac{e}{2\pi}\int_{-\pi}^{\pi} V^3(k, z)\, e^{-n\vartheta i}\, d\vartheta - k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right| < 8A_{19} k^{-3} n \log^3 n. \tag{133}$$

By (123), (130), (131), and (133),

$$\left| \frac{1}{2\pi i}\int_{A(h,k)} \frac{f^3(x)}{x^{n+1}}\, dx - \xi_k^{-hn} k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right| < A_{30} k^{-1} n \log^3 n (1+\log n - 2\log k). \tag{134}$$

5. We put, with Ramanujan†,

$$\sum_{\substack{h \le k \\ (h,k)=1}} \xi_k^{-hn} = c_k(n). \tag{135}$$

Then, by (134),

$$\left| \sum_{\substack{h \le k \\ (h,k)=1}} \frac{1}{2\pi i}\int_{A(h,k)} \frac{f^3(x)}{x^{n+1}}\, dx - c_k(n)\, k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right| < A_{30} n \log^3 n (1+\log n - 2\log k). \tag{136}$$

Now

$$\sum_{k=1}^{\sqrt{n}} \log k > \int_1^{[\sqrt{n}]} \log u\, du = [\sqrt{n}]\log[\sqrt{n}] + 1 - [\sqrt{n}] > \tfrac{1}{2}[\sqrt{n}]\log n - [\sqrt{n}];$$

hence $\sum_{k=1}^{\sqrt{n}} (1+\log n - 2\log k) < 3[\sqrt{n}]$; hence, by (116) and (136),

$$\left| \nu(n) - \sum_{k=1}^{\sqrt{n}} c_k(n)\, k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right| < 3A_{30} n^{\frac{3}{2}} \log^3 n. \tag{137}$$

* 0^0 is taken to mean 1.

† *Trans. Camb. Phil. Soc.*, 22 (1918), 259.

By (111) and (135)

$$\left| \sum_{k=[\sqrt{n}]+1}^{\infty} c_k(n)\, k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right|$$

$$< A_{31} n^2 \sum_{m=0}^{3} \log^{3-m} n \sum_{k=[\sqrt{n}]+1}^{\infty} k^{-2} \log^m k$$

$$< A_{32} n^{\frac{3}{2}} \log^3 n\,;$$

hence, by (137),

$$\left| r(n) - \sum_{k=1}^{\infty} c_k(n)\, k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m \right| < A_{33} n^{\frac{3}{2}} \log^3 n. \tag{138}$$

6. We put $F(s) = \sum_{k=1}^{\infty} c_k(n) k^{-s}$ $(\sigma > 1)$, and denote the m-th derivative of $F(s)$ by $F^{(m)}(s)$, the 0-th derivative meaning the function itself. Then

$$\sum_{k=1}^{\infty} c_k(n)\, k^{-3} \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}(-2\log k)^m = \sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}\, 2^m F^{(m)}(3). \tag{139}$$

We put, as usual, $\sigma_u(n) = \sum_{d \mid n} d^u$, and denote, by $\sigma_u^{(r)}(n)$, the r-th derivative of $\sigma_u(n)$ with respect to u :

$$\sigma_u^{(r)}(n) = \sum_{d \mid n} d^u \log^r d \quad (r \geqslant 0). \tag{140}$$

It has been proved by Ramanujan* and Hardy† that

$$F(s) = \zeta^{-1}(s)\, \sigma_{1-s}(n)\,;$$

hence

$$F^{(m)}(3) = \sum_{r=0}^{m} \binom{m}{r} a_{m-r} (-1)^r \sigma_{-2}^{(r)}(n), \tag{141}$$

where a_{m-r} is the $(m-r)$-th derivative of $\zeta^{-1}(s)$ for $s = 3$. By (111) and (141)

$$\sum_{m=0}^{3} \binom{3}{m} b_{3-m,n}\, 2^m F^{(m)}(3) = n^2 \sum_{m=0}^{3} \log^{3-m} n \sum_{r=0}^{m} c_{m,r}\, \sigma_{-2}^{(r)}(n), \tag{142}$$

where the constants $c_{m,r}$ can easily be calculated; in particular

$$c_{0,0} = \frac{1}{2\zeta(3)}. \tag{143}$$

* *Loc. cit.*, 264 (7 . 2).

† *Proc. Camb. Phil. Soc.*, 20 (1920), 266.

From (138), (139), (142), and (143), we obtain

The main theorem.

There are certain constants $c_{1,0}, \ldots, c_{3,3}$ such that

$$\nu(n) = \frac{1}{2\zeta(3)} n^2 \log^3 n \sigma_{-2}(n) + n^2 \sum_{m=1}^{3} \log^{3-m} n \sum_{r=0}^{m} c_{m,r} \sigma_{-2}^{(r)}(n) + O(n^2 \log^3 n).$$

Since $n^2 \sigma_{-2}(n) = \sigma_2(n)$, and since the functions $\sigma_{-2}^{(r)}(n)$ are bounded, we have

$$\nu(n) = \frac{1}{2\zeta(3)} \sigma_2(n) \log^3 n + O(n^2 \log^2 n).$$

A similar formula has been proved in an elementary way by Ingham* for the number of solutions of the equation $xy + zu = n$.

* *Journal London Math. Soc.*, 2 (1927), 207–208.

ON "RIEMANN" INTEGRATION WITH RESPECT TO AN ADDITIVE FUNCTION OF SETS

By ROSALIND CECILY YOUNG.

[Received 7 March, 1928.—Read 8 March, 1928.]

IN the present note I consider nominally sets of points of the field of Borel sets in space of n dimensions. The only idea, however, special to this type of set, actually involved is that of the *distance* between two elements (points), on which is based that of the span of a set, of the norm of a subdivision, as well as that of point of accumulation of a set and the consequent ideas of closed and open sets relatively to a containing set. The characteristic notions of product, sum, difference, limits* of sets are not geometrical, and are common to all types of sets. The field of sets considered may be restricted if desired, provided that we maintain the possibility of forming subdivisions of arbitrarily small norm of every set; of finding subsets of arbitrarily small span containing any element of any given set; and of proceeding to the limit of any sequence of sets, without leaving the field.

1. The term "additive function of sets" is employed as an abbreviation to designate a finite function of the variable set E, say $g(E)$, possessing the two properties of *additiveness* and *continuity*, in the special sense that

(1) If E be expressed in any manner as the *sum* of a finite number of subsets $E_1, E_2, \ldots, E_q$, no two of which have a point in common (or, as we shall say, *mutually distinct*), we have

$$g(E) = g(E_1)+g(E_2)+\ldots+g(E_q).$$

* Notions of upper and lower limits of a succession of sets are due (under the names of "ensemble limite complet, ensemble limite restreint") to Borel, *Fonctions des variables réelles* (Paris, 1905), 18.

(2) If E be expressed in any manner as the *limit* of a sequence of sets E_n, we have invariably

$$g(E) = \lim_{n\to\infty} g(E_n).$$

The finiteness of the function then involves its boundedness and hence its expression as the difference of two non-negative functions, also possessing the properties (1) and (2). This expression is insured as soon as we possess a non-negative function $g_1(E)$ satisfying the conditions (1) and (2), and nowhere less than $g(E)$; and such a function, in fact the smallest of all possible such functions everywhere, is defined by writing

$$g_1(E) = \text{upper bound of } g(e) \text{ in } E,$$

which, by (1) and (2), is necessarily greater than or equal to zero, and which, by a simple reasoning, now classical, again possesses the properties (1) and (2).

The expression of $g(E)$ as the difference of the two special non-negative additive functions $g_1(E)$ and $g_2(E) = g_1(E) - g(E)$ is then called its *canonical expression.*

2. A finite number q of sets E_i, mutually distinct, is said to form a *subdivision* of their sum E_0. The sets E_i will be called the *units* of the subdivision. The *span* of a set being defined as the upper bound of the distance between any two of its points, a *norm* of a subdivision is a number greater than the spans of all its units. A succession of subdivisions of a set E_0, of norms diminishing to zero, will be called a *sequence of subdivisions of* E_0.

[NOTE.—When a general theory of non-additive functions of sets is contemplated, it is necessary to regard a "subdivision" in the above sense as a new entity, and to designate it by a special symbol*: The characteristic process of such a theory consists, in fact, in forming the sum of the values of the given function for the units of any "subdivision" (thus defining properly an additive function of the variable subdivision) and in comparing this with the value for the sumset of these units. The subject of the present note involves only a special type of these *functions of subdivisions* (viz. Riemann and Darboux sums), and, starting from an additive function of sets which serves to construct these functions, is directed immediately to the deduction of an additive function from them.

* Cf. my memoir "On functions of Σ defined by addition, etc.", *Math. Zeitschrift* (1928), 29 (2/3), 171–216.

There is therefore no call to dwell unduly on the theory of non-additive functions of sets; an occasional circumlocution will suffice to make matters clear to any reader.]

3. Given a function of the variable point P, say $\xi(P)$, the process* of "Riemann" integration of $\xi(P)$ with respect to the additive function of sets $g(E)$, for a given set E_0, consists in forming, for each subdivision of E_0, the product of the value of $g(E)$ for a unit of the subdivision by an arbitrary value of $\xi(P)$ in that unit, summing over the various units of the subdivision, and then considering the limits of these sums [called the *Riemann sums* relative to $\xi(P)$ and $g(E)$ for the corresponding subdivisions], for all possible sequences of subdivisions of E_0. The greatest and least of these limits, when all possible choices of values of $\xi(P)$ are also taken into account, are called the *upper and lower integrals of* $\xi(P)$ *with respect to* $g(E)$, or, when they coincide and are finite, its "Riemann" integral with respect to $g(E)$ for the set E_0. We denote them by

$$\overline{\int}_{E_0} \xi(P)\, dg, \qquad \underline{\int}_{E_0} \xi(P)\, dg.$$

4. (*a*) If e be any subset of E_0, of span less than δ, we can take e to be a unit of a subdivision of E_0 of norm δ; the additional part of this subdivision may, in fact, be any subdivision of $E_0 - e$ of norm δ. Let S' be any one of the Riemann sums relative to $\xi(P)$ and $g(E)$ for this subdivision of $E_0 - e$.

Suppose, if possible, that $\xi(P)$ is unbounded in e and that $g(e) \neq 0$. We can then choose in e a value ξ_e of $\xi(P)$ larger in absolute value than

$$\frac{|S'| + (1/\delta)}{|g(e)|}.$$

The expression $$S = \xi_e g(e) + S'$$

is now a Riemann sum relative to $\xi(P)$ and $g(E)$ for a subdivision of E_0 of norm δ. Also

$$|S| \geqslant \big| |\xi_e g(e)| - |S'| \big| \geqslant \frac{1}{\delta}.$$

* I notice that Radon, in a memoir "Ueber die absoluten additiven Mengenfunktionen", *Sitzungsber. der k. Akad. d. Wissensch. in Wien*, 122, *math.-naturw. Kl.* (1913), 1322–4, considered a process of this type, for a uniformly continuous integrand. He does not deduce any of its properties and appears indeed to introduce it only to aid in the comprehension of the more general process of Lebesgue type which constitutes the chief subject of the memoir.

It follows that if, for every δ, however small, there exists a subset of E_0, of span less than δ, in which $\xi(P)$ is unbounded and for which $g(e) \neq 0$, then there are sequences of subdivisions of E_0 for which the Riemann sums relative to $\xi(P)$ and $g(E)$ diverge to $\pm\infty$; *i.e.* the upper and lower integrals of $\xi(P)$ with respect to $g(E)$ for E_0 cannot both be finite. Thus we have the statement :—

If the upper and lower integrals of $\xi(P)$ with respect to $g(E)$ for a given set E_0 are finite, $g(e)$ must vanish for all subsets of E_0, of span less than some positive δ_0, in which $\xi(P)$ is unbounded.

It follows that *$g(E)$ must vanish identically in each such subset e.* For, if e' be any subset of e, $\xi(P)$ is unbounded in e' or in the complementary subset $e-e'$ of e; either $g(e')$ or $g(e-e')$ must therefore vanish, hence both, since

$$g(e')+g(e-e') = g(e) = 0.$$

(*b*) Let us now assume that the above condition, necessary for the finiteness of the upper and lower integrals of $\xi(P)$ with respect to $g(E)$, is fulfilled for the given set E_0.

Let O_δ be the sum of all the subsets of E_0, of spans less than δ, in which $\xi(P)$ is unbounded; in the complementary subset, E_0-O_δ, $\xi(P)$ is then bounded. O_δ decreases with δ; and, in O_{δ_0}, $g(e)$ vanishes identically.

Let δ_1 be chosen less than $\frac{1}{2}\delta_0$. Every subset of E_0, of span less than δ_1, containing a point of O_{δ_1}, is a subset of O_{δ_0}; for that point is contained in a subset of E_0 of span less than δ_1 in which $\xi(P)$ is unbounded, and by adding the given subset we obtain a subset of E_0, of span less than $2\delta_1 < \delta_0$, in which $\xi(P)$ is unbounded, *i.e.* a subset of O_{δ_0}.

Let $\xi_1(P)$ be equal to $\xi(P)$ in O_{δ_1} and zero elsewhere. Form any subdivision of E_0 of norm less than δ_1. Since the units of the subdivision which contain points of O_{δ_1} are all subsets of O_{δ_0}, $g(e) = 0$ for each such unit. Their contribution to the Riemann sum relative to $\xi_1(P)$ and $g(E)$ is thus zero. The remaining units contribute nothing to this sum, since they contain no point of O_{δ_1}, the only set in which $\xi_1(P)$ differs from zero.

Thus every Riemann sum relative to $\xi_1(P)$ and $g(E)$, for subdivisions of E_0 of norm less than δ_1, is zero. In particular, their limit for sequences of subdivisions of E_0 is always zero : the upper and lower integrals of $\xi_1(P)$ with respect to $g(E)$ for E_0 are zero.

The function $\xi_2(P) = \xi(P)-\xi_1(P)$, equal to $\xi(P)$ outside O_{δ_1} and zero elsewhere, is bounded. Its upper and lower integrals for E_0 are certainly finite. The necessary condition for the finiteness of these upper and lower integrals of $\xi(P)$ for E_0 is thus also sufficient, and we have the following theorems.

THEOREM Ia. *A necessary and sufficient condition for a function $\xi(P)$ to have finite upper and lower integrals with respect to $g(E)$ for a given set E_0 is that $g(E)$ should vanish identically in every subset of E_0, of span less than some positive δ_0, in which $\xi(P)$ is unbounded.*

THEOREM Ib. *A necessary and sufficient condition for a function $\xi(P)$ to have finite upper and lower integrals with respect to $g(E)$ for E_0 is that $\xi(P)$ should be the sum of a bounded function and of a function whose upper and lower integrals with respect to $g(E)$ for E_0 vanish.*

5. We proceed to investigate the problem of "Riemann" integration of bounded functions.

The upper and lower integrals of a bounded function $\xi(P)$ with respect to an additive function $g(E)$ for the variable set E are finite functions of E with only half the property of additiveness. In fact, if $E_1, E_2, \ldots, E_q$ are mutually distinct,

$$(\alpha) \qquad \overline{\int}_{E_1+E_2+\ldots+E_q} \xi(P)\,dg \geqslant \overline{\int}_{E_1} \xi(P)\,dg + \overline{\int}_{E_2} \xi(P)\,dg + \ldots + \overline{\int}_{E_q} \xi(P)\,dg.$$

$$(\beta) \qquad \underline{\int}_{E_1+E_2+\ldots+E_q} \xi(P)\,dg \leqslant \underline{\int}_{E_1} \xi(P)\,dg + \underline{\int}_{E_2} \xi(P)\,dg + \ldots + \underline{\int}_{E_q} \xi(P)\,dg.$$

Only when they coincide do they define an additive function.

In order to investigate their coincidence for a particular given set E_0 we introduce the following functions of E, which are also non-additive:

$$\bar{\phi}(E) = \begin{cases} \bar{\xi}_E\, g(E) & \text{if} \quad g(E) \geqslant 0, \\ \underline{\xi}_E\, g(E) & \text{if} \quad g(E) < 0; \end{cases}$$

$$\underline{\phi}(E) = \begin{cases} \underline{\xi}_E\, g(E) & \text{if} \quad g(E) \geqslant 0, \\ \bar{\xi}_E\, g(E) & \text{if} \quad g(E) < 0; \end{cases}$$

$$\Omega(E) = \bar{\phi}(E) - \underline{\phi}(E) = (\bar{\xi}_E - \underline{\xi}_E)\,|\,g(E)\,| = \omega_E\,|\,g(E)\,|,$$

where $\bar{\xi}_E$ and $\underline{\xi}_E$ are the upper and lower bounds of $\xi(P)$ in E, and $\omega_E = \bar{\xi}_E - \underline{\xi}_E$ is the *oscillation* of $\xi(P)$ in E.

The sum of the values of $\bar{\phi}$ and $\underline{\phi}$ for the units of a subdivision of E_0 are called the *upper and lower Darboux sums* relative to $\xi(P)$ and $g(E)$ for that subdivision of E_0. It is obvious that we can form Riemann sums relative to $\xi(P)$ and $g(E)$, for the same subdivision of E_0, as near as we please in value to either of these Darboux sums, and, on the other hand, that every Riemann sum lies between the two Darboux sums. Accordingly,

The upper and lower integrals of a bounded $\xi(P)$ with respect to $g(E)$ for E_0 may be obtained as the greatest limit of the upper Darboux sums and the least limit of the lower Darboux sums for all sequences of subdivisions of E_0.

The difference between these limits is not less than the greatest limit of the differences between the Darboux sums, *i.e.* of the sums of $\Omega(E)$ for the units of the subdivisions.

These differences are non-negative, and it is easy to find a condition for their limit to be zero, *i.e.* a necessary condition for the coincidence of the upper and lower integrals of $\xi(P)$ with respect to $g(E)$ for E_0. Later we shall see that the condition is also sufficient. We obtain it as follows :—

Since* we can draw a network of "planes", parallel to the $(n-1)$-dimensional coordinate "planes", and at distances less than δ apart, in each of which $g(e)$ vanishes identically, we can determine a subdivision, of norm δ, of all space into *open intervals* and "*plane*" *sets*—hence of E into "*portions*" and "plane" sets—in the latter of which $g(e)$ vanishes identically. Let us call such a subdivision of E a *normal* subdivision of E.

Let E_ϵ be any set, closed relatively to E_0, of points of E_0 at which $\xi(P)$ has oscillation greater than or equal to ϵ in E_0; *i.e.* points each of which is contained in a subset of E_0 of span as small as we please in which $\xi(P)$ has oscillation greater than or equal to ϵ.

Let $\mathfrak{E}_{\epsilon,\delta}$ be the sum of those units E_i of a given normal subdivision of E_0, of norm δ, of which each contains a point of E_ϵ. In each of these units E_i which are portions of E, $\xi(P)$ has oscillation greater than or equal to ϵ, while for each of the others, $g(E_i) = 0$; hence, in all, they contribute not less than

$$\epsilon \, | \, g(\mathfrak{E}_{\epsilon,\delta}) \, |$$

to the sum of $\Omega(E)$ for the units of the subdivision. A necessary condition for this sum to tend to zero with δ is therefore

$$\lim_{\delta \to 0} g(\mathfrak{E}_{\epsilon,\delta}) = 0.$$

When δ describes any sequence of values tending to 0, $\mathfrak{E}_{\epsilon,\delta}$ contains E_ϵ for every δ, and hence so does

$$\varlimsup_{\delta \to 0} \mathfrak{E}_{\epsilon,\delta}.$$

* By the classical property that the "planes", parallel to a given one, in which $g(e)$ does not vanish identically, are countable.

Also every point of $\overline{\lim_{\delta \to 0}} \mathfrak{E}_{\epsilon, \delta}$

is contained in a subset of E_0 of span as small as we please containing a point of E_ϵ, *i.e.* is a point of E_ϵ, since this set is closed relatively to E_0. Hence

$$\lim_{\delta \to 0} \mathfrak{E}_{\epsilon, \delta} = E_\epsilon.$$

Thus the above limiting equation becomes

$$g(E_\epsilon) = 0.$$

Accordingly, we see that $g(E)$ must vanish for every set, closed relatively to E_0, of points of E_0 at which $\xi(P)$ has oscillation greater than ϵ in E_0, and hence, by transfinite induction and the monotone generation of sets, it must vanish for every (Borel) set of points of E_0 at which $\xi(P)$ has oscillation greater than ϵ in E_0.

Thus we see that $g(E)$ must vanish identically in the set E_ϵ of points of E_0 at which $\xi(P)$ has oscillation greater than ϵ in E_0. Letting $\epsilon \to 0$, when this subset increases and has for limit the set of all points of E_0 at which $\xi(P)$ is not continuous in E_0, we thus find ultimately that $g(E)$ must vanish identically in the set of points of E_0 at which $\xi(P)$ is not continuous in E_0. Thus we have :—

If the upper and lower integrals of a function $\xi(P)$, bounded in E_0, with respect to the additive function $g(E)$, for E_0, coincide, then $g(E)$ vanishes identically in the set E' of points of E_0 at which $\xi(P)$ is discontinuous in E_0.

If $$g(E) = g_1(E) - g_2(E)$$

be the canonical expression of $g(E)$ as the difference of two non-negative functions, the above is equivalent to

$$g_1(E') = g_2(E') = 0.$$

We proceed to show that this suffices to insure the coincidence of the upper and lower integrals of $\xi(P)$ with respect to $g_1(E)$ and $g_2(E)$ separately for E_0; and hence, since

$$\overline{\int_{E_0}} \xi(P)\, dg_1 - \underline{\int_{E_0}} \xi(P)\, dg_2 \geqslant \overline{\int_{E_0}} \xi(P)\, d(g_1 - g_2) \geqslant \underline{\int_{E_0}} \xi(P)\, d(g_1 - g_2)$$

$$\geqslant \underline{\int_{E_0}} \xi(P)\, dg_1 - \int_{E_0} \xi(P)\, dg_2,$$

the coincidence of those with respect to $g(E)$, with

$$\overline{\int}_{E_0} \xi(P)\, dg = \overline{\int}_{E_0} \xi(P)\, dg_1 - \underline{\int}_{E_0} \xi(P)\, dg_2.$$

6. We have to show that, *if* $g(E)$ *is* NON-NEGATIVE *and vanishes for the set* E' *of points of* E_0 *at which* $\xi(P)$ *is discontinuous in* E_0, *then* $\xi(P)$ *has coincident upper and lower integrals with respect to* $g(E)$ *for* E_0.

We first note that, when $g(E)$ is non-negative, the two functions $\overline{\phi}(E)$ and $\underline{\phi}(E)$ possess half the property of additiveness, *i.e.* when $E_1, E_2, \ldots, E_q$ are distinct,

$$(\alpha) \qquad \overline{\phi}(E_1+E_2+\ldots+E_q) \geqslant \overline{\phi}(E_1)+\overline{\phi}(E_2)+\ldots+\overline{\phi}(E_q),$$

$$(\beta) \qquad \underline{\phi}(E_1+E_2+\ldots+E_q) \leqslant \underline{\phi}(E_1)+\underline{\phi}(E_2)+\ldots+\underline{\phi}(E_q).$$

The theory of functions of these types may be developed in the same manner as that of the corresponding non-additive functions of intervals*.

In particular, we have the following properties :—

Let $\overline{S}_E(\phi)$, $\underline{S}_E(\phi)$ denote the largest and least possible limits of the sum of the values of $\phi(e)$ for the units of a subdivision of E whose norm tends to zero.

PROP. 1. *If* $\Phi(E)$ *is of type* (α), $\underline{S}_E(\Phi)$ *is additive*; *if* $\Phi(E)$ *is of type* (β), $\overline{S}_E(\Phi)$ *is additive.*

In fact, if $E_1, E_2, \ldots, E_q$ be the units of any subdivision of E_0, in general

$$\sum_{i=1}^{q} \underline{S}_{E_i}(\Phi) \geqslant \underline{S}_{E_0}(\Phi);$$

but in view of (α), for Φ, if on any given subdivision of E_0 we superpose the subdivision $E_1, E_2, \ldots, E_q$ (*i.e.* consider as separate units the parts which each unit of the given subdivision has in common with the separate sets E_i) we can only decrease the value of the sum of $\Phi(E)$ for the given subdivision; whence also

$$\sum_{i=1}^{q} \underline{S}_{E_i}(\Phi) \leqslant \underline{S}_{E_0}(\Phi);$$

and the result follows.

---

* For this, see my note on "Functions of $\mathfrak{A}$ defined by addition" already quoted.

Further, functions of these types also have the property that

PROP. 2. *If* $\Phi(E)$ *is of type* (α), $\bar{S}_E\{\bar{S}(\Phi)\} = \underline{S}_E\{\bar{S}(\Phi)\} = \underline{S}_E(\Phi)$; *if* $\Phi(E)$ *is of type* (β), $\bar{S}_E\{\underline{S}(\Phi)\} = \underline{S}_E\{\underline{S}(\Phi)\} = \bar{S}_E(\Phi)$.

In fact, on the one hand, by Prop. 1,

$$\bar{S}_E\{\bar{S}(\Phi)\} \geqslant \underline{S}_E\{\bar{S}(\Phi)\} \geqslant \underline{S}_E\{\underline{S}(\Phi)\} = \underline{S}_E(\Phi).$$

On the other hand, as $\{\Phi(E) - \bar{S}_E(\Phi)\}$ is then non-negative, so is

$$\underline{S}_E\{\Phi - \bar{S}(\Phi)\};$$

but this is less than or equal to

$$\underline{S}_E(\Phi) - \underline{S}_E\{\bar{S}(\Phi)\} = \underline{S}\{\underline{S}(\Phi)\} - \underline{S}\{\bar{S}(\Phi)\} \leqslant 0.$$

Therefore all three expressions are here zero, and proposition 2 follows.

As a particular *corollary*, we may state the following :—

If for E_0 *a function* $\Phi_1(E)$ *of type* (α) *and a function* $\Phi_2(E)$ *of type* (β) *have the same* $\bar{S}_{E_0}$ *and the same* $\underline{S}_{E_0}$, *then these two quantities are equal.*

Applying this to our functions $\bar{\Phi}(E)$ and $\underline{\Phi}(E)$, we see that the coincidence of the upper and lower integrals of $\xi(P)$ with respect to $g(E)$ for E_0, *i.e.*

$$\bar{S}_{E_0}(\bar{\Phi}) = \underline{S}_{E_0}(\underline{\Phi}),$$

is ensured if, simultaneously,

$$\begin{cases} \bar{S}_{E_0}(\bar{\Phi}) - \bar{S}_{E_0}(\underline{\Phi}) = 0, \\ \underline{S}_{E_0}(\bar{\Phi}) - \underline{S}_{E_0}(\underline{\Phi}) = 0. \end{cases}$$

Since $$\bar{S}_{E_0}(\bar{\Phi} - \underline{\Phi}) \geqslant \bar{S}_{E_0}(\bar{\Phi}) - \bar{S}_{E_0}(\underline{\Phi}) \geqslant 0$$

and also $$\bar{S}_{E_0}(\bar{\Phi} - \underline{\Phi}) \geqslant \underline{S}_{E_0}(\bar{\Phi}) - \underline{S}_{E_0}(\underline{\Phi}) \geqslant 0,$$

it suffices *a fortiori* that we should have

$$\bar{S}_{E_0}(\bar{\Phi} - \underline{\Phi}) = \bar{S}_{E_0}(\Omega) = 0.$$

The proof that the assumption $g(E') = 0$ for the set E' of points of E_0 at which $\xi(P)$ is discontinuous in E_0 ensures the above is almost immediate.

Let us consider, indeed, any subdivision of E_0, of norm δ, and let $\mathscr{E}'_{\delta,\epsilon}$ be the sum of those of its units in each of which $\xi(P)$ has oscillation greater than or equal to ϵ. Any point P belonging to $\mathscr{E}'_{\delta,\epsilon}$ for a

sequence of values of δ tending to zero, *i.e.* any point of

$$\overline{\lim_{\delta = 0}} \mathcal{E}'_{\delta, \epsilon},$$

is a point of E_0 at which $\xi(P)$ has oscillation greater than or equal to ϵ in E_0, *i.e.* certainly a point of E'. Thus, for that sequence,

$$\overline{\lim_{\delta \to 0}} \mathcal{E}'_{\delta, \epsilon} \subset E'.$$

As $g(E)$ is non-negative, this implies that

$$\overline{\lim_{\delta \to 0}} g(\mathcal{E}'_{\delta, \epsilon}) \leqslant g(E'),$$

which is zero. Since $\xi(P)$ is bounded, it follows that the contribution of the units in question to the sum of $\Omega(E)$ for all units of the subdivision tends to zero with δ. But in each of the remaining units, $\xi(P)$ has oscillation less than ϵ, and their contribution is therefore less than $\epsilon g(E_0)$.

We conclude that

$$\bar{S}_{E_0}(\Omega) \leqslant \epsilon g(E_0).$$

Since ϵ is arbitrary, this proves that the assumption

$$g(E') = 0$$

involves

$$\bar{S}_{E_0}(\Omega) = 0.$$

We thus have the following theorems :—

THEOREM IIA. *A necessary and sufficient condition for a function $\xi(P)$, bounded in E_0, to have coincident upper and lower integrals with respect to $g(E)$ for E_0, is that $g(E)$ should vanish identically in the set of points of E_0 at which $\xi(P)$ is discontinuous in E_0.*

THEOREM IIB. *A necessary and sufficient condition for a function $\xi(P)$, bounded in E_0, to have coincident upper and lower integrals with respect to $g(E)$ for E_0, is that this should be the case separately with respect to each of the two non-negative additive functions $g_1(E)$ and $g_2(E)$ whose difference furnishes the canonical expression of $g(E)$.*

More generally, combining these with Theorem I :—

THEOREM III. *A necessary and sufficient condition for a function $\xi(P)$ to have finite and coincident upper and lower integrals with respect to an additive function $g(E)$, for E_0, is that $\xi(P)$ should be continuous in E_0, except at most at points of a subset of E_0 in which $g(E)$ vanishes identically, and should be bounded in every subset of E_0 of span less than a certain δ_0 in which $g(E)$ does not vanish identically.*

7. In a recent paper, "On 'Riemann' integration with respect to a continuous increment"*, my object was to see what was the most special type of subdivision which would yield a simple theory of the "Riemann" process, and what might be the influence of different restrictions on the mode of subdivision. The outcome was that, in the widest measure, this influence appeared as inexistent and the condition of integrability remained unchanged, so that it would suffice, in order to investigate the case of a more general mode of subdivision, to examine that of a more special one, or *vice versa*.

My object in the present work has been the reverse; it was to see what occurs when every possible kind of subdivisions into sets of points is admitted and the "Riemann" process is applied to them.

A comparison will show that the ultimate conditions are still precisely the same in substance. They represent, in a more definite form, the extension of W. H. Young's criterion of Riemann integrability.

The chief distinction to be drawn between the two constructions is the following. In the case of the fields of sets (simple sets) considered in my former theory, every element of one field contained elements of any other, and could indeed be approximated to as much as we pleased by such contained elements. The condition in question, if satisfied in the first element, would be so equally in all the contained elements of every other field. Thus a function $\xi(P)$, Riemann integrable, with respect to a "continuous increment" generating $g(E)$, in a portion of the field of this increment, would automatically be Riemann integrable, with respect to any other continuous increment generating $g(E)$, in a portion of the field of the latter increment. And at the same time, as we now perceive, it would possess a "Riemann" integral with respect to $g(E)$ itself for every subset of an element of that portion of the field.

An arbitrary Borel set may, however, have no "simple" subset whatever. Thus a function $\xi(P)$ may possess a "Riemann" integral with respect to $g(E)$ for a portion of the field of all Borel sets without being Riemann integrable anywhere with respect to any "increment" generating $g(E)$. In this form the remark is almost trivial, since we may take, for instance, a single P_0 as our set E. Any function $\xi(P)$ has a "Riemann" integral with respect to $g(E)$ for a set $E = \{P_0\}$ containing only one point, and this is

$$\xi(P_0)\, g(\{P_0\}).$$

* *Math. Zeitschrift*, *loc. cit.*, 217–233.

DETERMINANTAL FORM OF THE FORMULA FOR EXPRESSING A RATIONAL INTEGRAL FUNCTION OF A SINGLE VARIABLE IN TERMS OF TWO OTHER SUCH FUNCTIONS WHICH ARE MUTUALLY PRIME.

By M. J. M. HILL.

[Received 25 August, 1927; revised 19 March, 1928; read 14 June, 1928.]

1. It is a familiar problem in the theory of partial fractions to express a polynomial $f(x)$ of degree n in terms of two given polynomials $g(x)$ and $h(x)$ which have no common factor and which are of degrees s and t respectively.

2. If $n < s+t$, polynomials $\lambda(x)$ and $\mu(x)$, of degrees $t-1$ and $s-1$ at most respectively, can be found so that

$$f(x) = \lambda(x)\,g(x)+\mu(x)\,h(x).$$

If $n \geqslant s+t$, polynomials $\lambda(x)$, $\mu(x)$, $\nu(x)$, of degrees at most $t-1$, $s-1$, $n-s-t$ respectively, can be found so that

$$f(x) = \lambda(x)\,g(x)+\mu(x)\,h(x)+\nu(x)\,g(x)\,h(x).$$

Various methods have been given for determining these polynomials, but so far as I know the method given below is the only one which gives *explicit* forms for the polynomials $\lambda(x)$, $\mu(x)$, $\nu(x)$ of lowest degree satisfying the required conditions.

3. *Case* I. If $n < s+t$, construct the determinant $D(x)$ of order $s+t+1$ and of degree $s+t-1$ in x, thus: the first row consists of

(I) $\quad f(x);\quad g(x),\ xg(x),\ \ldots,\ x^{t-1}g(x);\quad h(x),\ xh(x),\ \ldots,\ x^{s-1}h(x);$

the second row consists of the absolute terms in the above functions; the $(q+2)$-th row consists of the coefficients of x^q in the above functions.

On multiplying the second row by 1 and subsequent rows by appropriate powers of x and subtracting from the first, it is seen that $D(x)$ vanishes identically. If we now expand $D(x)$ in cofactors of the first row, we have

$$Ef(x)+L(x)g(x)+M(x)h(x)=0,$$

where E is the eliminant of $g(x)=0$ and $h(x)=0$; $L(x)$ is the determinant obtained from $D(x)$ by putting in its first row zero for $f(x)$ and $h(x)$ and unity for $g(x)$; $M(x)$ is the determinant obtained from $D(x)$ by putting in its first row zero for $f(x)$ and $g(x)$ and unity for $h(x)$.

Since $g(x)$ and $h(x)$ have no common factor, $E\neq 0$, and $f(x)$ is expressed in the required form. Clearly $L(x)$ and $M(x)$ are of degrees $t-1$ and $s-1$ in x at most respectively.

4. *Case* II. If $n\geqslant s+t$, let

$$p(x)=g(x)h(x),$$

and form the determinant $D(x)$ of order $n+2$ whose first row is as follows:

$$\text{(II)}\quad f(x);\quad g(x),\ xg(x),\ \ldots,\ x^{t-1}g(x);\quad h(x),\ xh(x),\ \ldots,\ x^{s-1}h(x);$$
$$p(x),\ xp(x),\ \ldots,\ x^{n-s-t}p(x),$$

the constituents of the remaining rows being formed by the same rule as in the case of Determinant I. As in Case I, $D(x)=0$, and on expanding in cofactors of the first row we have

$$E'f(x)+L(x)g(x)+M(x)h(x)+N(x)p(x)=0.$$

On writing out E' it is seen that it is the product of the eliminant of $g(x)=0$ and $h(x)=0$ and the $(n-s-t+1)$-th power of the coefficient of the highest power of x in $p(x)$, and so it does not vanish. $L(x)$, $M(x)$, $N(x)$ are given by determinants formed from $D(x)$ by rules similar to those obtaining in Case I, and they are of degree in x, $t-1$, $s-1$, $n-s-t$ respectively. For example, $N(x)$ is the determinant formed from $D(x)$ by putting in its first row zero for $f(x)$, $g(x)$, $h(x)$ and unity for $p(x)$. Thus $f(x)$ is expressed in the required form and the highest power of x occurring is the n-th.

5. Now the form last given is symmetrical with regard to $g(x)$ and $h(x)$, but it requires the calculation of $p(x)$, which is the product $g(x)\,.\,h(x)$, and, as will now be shown, it contains a constant factor which may be removed. Let h_α denote the coefficient of x^α in $h(x)$, with similar mean

ings for g_α, p_α. Multiply the constituents in the columns headed by $g(x)$, $xg(x)$, ..., $x^{t-1}g(x)$ by h_0, h_1, ..., h_{t-1} respectively, and subtract from the column headed by $p(x)$, *i.e.* $g(x) \,.\, h(x)$. There will be left at the head of this column the expression $h_t x^t g(x)$ and in the rows beneath this there will be left the coefficients of various powers of x in $h_t x^t g(x)$, so that the column can be divided by h_t and there remains at the head of the column $x^t g(x)$.

Now multiply the columns headed by $xg(x)$, $x^2 g(x)$, ..., $x^t g(x)$ by h_0, h_1, ..., h_{t-1} respectively, and subtract from the column headed by $xp(x)$, *i.e.* $xh(x) \,.\, g(x)$. Then there will remain at the head of this column $h_t x^{t+1} g(x)$, and in the rows beneath this there will be left the coefficients of the various powers of x in $h_t x^{t+1} g(x)$, so that this column is divisible by h_t and there remains at the head of it the expression $x^{t+1} g(x)$.

Proceeding in like manner, we see that we can reduce the columns headed by $x^2 p(x)$, $x^3 p(x)$, ..., $x^{n-s-t} p(x)$, *i.e.*

$$x^2 g(x) \,.\, h(x), \quad x^3 g(x) \,.\, h(x), \quad \ldots, \quad x^{n-s-t} g(x) \,.\, h(x).$$

We shall replace them by columns headed by

$$x^{t+2} g(x), \quad x^{t+3} g(x), \quad \ldots, \quad x^{n-s} g(x),$$

and we shall have divided the determinant by the factor $h_t^{n-s-t+1}$. Thus, removing this factor, we get, after rearranging the columns, a determinant of order $n+2$, in which the terms in the first row are

(III) $\quad f(x); \quad g(x), \; xg(x), \; \ldots, \; x^{n-s} g(x); \quad h(x), \; xh(x), \; \ldots, \; x^{s-1} h(x),$

the constituents of the remaining rows being formed as in the first determinant. This determinant is of degree n in x. The coefficient of $g(x)$ is a determinant of order $n+2$ and of degree $n-s$ in x. The coefficient of $h(x)$ is of degree $s-1$ in x. With these restrictions as to the degree in x of the coefficients of $g(x)$ and $h(x)$, this form is unique.

Just as (III) was obtained from (II) by rearranging and dividing by $h_t^{n-s-t+1}$, so by symmetry, if we divide (II) by $g_s^{n-s-t+1}$, we get the following determinant of order $n+2$ in which the terms in the first row are

(IV) $\quad f(x); \quad h(x), \; xh(x), \; \ldots, \; x^{n-t} h(x); \quad g(x), \; xg(x), \; \ldots, \; x^{t-1} g(x).$

(III) and (IV) are symmetrical with regard to $g(x)$ and $h(x)$, *i.e.* one can be obtained from the other by interchanging $g(x)$ with $h(x)$ and s with t.

[Hence it would seem, at first sight, that (II) could be divided by $(h_t \,.\, g_s)^{n-s-t+1}$ and a still simpler form obtained. This is not the case. For

take the form (III), in which the terms in the first row are

$$f(x),\ g(x),\ xg(x),\ \ldots,\ x^{n-s}g(x);\quad h(x),\ xh(x),\ \ldots,\ x^{s-1}h(x),$$

and expand in cofactors of the first *column*. It will be found that the coefficient of $f(x)$ is $E\,.\,g_s^{n-s-t+1}$, and the coefficient of any other term, say f_r, can be reduced to $E\,.\,x^r g_s^{n-s-t+1}$, so that we get only

$$\left[f(x)-\sum_0^n x^r f_r\right].E\,.\,g_s^{n-s-t+1},$$

which gives only the identity $0=0$, but the reduction of the coefficient of f_r to this form involves the breaking up of $g(x)$ and $h(x)$ and so destroys the utility of the form.]

6. We can apply the first determinantal form to the integration of the rational fraction,

$$\frac{f(x)}{\{g(x)\}^r},$$

where $f(x)$ is of degree lower than $g(x)$, and $g(x)$ contains no repeated factors.

If in Case (I) of § 1 we take $h(x)=(d/dx)g(x)=g'(x)$, so that $t=s-1$, then the determinant $D(x)$ so formed is of order $2s$, and

$$E\,.\,f(x)+L(x)\,.\,g(x)+M(x)\,.\,g'(x)=0,$$

where E is the eliminant of $g(x)=0$ and $g'(x)=0$, and $L(x)$ and $M(x)$ are expressed as determinants in the manner already explained.

Dividing out by $\{g(x)\}^r$ we get,

$$E\,\frac{f(x)}{\{g(x)\}^r}+\frac{L(x)}{\{g(x)\}^{r-1}}+M(x)\,\frac{g'(x)}{\{g(x)\}^r}=0.$$

Therefore

$$E\,\frac{f(x)}{\{g(x)\}^r}+\left[L(x)+\frac{1}{r-1}\,\frac{d}{dx}\,M(x)\right]\frac{1}{\{g(x)\}^{r-1}}-\frac{d}{dx}\left[\frac{1}{r-1}\,\frac{M(x)}{\{g(x)\}^{r-1}}\right]=0.$$

Now the expression

$$L(x)+\frac{1}{r-1}\,\frac{d}{dx}\,M(x)$$

is obtainable from the determinant $D(x)$ by replacing the constituents in the first row by the following:

$$0,\ 1,\ x,\ \ldots,\ x^{s-2};\quad 0,\ \frac{1}{r-1},\ \frac{2x}{r-1},\ \ldots,\ \frac{(s-1)x^{s-2}}{r-1}.$$

The constituents in the remaining rows are *the same* as those in the determinant $D(x)$ of this article.

If we call this determinant $P(x)$, we get

$$E\frac{f(x)}{\{g(x)\}^r}+\frac{P(x)}{\{g(x)\}^{r-1}}-\frac{d}{dx}\left[\frac{1}{r-1}\,\frac{M(x)}{\{g(x)\}^{r-1}}\right]=0.$$

Integrating, we get the reduction formula

$$E\int\frac{f(x)}{\{g(x)\}^r}\,dx+\int\frac{P(x)}{\{g(x)\}^{r-1}}\,dx-\left[\frac{1}{r-1}\,\frac{M(x)}{\{g(x)\}^{r-1}}\right]=\text{Constant}.$$

Thus every term in the reduction formula is given explicitly in terms of the coefficients of $f(x)$ and $g(x)$.

TRANSFORMATIONS OF GENERALIZED HYPERGEOMETRIC SERIES

By W. N. BAILEY.

[Received 25 January, 1928.—Revised 17 March, 1928.—Read 16 February, 1928.]

1. The object of this paper is to show how a certain simple process can be used to discover transformations of generalized hypergeometric series.

Suppose that we have a relation*

$$(1.1)\qquad {}_3F_2\left[\begin{matrix}a_1,\ a_2,\ a_3\ ;\\ \rho_1,\ \rho_2\ ;\end{matrix}\ cx\right] = (1-x)^{-\kappa}\,{}_3F_2\left[\begin{matrix}\beta_1,\ \beta_2,\ \beta_3\ ;\\ \sigma_1,\ \sigma_2\ ;\end{matrix}\ \frac{kx^\lambda}{(1-x)^\mu}\right],$$

where λ is a positive integer and μ any integer. If we multiply this relation by $(1-x)^{\kappa-p}$ and equate coefficients of x^n on the two sides, we obtain a transformation involving generalized hypergeometric series. We thus find that

$$\frac{(p-\kappa)_n}{(p)_n}\,{}_4F_3\left[\begin{matrix}a_1,\ a_2,\ a_3,\ -n\ ;\\ \rho_1,\ \rho_2,\ 1-p+\kappa-n\ ;\end{matrix}\ c\right]$$

is equal to

$$(1.2)\qquad {}_{\mu+3}F_{\mu+2}\left[\begin{matrix}\beta_1,\ \beta_2,\ \beta_3,\ \dfrac{-n}{\lambda},\ \dfrac{1-n}{\lambda},\ \ldots,\ \dfrac{\lambda-1-n}{\lambda},\ \dfrac{n+p}{\mu-\lambda},\ \dfrac{n+p+1}{\mu-\lambda},\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots,\ \dfrac{n+p+\mu-\lambda-1}{\mu-\lambda}\ ;\\ \sigma_1,\ \sigma_2,\ \dfrac{p}{\mu},\ \dfrac{p+1}{\mu},\ \ldots,\ \dfrac{p+\mu-1}{\mu}\ ;\end{matrix}\ \frac{(-\lambda)^\lambda(\mu-\lambda)^{\mu-\lambda}\,k}{\mu^\mu}\right]$$

if $\mu \geqslant \lambda$; to

$$(1.3)\qquad {}_{\lambda+3}F_{\lambda+2}\left[\begin{matrix}\beta_1,\ \beta_2,\ \beta_3,\ \dfrac{-n}{\lambda},\ \dfrac{1-n}{\lambda},\ \ldots,\ \dfrac{\lambda-1-n}{\lambda}\ ;\\ \sigma_1,\ \sigma_2,\ \dfrac{p}{\lambda},\ \dfrac{p+1}{\lambda},\ \ldots,\ \dfrac{p+\lambda-1}{\lambda}\ ;\end{matrix}\ (-1)^\lambda k\right]$$

* The series ${}_3F_2$ may be replaced by series of a different order with obvious modifications.

if $\mu = \lambda$; to

$$(1.4)\quad {}_{\lambda+3}F_{\lambda+2}\left[\begin{matrix}\beta_1, \beta_2, \beta_3, \dfrac{-n}{\lambda}, \dfrac{1-n}{\lambda}, \ldots, \dfrac{\lambda-1-n}{\lambda}; \dfrac{(-1)^{\mu}(\lambda)^{\lambda} k}{\mu^{\mu}(\lambda-\mu)^{\lambda-\mu}} \\ \sigma_1, \sigma_2, \dfrac{p}{\mu}, \dfrac{p+1}{\mu}, \ldots, \dfrac{p+\mu-1}{\mu}, \dfrac{p+1-n}{\lambda-\mu}, \\ \qquad\qquad \ldots, \dfrac{p+\lambda-\mu-n}{\lambda-\mu};\end{matrix}\right]$$

if $0 < \mu < \lambda$; and to

$$(1.5)\quad {}_{\lambda+\nu+3}F_{\lambda+\nu+2}\left[\begin{matrix}\beta_1, \beta_2, \beta_3, \dfrac{-n}{\lambda}, \dfrac{1-n}{\lambda}, \ldots, \dfrac{\lambda-1-n}{\lambda}, \dfrac{1-p}{\nu}, \dfrac{2-p}{\nu}, \\ \qquad\qquad \ldots, \dfrac{\nu-p}{\nu}; \dfrac{\lambda^{\lambda}\nu^{\nu} k}{(\lambda+\nu)^{\lambda+\nu}} \\ \sigma_1, \sigma_2, \dfrac{1-p-n}{\lambda+\nu}, \ldots, \dfrac{\lambda+\nu-p-n}{\lambda+\nu};\end{matrix}\right]$$

if $\mu = -\nu < 0$.

In § 2 we obtain a number of transformations involving nearly-poised series, and in §§ 3 and 4 some miscellaneous formulae are obtained, including the sum of a series of the type ${}_6F_5(1)$, the generalization of which gives the sum of a series of the type ${}_{2r+2}F_{2r+1}(1)$.

2. We shall now apply the method indicated in § 1 to some known formulae of the type (1.1). A list of such formulae is given in another paper*.

First consider the relation

$$(2.1)\quad {}_3F_2\left[\begin{matrix}x, y, z; \xi \\ 1+x-y, 1+x-z\end{matrix}\right] = (1-\xi)^{-x}\,{}_3F_2\left[\begin{matrix}1+x-y-z, \frac{1}{2}x, \frac{1}{2}+\frac{1}{2}x; -4\xi/(1-\xi)^2 \\ 1+x-y, 1+x-z\end{matrix}\right],$$

which was given by Whipple†. Using (1.2) and putting $p = 1-n-w+x$ we are led to

$$(2.11)\quad {}_4F_3\left[\begin{matrix}x, y, z, -n \\ 1+x-y, 1+x-z, w\end{matrix}\right] = \frac{\Gamma(w)\,\Gamma(w-x+n)}{\Gamma(w+n)\,\Gamma(w-x)} \times {}_5F_4\left[\begin{matrix}1+x-w, \frac{1}{2}(1+x), \frac{1}{2}x, 1+x-y-z, -n \\ 1+x-y, 1+x-z, 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n)\end{matrix}\right],$$

* W. N. Bailey, "Products of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 28 (1929), 242-254.

† F. J. W. Whipple, "Some transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 26 (1927), 257-272 (7.1).

which is the most general transformation of a nearly-poised series obtained by Whipple.* By reversing the series in (2.11) we get†, with a change of notation,

$$(2.12)\quad {}_4F_3\left[\begin{matrix} a, b, c, d \\ \kappa-b, \kappa-c, \kappa-d \end{matrix}\right] = \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)\,\Gamma(\kappa-b-c-d)}{\Gamma(\kappa-c-d)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa-b-d)\,\Gamma(\kappa)} \times {}_5F_4\left[\begin{matrix} b, c, d, \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ \kappa-a, \frac{1}{2}\kappa, \frac{1}{2}+\frac{1}{2}\kappa, b+c+d-\kappa+1 \end{matrix}\right],$$

provided that the series terminate.

Whipple first proved (2.12) and deduced (2.11) by reversing the series. He then deduced the relation (2.1). The formula (2.1) can, however, be proved‡ by comparing coefficients of ζ^n and using Saalschütz's theorem§, and so we have an alternative method of arriving at these results.

Now consider the relations

$$(2.2)\quad {}_2F_1\left[\begin{matrix} x, y \\ 1+x-y \end{matrix}; \zeta\right] = (1-\zeta)^{-x}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}x-y, \frac{1}{2}x \\ 1+x-y \end{matrix}; -4\zeta/(1-\zeta)^2\right],$$

$$(2.3)\quad {}_3F_2\left[\begin{matrix} x-1, y-1, \frac{1}{2}+\frac{1}{2}x \\ 1+x-y, \frac{1}{2}x-\frac{1}{2} \end{matrix}; \zeta\right] = (1-\zeta)^{1-x}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}x-y, \frac{1}{2}x \\ 1+x-y \end{matrix}; -4\zeta/(1-\zeta)^2\right],$$

which are special cases of (2.1)‖.

From these we evidently have

$$(2.4)\quad (1-\zeta)\,{}_2F_1\left[\begin{matrix} x, y \\ 1+x-y \end{matrix}; \zeta\right] = {}_3F_2\left[\begin{matrix} x-1, y-1, \frac{1}{2}+\frac{1}{2}x \\ 1+x-y, \frac{1}{2}x-\frac{1}{2} \end{matrix}; \zeta\right].$$

Multiplying by $(1-\zeta)^{w+n-2}$ and equating coefficients of x^n, we get the formula

$$(2.41)\quad {}_3F_2\left[\begin{matrix} x, y, -n \\ 1+x-y, w \end{matrix}; 1\right] = \frac{w-1}{w+n-1}\,{}_4F_3\left[\begin{matrix} x-1, \frac{1}{2}(x+1), y-1, -n \\ \frac{1}{2}(x-1), 1+x-y, w-1 \end{matrix}; 1\right].$$

The interest in this formula lies in the fact that it expresses any terminating nearly-poised series of the type ${}_3F_2$ in terms of a nearly-poised

* Whipple, *loc. cit.*, (6.5).

† Whipple, *loc. cit.* (3.5).

‡ See Bailey, *loc. cit.* (4.12).

§ L. Saalschütz, "Eine Summationsformel", *Zeitschrift für Math. u. Phys.*, 35 (1890), 186–188.

‖ Cf. Bailey, *loc. cit.* (4.07) and (4.03).

series of the type ${}_4F_3$, whereas Whipple's formulae express nearly-poised series in terms of Saalschützian series. The formula can be deduced by using (2 . 11) to transform the series on the right, and by using (2 . 11) with $z = \frac{1}{2}(1+x)$ to transform the series on the left. The two Saalschützian series obtained are seen to be identical.

By reversing the series in (2 . 41) we obtain the equivalent formula

$$(2.42)\quad {}_3F_2\left[\begin{matrix} a, & b, & -n \\ \kappa-b, & \kappa+n & \end{matrix}\right] = \frac{\kappa(\kappa-b+n)}{(\kappa-b)(\kappa+n)}\,{}_4F_3\left[\begin{matrix} a+1, & b, & 1+\frac{1}{2}\kappa, & -n \\ \kappa+1-b, & \frac{1}{2}\kappa, & \kappa+1+n & \end{matrix}\right]$$

connecting two nearly-poised series in which the breakdown in the equality of sums of pairs of parameters occurs with the first pair.

From (2 . 3) it is easily shown that

$$(2.5)\quad (1-\xi)^{2a-1}\,{}_3F_2\left[\begin{matrix} 2a-1, & a+\frac{1}{2}, & a-b-\frac{1}{2}\,; \\ a-\frac{1}{2}, & a+b+\frac{1}{2} & \end{matrix}\ \xi\right] = (1-\xi)^{2b-1}\,{}_3F_2\left[\begin{matrix} 2b-1, & b+\frac{1}{2}, & b-a-\frac{1}{2}\,; \\ b-\frac{1}{2}, & a+b+\frac{1}{2} & \end{matrix}\ \xi\right].$$

The usual process applied to this formula leads to the transformation

$$(2.51)\quad (2a+p)_n\,{}_4F_3\left[\begin{matrix} 2a, & a+1, & a-b-\frac{1}{2}, & -n \\ a, & a+b+\frac{3}{2}, & 2a+p & \end{matrix}\right] = (2b+p)_n\,{}_4F_3\left[\begin{matrix} 2b, & b+1, & b-a-\frac{1}{2}, & -n \\ b, & a+b+\frac{3}{2}, & 2b+p & \end{matrix}\right],$$

which is a relation between two nearly-poised series and can be deduced from (2 . 11).

3. We proceed to investigate the formulae which can be derived from the two cubic transformations*

$$(3.1)\quad (1-x)^{-a}\,{}_3F_2\left[\begin{matrix} \frac{1}{3}a, & \frac{1}{3}(a+1), & \frac{1}{3}(a+2)\,; \\ b, & c\,; & \end{matrix}\ -27x/4(1-x)^3\right] = {}_3F_2\left[\begin{matrix} a, & \frac{1}{2}+b-c, & \frac{1}{2}-b+c\,; \\ b, & c\,; & \end{matrix}\ \frac{1}{4}x\right],$$

where $a = b+c-\frac{3}{2}$, and

$$(3.2)\quad (1-x)^{\frac{1}{2}-a-b}\,{}_3F_2\left[\begin{matrix} \frac{1}{3}(a+b-\frac{1}{2}), & \frac{1}{3}(a+b+\frac{1}{2}), & \frac{1}{3}(a+b+\frac{3}{2})\,; \\ a+\frac{1}{2}, & b+\frac{1}{2} & \end{matrix}\ 27x^2/4(1-x)^3\right] = {}_3F_2\left[\begin{matrix} a+b-\frac{1}{2}, & a, & b\,; \\ 2a, & 2b\,; & \end{matrix}\ 4x\right].$$

* Bailey, *loc. cit.* (4.05) and (4.06).

These relations may be proved by comparing coefficients of x^n and using Saalschütz's theorem.

By the usual process we deduce the identities

$$(3.11)\quad {}_4F_3\left[\begin{matrix} b+c-\frac{3}{2},\ b-c+\frac{1}{2},\ c-b+\frac{1}{2},\ -n \\ b,\ c,\ b+c-\frac{1}{2}-p-n \end{matrix};\ \frac{1}{4}\right] = \frac{(p)_n^{\frac{1}{2}}}{(p-a)_n}\ {}_6F_5\left[\begin{matrix} \frac{1}{3}a,\ \frac{1}{3}(a+1),\ \frac{1}{3}(a+2),\ \frac{1}{2}(n+p),\ \frac{1}{2}(n+p+1),\ -n \\ b,\ c,\ \frac{1}{3}p,\ \frac{1}{3}(p+1),\ \frac{1}{3}(p+2) \end{matrix};\ 1\right],$$

where $a = b+c-\frac{3}{2}$, and

$$(3.21)\quad {}_6F_5\left[\begin{matrix} \frac{1}{3}(a+b-\frac{1}{2}),\ \frac{1}{3}(a+b+\frac{1}{2}),\ \frac{1}{3}(a+b+\frac{3}{2}),\ -\frac{1}{2}n,\ \frac{1}{2}(1-n),\ n+p \\ a+\frac{1}{2},\ b+\frac{1}{2},\ \frac{1}{3}p,\ \frac{1}{3}(p+1),\ \frac{1}{3}(p+2) \end{matrix};\ 1\right] = \frac{(p-a-b+\frac{1}{2})_n}{(p)_n}\ {}_4F_3\left[\begin{matrix} a,\ b,\ a+b-\frac{1}{2},\ -n \\ 2a,\ 2b,\ \frac{1}{2}-n-p+a+b \end{matrix};\ 4\right].$$

The series on the right of (3.11) is Saalschützian.

If in (3.1) and (3.11) we put $c = b+\frac{1}{2}$, we get the formulae

$$(3.12)\quad {}_3F_2\left[\begin{matrix} \frac{1}{3}a,\ \frac{1}{3}(a+1),\ \frac{1}{3}(a+2) \\ \frac{1}{2}(a+1),\ \frac{1}{2}(a+2) \end{matrix};\ -27x/4(1-x)^3\right] = (1-x)^a$$

and

$$(3.13)\quad {}_6F_5\left[\begin{matrix} \frac{1}{3}a,\ \frac{1}{3}(a+1),\ \frac{1}{3}(a+2),\ \frac{1}{2}(n+p),\ \frac{1}{2}(n+p+1),\ -n \\ \frac{1}{2}(a+1),\ \frac{1}{2}(a+2),\ \frac{1}{3}p,\ \frac{1}{3}(p+1),\ \frac{1}{3}(p+2) \end{matrix};\ 1\right] = \frac{(p-a)_n}{(p)_n}.$$

The relation (3.12) can be expressed more simply in the form

$$(3.14)\quad 1-\frac{a+5}{2!}\frac{x}{(1-x)^3}+\frac{(a+7)(a+8)}{3!}\frac{x^2}{(1-x)^6}-\ldots = \frac{(1-x)^3[1-(1-x)^a]}{ax}.$$

On reversing the series in (3.13) we obtain a formula equivalent to

$$(3.15)\quad {}_6F_5\left[\begin{matrix} \frac{1}{3}a,\ \frac{1}{3}(a+1),\ \frac{1}{3}(a+2),\ \frac{1}{2}(n+q-1),\ \frac{1}{2}(n+q),\ -n \\ \frac{1}{3}q,\ \frac{1}{3}(q+1),\ \frac{1}{3}(q+2),\ \frac{1}{2}a,\ \frac{1}{2}(a+1) \end{matrix};\ 1\right] = \frac{(q-a)_n}{(q-3n+1)(q)_{n-1}},$$

and by making $n,\ q\to\infty$, and $n/q\to x/(1-x)$ we are led to the relation*

$$(3.3)\quad {}_3F_2\left[\begin{matrix} \frac{1}{3}a,\ \frac{1}{3}(a+1),\ \frac{1}{3}(a+2) \\ \frac{1}{2}a,\ \frac{1}{2}(a+1) \end{matrix};\ -27x/4(1-x)^3\right] = \frac{(1-x)^a}{1+2x},$$

or, more simply,

$$(3.31)\quad 1-\frac{a+2}{1!}\frac{x}{(1-x)^3}+\frac{(a+4)(a+5)}{2!}\frac{x^2}{(1-x)^6}-\ldots = \frac{(1-x)^a}{1+2x}.$$

* A rigorous proof of (3.3) is included in the proof of (3.5) below.

The formulae (3 . 12) and (3 . 3) can be generalized, the results being

$$(3.4)\quad {}_{r+1}F_r\left[\begin{matrix}\frac{a}{r+1}, & \frac{a+1}{r+1}, & \cdots, & \frac{a+r}{r+1}; & \frac{-(r+1)^{r+1}x}{r^r(1-x)^{r+1}}\\ & \frac{a+1}{r}, & \frac{a+2}{r}, \cdots, & \frac{a+r}{r}; & \end{matrix}\right] = (1-x)^a,$$

$$(3.5)\quad {}_{r+1}F_r\left[\begin{matrix}\frac{a}{r+1}, & \frac{a+1}{r+1}, & \cdots, & \frac{a+r}{r+1}; & \frac{-(r+1)^{r+1}x}{r^r(1-x)^{r+1}}\\ & \frac{a}{r}, & \frac{a+1}{r}, \cdots, & \frac{a+r-1}{r}; & \end{matrix}\right] = \frac{(1-x)^a}{1+rx}.$$

or, more simply,

$$(3.41)\quad \sum_{n=0}^{\infty} \frac{(a+r+1+nr)_n}{(n+1)!}\,\frac{(-x)^n}{(1-x)^{n(r+1)}} = \frac{\{1-(1-x)^a\}(1-x)^{r+1}}{ax}$$

and

$$(3.51)\quad \sum_{n=0}^{\infty} \frac{(a+nr)_n}{n!}\,\frac{(-x)^n}{(1-x)^{n(r+1)}} = \frac{(1-x)^a}{1+rx}.$$

The formula (3 . 51) can be proved* by multiplying by $(1-x)^{-a}$ and noticing that the coefficient of x^t on the left is the same as the coefficient of x^t in $(1-x)^{-a}[1-(1-x)^{-r}]^t$, which is $(-r)^t$. To prove (3 . 41), multiply by $(1-x)^{-(a+r+1)}$, and we find that the coefficient of x^t on the left is the same as that in

$$(1-x)^{-(a+1)}[1-\{1-(1-x)^{-r}\}^{t+1}]/(t+1),$$

i.e. in $(1-x)^{-(a+1)}/(t+1)$, and the result easily follows.

From (3 . 4) we deduce by the usual method of this paper

$$(3.42)\quad {}_{2r+2}F_{2r+1}\left[\begin{matrix}\frac{a}{r+1}, & \frac{a+1}{r+1}, & \cdots, & \frac{a+r}{r+1}, & \frac{n+p}{r}, & \frac{n+p+1}{r}, & \\ & & & & \cdots, & \frac{n+p+r-1}{r}, & -n\\ \frac{a+1}{r}, & \frac{a+2}{r}, & \cdots, & \frac{a+r}{r}, & \frac{p}{r+1}, & \frac{p+1}{r+1}, \cdots, & \frac{p+r}{r+1}\end{matrix}\right] = \frac{(p-a)_n}{(p)_n},$$

* I am indebted to the referee for pointing out this short proof.

and from (3.5) we obtain a relation between two series of the type ${}_{2r+2}F_{2r+1}$.

When the series in (3.42) is reversed we get a formula equivalent to

$$(3.43)\quad {}_{2r+2}F_{2r+1}\left[\begin{matrix} \frac{a}{r+1}, & \frac{a+1}{r+1}, & \cdots, & \frac{a+r}{r+1}, & \frac{n+q-1}{r}, & \frac{n+q}{r}, & & \\ & & & & & \cdots, & \frac{n+q+r-2}{r}, & -n \\ \frac{q}{r+1}, & \frac{q+1}{r+1}, & \cdots, & \frac{q+r}{r+1}, & \frac{a}{r}, & \frac{a+1}{r}, & \cdots, & \frac{a+r-1}{r} \end{matrix}\right] = \frac{(q-a)_n}{\{q+n(r+1)-1\}(q)_{n-1}}.$$

It is perhaps worth while noting that, for $p=-(a+3n)$, (3.13) gives the sum of a "half-poised" series of the type ${}_6F_5$, and, for $b=a+\frac{1}{2}$, $p=a-\frac{3}{2}n$, (3.21) gives a relation between a "half-poised" series ${}_6F_5$ and a series of the type ${}_3F_2(4)$.

4. Some other known relations are*

$$(4.1)\quad (1-x)^{-a}\,{}_2F_1\left[a,\ \beta;\ a+\beta+\tfrac{1}{2};\ \frac{-x^2}{4(1-x)}\right] = {}_2F_1[2a,\ a+\beta;\ 2a+2\beta;\ x],$$

$$(4.2)\quad (1-x)^{a+\beta-\frac{1}{2}}\,{}_2F_1[a,\ \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)] = {}_2F_1[a-\beta+\tfrac{1}{2},\ \beta-a+\tfrac{1}{2};\ a+\beta+\tfrac{1}{2};\ x],$$

$$(4.3)\quad {}_2F_1[a,\ \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)] = {}_2F_1[2a,\ 2\beta;\ a+\beta+\tfrac{1}{2};\ x],$$

$$(4.4)\quad (1-x)^{-2a}\,{}_2F_1\left[a,\ a+\tfrac{1}{2};\ \beta+\tfrac{1}{2};\ \frac{x^2}{(1-x)^2}\right] = {}_2F_1[2a,\ \beta;\ 2\beta;\ 2x].$$

These relations may be proved by comparing coefficients and using Saalschütz's formula in the first three and Vandermonde's in the last. From these formulae we derive the transformations

$$(4.11)\quad {}_4F_3\left[\begin{matrix} a,\ \beta,\ -\frac{1}{2}n,\ \frac{1}{2}(1-n) \\ a+\beta+\frac{1}{2},\ p,\ 1-p-n \end{matrix}\right] = \frac{(p-a)_n}{(p)_n}\,{}_3F_2\left[\begin{matrix} 2a,\ a+\beta,\ -n \\ 2a+2\beta,\ 1-p+a-n \end{matrix}\right],$$

* (4.3) is due to Gauss, "Disquisitiones generales circa seriem infinitam, etc.", *Werke*, 3 (102). (4.4) is given by Kummer, *Journal für Math.*, 15 (1836), 78 (52). For (4.1) and (4.2), see Bailey, *loc. cit.* (4.22) and (4.23).

$$(4.21)\quad {}_4F_3\left[\begin{matrix}\alpha, \beta, 2\gamma+n, -n\\ \alpha+\beta+\frac{1}{2}, \gamma, \gamma+\frac{1}{2}\end{matrix}\right] = \frac{(2\gamma-\alpha-\beta+\frac{1}{2})_n}{(2\gamma)_n}\,{}_3F_2\left[\begin{matrix}\alpha-\beta+\frac{1}{2}, \beta-\alpha+\frac{1}{2}, -n\\ \alpha+\beta+\frac{1}{2}, 2\gamma+\frac{1}{2}-\alpha-\beta\end{matrix}\right],$$

$$(4.31)\quad {}_4F_3\left[\begin{matrix}\alpha, \beta, 2\gamma+n, -n\\ \alpha+\beta+\frac{1}{2}, \gamma, \gamma+\frac{1}{2}\end{matrix}\right] = {}_3F_2\left[\begin{matrix}2\alpha, 2\beta, -n\\ \alpha+\beta+\frac{1}{2}, 2\gamma\end{matrix}\right],$$

$$(4.41)\quad {}_4F_3\left[\begin{matrix}\alpha, \alpha+\frac{1}{2}, -\frac{1}{2}n, \frac{1}{2}(1-n)\\ \beta+\frac{1}{2}, \gamma, \gamma+\frac{1}{2}\end{matrix}\right] = \frac{(2\gamma-2\alpha)_n}{(2\gamma)_n}\,{}_3F_2\left[\begin{matrix}2\alpha, \beta, -n; 2\\ 2\beta, 1-n+2\alpha-2\gamma\end{matrix}\right].$$

The series on the left of (4 . 11), (4 . 21) and (4 . 31) are Saalschützian. The formulae (4 . 21) and (4 . 31) are related to our previous result (2 . 11), which reduces to a relation between ${}_3F_2$ and ${}_4F_3$ when $z = \frac{1}{2}(1+x)$*, giving

$$(A)\quad {}_4F_3\left[\begin{matrix}\alpha, \beta, 2\gamma+n, -n\\ \alpha+\beta+\frac{1}{2}, \gamma, \gamma+\frac{1}{2}\end{matrix}\right] = \frac{(2\gamma-2\alpha)_n}{(2\gamma)_n}\,{}_3F_2\left[\begin{matrix}2\alpha, \frac{1}{2}+\alpha-\beta, -n\\ \frac{1}{2}+\alpha+\beta, 1+2\alpha+2\gamma-n\end{matrix}\right],$$

the ${}_3F_2$ being a nearly-poised series. There are 17 other terminating ${}_3F_2$'s into which a given terminating ${}_3F_2$ can be transformed. It can be verified† that the series occurring in (4 . 21) and (4 . 31) are two of those which can be obtained in this way from that in (A).

The relation (4 . 41) has been obtained previously‡ by a different method.

5. In conclusion, I should like to state that the process used in this paper, of multiplying a known relation by a power of $(1-x)$ and then equating coefficients, has been used by Whipple§ to obtain a relation involving a half-poised series of the type ${}_4F_3$. Some other identities can be obtained by using other formulae given in my paper on products of hypergeometric series, but I think that I have given all that are of any interest.

My acknowledgements are due to the referee, particularly for his suggestions as to the form in which the paper should be presented.

* Cf. Whipple, *loc. cit.* (6.52).

† In Whipple's notation $Fp(0; 4, 5) \sim Fn(4; 0, 2) \sim Fn(4; 0, 1)$. See Whipple, "A group of generalized hypergeometric series, etc.", *Proc. London Math. Soc.* (2), 23 (1924), 104–114.

‡ Bailey, *loc. cit.* (5.41).

§ Whipple, *loc. cit.*, §8.

SOME IDENTITIES INVOLVING GENERALIZED HYPERGEOMETRIC SERIES

By W. N. BAILEY.

[Received 19 April, 1928.—Read 26 April, 1928.]

1. In a recent paper* on generalized hypergeometric series I showed how transformations of such series could be obtained by a certain process. In this paper a much more powerful method is given for obtaining such transformations. It will be seen, for example, that, assuming only known results for series of the type ${}_3F_2(1)$, one application of the method is sufficient to discover the most general known transformation of nearly-poised series (as in the case of the simpler method of the paper already referred to), while two applications are sufficient to give the most general known transformation of terminating well-poised series†. Of the new results obtained perhaps the most interesting is (7.21), which gives a relation between two well-poised series of the type ${}_9F_8$ and includes as a limiting case the transformation of well-poised series mentioned above (and also Dougall's theorem). Several formulae are obtained which express nearly-poised series in terms of well-poised series of the type ${}_9F_8$.

2. Suppose that we have a relation‡

$$(2.1)\quad {}_3F_2\left[\begin{matrix}\rho_1, \rho_2, \rho_3; \\ \sigma_1, \sigma_2;\end{matrix} bX\right] = (1-X)^{-\kappa}\, {}_3F_2\left[\begin{matrix}a_1, a_2, a_3; \\ \beta_1, \beta_2;\end{matrix} cX^\lambda/(1-X)^\mu\right],$$

where λ and μ are integers, λ being positive.

For X put $xy/(1-x)(1-y)$, and multiply by $(1-x)^{-p}(1-y)^{-q}$.

* Bailey, **1**.

† See the foot-note to (5.4) below.

‡ A list of such relations is given in Bailey, **2**, § 4.

Equating coefficients of $x^m y^n$ on the two sides, we have, after some reductions,

$$(2.2)\quad {}_5F_4\left[\begin{matrix}\rho_1, \rho_2, \rho_3, -m, -n; \\ \sigma_1, \sigma_2, p, q;\end{matrix}\ b\right]$$
$$= \frac{(p-\kappa)_m}{(p)_m} \sum_r \frac{(-1)^{\lambda r} c^r (a_1)_r (a_2)_r (a_3)_r (-m)_{\lambda r} (-n)_{\lambda r} (1-p+\kappa)_{r(\mu-\lambda)}}{r!\,(\beta_1)_r (\beta_2)_r (1-p+\kappa-m)_{\mu r} (q)_{\lambda r}}$$
$$\times {}_3F_2\left[\begin{matrix}\kappa+\mu r, q+n, -m+\lambda r \\ 1-p+\kappa-m+\mu r, q+\lambda r\end{matrix}\right].$$

This formula, with any real number a replacing the positive integer n, may be proved as follows. Making use of the fact that the coefficients of X^s in (2.1) are equal, we have

$${}_5F_4\left[\begin{matrix}\rho_1, \rho_2, \rho_3, a, -m; \\ \sigma_1, \sigma_2, p, q;\end{matrix}\ b\right] = \sum_{s=0}^{m} \frac{(\rho_1)_s (\rho_2)_s (\rho_3)_s (a)_s (-m)_s \cdot b^s}{s!\,(\sigma_1)_s (\sigma_2)_s (p)_s (q)_s}$$
$$= \sum_{s=0}^{m} \sum_{r=0}^{[s/\lambda]} \frac{(-1)^{\lambda r} c^r (a_1)_r (a_2)_r (a_3)_r (-s)_{\lambda r} (\kappa+\mu r)_{s-\lambda r} (a)_s (-m)_s}{r!\,(\beta_1)_r (\beta_2)_r\, s!\,(p)_s (q)_s}$$
$$= \sum_{s=0}^{m} \sum_{r=0}^{[s/\lambda]} \frac{c^r (a_1)_r (a_2)_r (a_3)_r (\kappa+\mu r)_{s-\lambda r} (a)_s (-m)_s}{r!\,(\beta_1)_r (\beta_2)_r (p)_s (q)_s (s-\lambda r)!}.$$

Put $s-\lambda r = t$, and we get

$$\sum_{r=0}^{[m/\lambda]} \sum_{t=0}^{m-\lambda r} \frac{c^r (a_1)_r (a_2)_r (a_3)_r (a)_{\lambda r} (-m)_{\lambda r}}{r!\,(\beta_1)_r (\beta_2)_r (p)_{\lambda r} (q)_{\lambda r}} \cdot \frac{(\kappa+\mu r)_t (a+\lambda r)_t (\lambda r-m)_t}{t!\,(p+\lambda r)_t (q+\lambda r)_t}.$$

Thus

$$(2.3)\quad {}_5F_4\left[\begin{matrix}\rho_1, \rho_2, \rho_3, a, -m; \\ \sigma_1, \sigma_2, p, q;\end{matrix}\ b\right]$$
$$= \sum_{r=0}^{[m/\lambda]} \frac{c^r (a_1)_r (a_2)_r (a_3)_r (a)_{\lambda r} (-m)_{\lambda r}}{r!\,(\beta_1)_r (\beta_2)_r (p)_{\lambda r} (q)_{\lambda r}} {}_3F_2\left[\begin{matrix}\kappa+\mu r, a+\lambda r, -m+\lambda r \\ p+\lambda r, q+\lambda r\end{matrix}\right].$$

This is easily seen to be equivalent to (2.2), n not necessarily being an integer, by using one of the known relations between series of the type ${}_3F_2(1)$. It will be seen* that the form (2.3) is more suitable for purposes of generalization.

We may, in this identity, take special values of the parameters such as $p = a$, and so reduce the number of parameters in the F's which occur. The examples of interest are, of course, those in which the series on the right of the identities reduce to hypergeometric series.

* In §4.

3. We shall now give an example of the process outlined above. Consider the formula *

$$(3.1)\quad {}_3F_2\left[\begin{matrix} a, b, c\,; & x \\ 1+a-b, & 1+a-c \end{matrix}\right] = (1-x)^{-a}\,{}_3F_2\left[\begin{matrix} 1+a-b-c, \tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a\,; & -4x/(1-x)^2 \\ 1+a-b, & 1+a-c\,; \end{matrix}\right].$$

Then using (2.3) we have

$$(3.2)\quad {}_5F_4\left[\begin{matrix} a, b, c, d, & -m \\ 1+a-b, & 1+a-c, p, q \end{matrix}\right] = \frac{(p-a)_m}{(p)_m}\sum_{r=0}^{m}\frac{(-4)^r(1+a-b-c)_r(\frac{1}{2}a)_r(\frac{1}{2}+\frac{1}{2}a)_r(d)_r(-m)_r}{r!\,(1+a-b)_r(1+a-c)_r(p)_r(q)_r} \times {}_3F_2\left[\begin{matrix} a+2r, & d+r, & -m+r \\ & p+r, & q+r \end{matrix}\right].$$

In order to make the series on the right a hypergeometric series we must choose our parameters so that the ${}_3F_2$ reduce to a single term of gamma functions. This is the case when $p = 1+a-d$, $q = 1+a+m$, for then Dixon's formula† sums the ${}_3F_2$ and we get‡

$$(3.3)\quad {}_5F_4\left[\begin{matrix} a, b, c, d, & -m \\ 1+a-b, & 1+a-c, 1+a-d, 1+a+m \end{matrix}\right] = \frac{(1+a)_m(1+\frac{1}{2}a-d)_m}{(1+\frac{1}{2}a)_m(1+a-d)_m}\,{}_4F_3\left[\begin{matrix} 1+a-b-c, & \tfrac{1}{2}a, & d, & -m \\ 1+a-b, & 1+a-c, & d-\tfrac{1}{2}a-m & \end{matrix}\right],$$

the series on the left being well-poised and that on the right Saalschützian.

If, however, we first reduce the number of parameters so that the ${}_3F_2$ become ${}_2F_1$, we can apply Gauss's theorem instead of Dixon's. Thus, with $d = q$ we get

$$(3.4)\quad {}_4F_3\left[\begin{matrix} a, b, c, & -m \\ 1+a-b, & 1+a-c, p \end{matrix}\right] = \frac{(p-a)_m}{(p)_m}\,{}_5F_4\left[\begin{matrix} 1+a-p, & \frac{1}{2}(1+a), & \frac{1}{2}a, & 1+a-b-c, & -m \\ 1+a-b, & 1+a-c, & 1+\frac{1}{2}(a-q-m), & \frac{1}{2}(1+a-q-m) & \end{matrix}\right],$$

which is due to Whipple§ and is the most general known transformation of a nearly-poised series.

---

* Whipple, **1** (7.1). See also Bailey, **2** (4.12) for an alternative proof.

† Dixon, **1**.

‡ (3.3) is the particular case of (7.7) in Whipple, **2**, when $g = \frac{1}{2}a$.

§ Whipple, **1** (6.5).

A particular case of (3 . 3) is given by putting $b = 1+\frac{1}{2}a$, when the series on the right reduces to a ${}_3F_2$ which can be summed by Saalschütz's theorem *. We thus obtain †

$$(3\,.\,5)\qquad {}_5F_4\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ c,\ d,\ -m \\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a+m \end{matrix}\right] = \frac{(1+a)_m(1+a-c-d)_m}{(1+a-c)_m(1+a-d)_m};$$

a result which we shall use later.

4. The formula (2 . 1) from which we started is equivalent to the formula

$$(4\,.\,1)\qquad \sum_{r=0}^{[n/\lambda]} \frac{(a_1)_r(a_2)_r(a_3)_r(-n)_{\lambda r}(\kappa+n)_{r(\mu-\lambda)}(-1)^{\lambda r}c^r}{r!\,(\beta_1)_r(\beta_2)_r(\kappa)_{\mu r}} = \frac{(\rho_1)_n(\rho_2)_n(\rho_3)_n b^n}{(\sigma_1)_n(\sigma_2)_n(\kappa)_n},$$

this being the result obtained by equating coefficients of x^n on the two sides of (2 . 1). Hence the main result of § 2 states that, if (4 . 1) is true, so is (2 . 3).

This result can easily be generalized. In fact, exactly the same kind of proof as that used for (2 . 3) shows that, if

$$(4\,.\,2)\qquad \sum_{r=0}^{[n/\lambda]} \frac{(a_1)_r(a_2)_r(a_3)_r(-n)_{\lambda r}(\kappa_1+n)_{r(\mu_1-\lambda)}(\kappa_2+n)_{r(\mu_2-\lambda)}(l_1)_{\nu_1 r}(l_2)_{\nu_2 r}(-1)^{\lambda r}c_r}{r!\,(\beta_1)_r(\beta_2)_r(l_1+n)_{r(\nu_1-\lambda)}(l_2+n)_{r(\nu_2-\lambda)}(\kappa_1)_{\mu_1 r}(\kappa_2)_{\mu_2 r}}$$

$$= \frac{(\rho_1)_n(\rho_2)_n(\rho_3)_n(l_1)_n(l_2)_n b^n}{(\sigma_1)_n(\sigma_2)_n(\kappa_1)_n(\kappa_2)_n},$$

then

$$(4\,.\,3)\qquad F\left[\begin{matrix} \rho_1,\ \rho_2,\ \rho_3,\ a_1,\ a_2,\ -m\,;\ b \\ \sigma_1,\ \sigma_2,\ p_1,\ p_2,\ p_3\,; \end{matrix}\right]$$

$$= \sum_{r=0}^{[m/\lambda]} \frac{c^r(a_1)_r(a_2)_r(a_3)_r(a_1)_{\lambda r}(a_2)_{\lambda r}(-m)_{\lambda r}}{r!\,(\beta_1)_r(\beta_2)_r(p_1)_{\lambda r}(p_2)_{\lambda r}(p_3)_{\lambda r}}$$

$$\times F\left[\begin{matrix} \kappa_1+\mu_1 r,\ \kappa_2+\mu_2 r,\ a_1+\lambda r,\ a_2+\lambda r,\ -m+\lambda r \\ l_1+\nu_1 r,\ l_2+\nu_2 r,\ p_1+\lambda r,\ p_2+\lambda r,\ p_3+\lambda r \end{matrix}\right].$$

In these formulae there may be any number of the quantities a, β, ρ, σ, κ, l, a, p. It is supposed that λ is a positive integer, but the μ's and ν's may be positive or negative provided that we replace $(a)_{-n}$ by $(-1)^n/(1-a)_n$. The series on the left of (4 . 2) can be expressed as a generalized hypergeometric series. If we can sum the F on the right of (4 . 3) in terms of gamma functions, the right-hand side of (4 . 3)

* Saalschütz, **1**.

† Whipple, **2** (5.2), and Dougall, **1** (9).

can also be expressed as a generalized hypergeometric series. Thus, from a known sum of a generalized hypergeometric series, we have a method of deducing transformations of such series.

5. In applying the method of § 4 it is natural to begin with Saalschütz's formula for the sum of a ${}_3F_2$. This formula can be written

$$(5.1)\qquad {}_3F_2\left[\begin{matrix}1+a-b-c,\ a+n,\ -n\\ 1+a-b,\ 1+a-c\end{matrix}\right]=\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n},$$

the form in which it is used to prove (3.1). From this we have

$$\sum_{r=0}^{n}\frac{(1+a-b-c)_r(a+n)_r(-n)_r}{r!\,(1+a-b)_r(1+a-c)_r}=\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n},$$

which can be written

$$(5.2)\qquad \sum_{r=0}^{n}\frac{(a+n)_r(-n)_r(-1)^r}{(a)_{2r}}\cdot\frac{(-4)^r(1+a-b-c)_r(\tfrac12 a)_r(\tfrac12+\tfrac12 a)_r}{r!\,(1+a-b)_r(1+a-c)_r}$$
$$=\frac{(a)_n(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n}\cdot\frac{1}{(c)_n}.$$

This is equivalent to (4.2) with $\lambda=1$, $\kappa_1=a$, $\mu_1=2$, and so we have, from (4.3),

$$(5.3)\qquad {}_{s+4}F_{s+3}\left[\begin{matrix}a,\ b,\ c,\ a_1,\ a_2,\ \ldots,\ a_s,\ -m\\ 1+a-b,\ 1+a-c,\ p_1,\ p_2,\ \ldots,\ p_{s+1}\end{matrix}\right]$$
$$=\sum_{r=0}^{m}\frac{(-4)^r(1+a-b-c)_r(\tfrac12 a)_r(\tfrac12+\tfrac12 a)_r(a_1)_r(a_2)_r\ldots(a_s)_r(-m)_r}{r!\,(1+a-b)_r(1+a-c)_r(p_1)_r(p_2)_r\ldots(p_{s+1})_r}$$
$$\times{}_{s+2}F_{s+1}\left[\begin{matrix}a+2r,\ a_1+r,\ a_2+r,\ \ldots.\ a_s+r,\ -m+r\\ p_1+r,\ p_2+r,\ \ldots,\ p_{s+1}+r\end{matrix}\right].$$

If we take $s=0$ or 1, we are simply led to (3.3) and (3.4). Now take $s=3$, $a_3=1+\frac12 a$, $p_4=\frac12 a$, and $1+a=a_1+p_1=a_2+p_2=-m+p_3$, so that the ${}_5F_4$ on the right of (5.3) can be summed by (3.5). We are thus led to the transformation

$$(5.4)\qquad {}_7F_6\left[\begin{matrix}a,\ 1+\frac12 a,\ b,\ c,\ a_1,\ a_2,\ -m\\ \frac12 a,\ 1+a-b,\ 1+a-c,\ 1+a-a_1,\ 1+a-a_2,\ 1+a+m\end{matrix}\right]$$
$$=\frac{(1+a)_m(1+a-a_1-a_2)_m}{(1+a-a_1)_m(1+a-a_2)_m}$$
$$\times{}_4F_3\left[\begin{matrix}1+a-b-c,\ a_1,\ a_2,\ -m\\ 1+a-b,\ 1+a-c,\ a_1+a_2-a-m\end{matrix}\right],$$

which expresses a well-poised series of the type ${}_7F_6(1)$ in terms of a Saalschützian ${}_4F_3$. This transformation is the most general transformation obtained by Whipple* for a terminating well-poised series.

If $1+2a = b+c+a_1+a_2-m$, the series on the right reduces to a Saalschützian ${}_3F_2$ and can be summed. We are thus led to Dougall's theorem†

$$(5.41)\quad {}_7F_6\left[\begin{matrix} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & e, & -m \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m & \end{matrix}\right]$$
$$= \frac{(1+a)_m(1+a-b-c)_m(1+a-b-d)_m(1+a-c-d)_m}{(1+a-b-c-d)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m},$$

provided that $1+2a = b+c+d+e-m$.

6. Now write Saalschütz's formula in the form

$$(6.1)\quad {}_3F_2\left[\begin{matrix} a, & b+n, & -n \\ c+n, & 1+a+b-c-n & \end{matrix}\right] = \frac{(c)_n(c-a)_{2n}(c-b)_n}{(c)_{2n}(c-a)_n(c-a-b)_n},$$

which can be written

$$(6.2)\quad \sum_{r=0}^{n} \frac{(-n)_r(b+n)_r(c-a-b+n)_{-r}(c)_{2r}(-1)^r}{(c+n)_r(b)_{2r}} \frac{(a)_r(\tfrac{1}{2}b)_r(\tfrac{1}{2}+\tfrac{1}{2}b)_r}{r!\,(\tfrac{1}{2}c)_r(\tfrac{1}{2}+\tfrac{1}{2}c)_r}$$
$$= \frac{(c)_n}{(b)_n(c-a-b)_n} \cdot \frac{(b)_n(c-b)_n(\tfrac{1}{2}c-\tfrac{1}{2}a)_n(\tfrac{1}{2}+\tfrac{1}{2}c-\tfrac{1}{2}a)_n}{(c-a)_n(\tfrac{1}{2}c)_n(\tfrac{1}{2}+\tfrac{1}{2}c)_n}.$$

Using (4.3) with $\lambda = 1$, $\kappa_1 = b$, $\mu_1 = 2$, $\kappa_2 = c-a-b$, $\mu_2 = 0$, $l_1 = c$, $\nu_1 = 2$, we get

$$(6.3)\quad {}_{s+5}F_{s+4}\left[\begin{matrix} b, & c-b, & \tfrac{1}{2}c-\tfrac{1}{2}a, & \tfrac{1}{2}+\tfrac{1}{2}c-\tfrac{1}{2}a, & a_1, & a_2, & \ldots, & a_s, & -m \\ & c-a, & \tfrac{1}{2}c, & \tfrac{1}{2}+\tfrac{1}{2}c, & p_1, & p_2, & \ldots, & p_{s+1} & \end{matrix}\right]$$
$$= \sum_{r=0}^{m} \frac{(a)_r(\tfrac{1}{2}b)_r(\tfrac{1}{2}+\tfrac{1}{2}b)_r(a_1)_r(a_2)_r\ldots(a_s)_r(-m)_r}{r!\,(\tfrac{1}{2}c)_r(\tfrac{1}{2}+\tfrac{1}{2}c)_r(p_1)_r\ldots(p_{s+1})_r}$$
$$\times {}_{s+3}F_{s+2}\left[\begin{matrix} b+2r, & c-a-b, & a_1+r, & \ldots, & a_s+r, & -m+r \\ & c+2r, & p_1+r, & \ldots, & p_{s+1}+r & \end{matrix}\right].$$

First take $s = 0$, $p_1 = 1-a-m$, in which case we can sum the ${}_3F_2$ on

* Whipple, **2** (7.7). See also Whipple, **4**. Whipple proves that the formula is also true when m is not integral [replacing an expression such as $(a)_m$ by $\Gamma(a+m)/\Gamma(a)$ on the right], provided that $1+a-b-c$ is a negative integer, *i.e.* the formula is true provided only that the series ${}_4F_3$ is terminating.

† Dougall, **1** (6). See also Whipple, **2** (8.2).

the right by Saalschütz's formula, and we are led to

$$(6.4)\quad {}_5F_4\left[\begin{matrix} b,\ c-b,\ \frac{1}{2}(c-a),\ \frac{1}{2}(1+c-a),\ -m \\ c-a,\ \frac{1}{2}c,\ \frac{1}{2}(1+c),\ 1-a-m \end{matrix}\right]$$
$$= \frac{(a+b)_m (c-b)_m}{(a)_m (c)_m}\, {}_5F_4\left[\begin{matrix} a,\ \frac{1}{2}b,\ \frac{1}{2}(1+b),\ a+b+m,\ -m \\ c+m,\ \frac{1}{2}(a+b),\ \frac{1}{2}(1+a+b),\ 1+b-c-m \end{matrix}\right].$$

Now take $s=4$ in (6.3) and choose the constants so that the ${}_7F_6$ on the right can be summed by Dougall's theorem, and we are led to

$$(6.5)\quad {}_5F_4\left[\begin{matrix} 2c-2b-1,\ a_1,\ a_2,\ a_3,\ -m \\ c-a_1,\ c-a_2,\ c-a_3,\ c+m \end{matrix}\right]$$
$$= \frac{(c)_m (1+b-a_1)_m (1+b-a_2)_m (1+b-a_3)_m}{(1+b)_m (c-a_1)_m (c-a_2)_m (c-a_3)_m}$$
$$\times {}_9F_8\left[\begin{matrix} b,\ 1+\frac{1}{2}b,\ a_1,\ a_2,\ a_3,\ 1+b-\frac{1}{2}c,\ \frac{1}{2}+b-\frac{1}{2}c,\ c-b,\ -m \\ \frac{1}{2}b,\ 1+b-a_1,\ 1+b-a_2,\ 1+b-a_3,\ \frac{1}{2}c,\ \frac{1}{2}+\frac{1}{2}c,\ 1+2b-c,\ 1+b+m \end{matrix}\right],$$

provided that $a_1+a_2+a_3 = b+c+m$.

This formula expresses a Saalschützian nearly-poised series ${}_5F_4$ as a well-poised series of the type ${}_9F_8$. If the series are reversed we obtain a formula equivalent to

$$(6.51)\quad {}_5F_4\left[\begin{matrix} a,\ b,\ c,\ d,\ -m \\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w \end{matrix}\right] = \frac{(1-a+2k)_m (1-a+k)_m}{(1+k)_m (1+2k-2a)_m}$$
$$\times {}_9F_8\left[\begin{matrix} k,\ 1+\frac{1}{2}k,\ \frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}a,\ k+b-a,\ k+c-a, \\ \qquad\qquad k+d-a,\ 1+a-w,\ -m \\ \frac{1}{2}k,\ 1+k-\frac{1}{2}a,\ \frac{1}{2}+k-\frac{1}{2}a,\ 1+a-b, \\ \qquad 1+a-c,\ 1+a-d,\ k-a+w,\ 1+k+m \end{matrix}\right],$$

where $k = 1+2a-b-c-d$, and the series on the left is restricted to be Saalschützian. The series on the left of (6.51) is still nearly-poised, but the breakdown in the equality of sums of pairs of parameters occurs with the last pair.

7. We now turn to Dougall's formula (5.41), which can be written

$$(7.1)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ c,\ d,\ e,\ k+n,\ -n \\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-k-n,\ 1+a+n \end{matrix}\right]$$
$$= \frac{(1+a)_n (1+a-c-d)_n (1+a-c-e)_n (1+a-d-e)_n}{(1+a-c)_n (1+a-d)_n (1+a-e)_n (1+a-c-d-e)_n},$$

where $k = 1+2a-c-d-e$.

From this we have, by our usual process,

$$(7.2)\quad {}_{s+5}F_{s+4}\left[\begin{matrix}1+a-d-e,\ 1+a-c-e,\ 1+a-c-d,\ k,\ a_1,\ \ldots,\ a_s,\ -m\\ 1+a-c,\ 1+a-d,\ 1+a-e,\ p_1,\ \ldots,\ p_{s+1}\end{matrix}\right]$$

$$=\sum_{r=0}^{m}\frac{(a)_r(c)_r(d)_r(e)_r(\tfrac{1}{2}k)_r(\tfrac{1}{2}+\tfrac{1}{2}k)_r(a_1)_r\ldots(a_s)_r(-m)_r}{r!\,(\tfrac{1}{2}a)_r(\tfrac{1}{2}+\tfrac{1}{2}a)_r(1+a-c)_r(1+a-d)_r(1+a-e)_r(p_1)_r\ldots(p_{s+1})_r}$$

$$\times{}_{s+3}F_{s+2}\left[\begin{matrix}k+2r,\ k-a,\ a_1+r,\ \ldots,\ a_s+r,\ -m+r\\ 1+a+2r,\ p_1+r,\ \ldots,\ p_{s+1}+r\end{matrix}\right],$$

where k has the value given above.

If $s=0$, $p=2k-2a-m$, the ${}_3F_2$ on the right can be summed by Saalschütz's formula, and we are again led to (6.51). Now take $s=4$ and choose the constants so that we can sum the ${}_7F_6$ on the right by Dougall's theorem. Then we get

$$(7.21)\quad {}_9F_8\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ c,\ d,\ e,\ a_1,\ a_2,\ a_3,\ -m\\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\\ 1+a-a_1,\ 1+a-a_2,\ 1+a-a_3,\ 1+a+m\end{matrix}\right]$$

$$=\frac{(1+a)_m(1+k-a_1)_m(1+k-a_2)_m(1+k-a_3)_m}{(1+k)_m(1+a-a_1)_m(1+a-a_2)_m(1+a-a_3)_m}$$

$$\times{}_3F_8\left[\begin{matrix}k,\ 1+\tfrac{1}{2}k,\ k+c-a,\ k+d-a,\\ k+e-a,\ a_1,\ a_2,\ a_3,\ -m\\ \tfrac{1}{2}k,\ 1+a-c,\ 1+a-d,\ 1+a-e,\\ 1+k-a_1,\ 1+k-a_2,\ 1+k-a_3,\ 1+k+m\end{matrix}\right],$$

where $k=1+2a-c-d-e$, and the parameters are subject to the restriction

$$a_1+a_2+a_3+c+d+e-m=2+3a.$$

This formula is a relation between two well-poised series, either of which is of general type (except for the second parameter) but is subject to the restriction that the sum of the denominator parameters exceeds the sum of the numerator parameters by two. The formula contains several interesting particular cases. If we substitute for k and a_3 and let $e\to\infty$, we derive (5.4), Whipple's transformation of a well-poised series ${}_7F_6$. If we put $k=a-e$, so that $c+d=1+a$, the formula reduces to Dougall's theorem. If we substitute for a_3 and let $a_2\to\infty$, or substitute for e and let $d\to\infty$, we get relations between terminating well-poised series given by Whipple*. Evidently (7.21) could be used to write down a number

* Whipple, **3**. In Whipple's notation the formulae are equivalent to $W(1;4)=W(2;4)$ and $W(1;4)=W(1;5)$.

of relations between well-poised series ${}_9F_8$ by taking the parameters in different sets of three.

Now take Dougall's theorem in the form

$$(7.3)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ d+n,\ e-n,\ k+n,\ -n \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d-n,\ 1+a-e+n,\ 1+a-k-n,\ 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n\,(c+d-a)_n\,(1+a-e)_n\,(1+a-c-e)_{2n}\,(1+a-d-e)_n}{(1+a-c)_n\,(1+a-e)_{2n}\,(d-a)_n\,(1+a-c-e)_n\,(1+a-c-d-e)_n},$$

where k has the same value as before. The usual process gives, in particular,

$$(7.4)\quad {}_9F_8\left[\begin{matrix} c+d-a,\ 1+a-d-e,\ \tfrac{1}{2}(1+a-c-e), \\ \qquad\qquad 1+\tfrac{1}{2}(a-c-e),\ d,\ k,\ a_1,\ a_2,\ -m \\ 1-e,\ 1+a-c-e,\ 1+a-c,\ \tfrac{1}{2}(1+a-e), \\ \qquad\qquad 1+\tfrac{1}{2}(a-e),\ p_1,\ p_2,\ p_3 \end{matrix}\right]$$

$$= \sum_{r=0}^{m} \frac{(a)_r\,(1+\tfrac{1}{2}a)_r\,(c)_r\,(d)_{2r}\,(k)_{2r}\,(a_1)_r\,(a_2)_r\,(-m)_r}{r!\,(1+a-e)_{2r}\,(1+a)_{2r}\,(\tfrac{1}{2}a)_r\,(1+a-c)_r\,(p_1)_r\,(p_2)_r\,(p_3)_r}$$

$$\times {}_7F_6\left[\begin{matrix} d+2r,\ k+2r,\ d-a,\ k-a,\ a_1+r,\ a_2+r,\ -m+r \\ 1-e,\ 1+a+2r,\ 1+a-e+2r,\ p_1+r,\ p_2+r,\ p_3+r \end{matrix}\right].$$

Choosing the constants so that the ${}_7F_6$ can be summed by Dougall's theorem, we find that

$$(7.41)\quad {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ \tfrac{1}{2}k,\ \tfrac{1}{2}+\tfrac{1}{2}k,\ a_1,\ a_1-d-m, \\ \qquad\qquad k-d-m,\ 1+2a-k+m,\ -m \\ \tfrac{1}{2}a,\ 1+a-\tfrac{1}{2}k,\ \tfrac{1}{2}+a-\tfrac{1}{2}k,\ 1+a-a_1,\ 1+a-a_1+d+m, \\ \qquad\qquad 1+a-k+d+m,\ k-a-m,\ 1+a+m \end{matrix}\right]$$

$$= \frac{(1+a)_m\,(1-k+d)_m\,(1+d-k+a)_m\,(1+d-a_1)_m}{(1+d)_m\,(1-k+a)_m\,(1+2a-k)_m\,(1+a-a_1)_m}$$

$$\times {}_9F_8\left[\begin{matrix} d,\ 1+\tfrac{1}{2}d,\ k,\ a_1,\ 1+d+a-2k, \\ \qquad \tfrac{1}{2}(1+d+k-a),\ \tfrac{1}{2}(d+k-a),\ 1+a-k,\ -m \\ \tfrac{1}{2}d,\ 1+d-k,\ 1+d-a_1,\ 2k-a,\ \tfrac{1}{2}(1+d-k+a), \\ \qquad\qquad 1+\tfrac{1}{2}(d-k+a),\ d+k-a,\ 1+d+m \end{matrix}\right]$$

provided that $a_1 = 1+2a-2k+d+m$.

This is a relation between two well-poised series (though not of general type); it is not a particular case of (7.21).

If, in (7.4), we take $p_1 = a_1$, $p_2 = a_2$, $p_3 = 2d-2a-m$, $k = 1+a-e$, so that $c = a-d$, the series on the right reduces to a Saalschützian

${}_3F_2$. Summing this series we get

$$(7.42)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ \tfrac{1}{2}d,\ \tfrac{1}{2}+\tfrac{1}{2}d,\ a-d,\ 1+2a-d+m,\ -m \\ \tfrac{1}{2}a,\ 1+a-\tfrac{1}{2}d,\ \tfrac{1}{2}+a-\tfrac{1}{2}d,\ 1+d,\ d-a-m,\ 1+a+m \end{matrix}\right] = \frac{(1+a)_m(1+2a-2d)_m}{(1+a-d)_m(1+2a-d)_m},$$

the sum of a particular well-poised series ${}_7F_6$. It is not a particular case of Dougall's theorem.

8. I now gather together the main formulae involving standard types of series. In this list are included several formulae not given in the text. These are derived by the use of the method given above from the formula (7.42) and the following* :

$$(A)\quad {}_3F_2\left[\begin{matrix} 2\kappa-2b,\ b+n,\ -n \\ 1+\kappa-b-n,\ 1+\kappa+n \end{matrix}\right] = \frac{(1+\kappa)_n(\kappa-b)_n(\tfrac{1}{2}+b-\tfrac{1}{2}\kappa)_n(b-\tfrac{1}{2}\kappa)_n}{(\tfrac{1}{2}+\tfrac{1}{2}\kappa)_n(1+\tfrac{1}{2}\kappa)_n(b-\kappa)_n(1+2b-\kappa)_n},$$

$$(B)\quad {}_4F_3\left[\begin{matrix} 2\kappa-2b,\ 1+\tfrac{1}{2}\kappa,\ b+n,\ -n \\ \tfrac{1}{2}\kappa,\ 1+\kappa-b-n,\ 1+\kappa+n \end{matrix}\right] = \frac{(1+\kappa)_n(\kappa-b)_n(1+2b-\kappa)_{2n}}{(\kappa)_{2n}(1+2b-\kappa)_n(b-\kappa)_n},$$

$$(C)\quad {}_4F_3\left[\begin{matrix} \tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ b+n,\ -n \\ \tfrac{1}{2}b,\ \tfrac{1}{2}+\tfrac{1}{2}b,\ 1+a \end{matrix}\right] = \frac{(b-a)_n}{(b)_n}.$$

The formulae (8.2), (8.3), (8.4) and (8.5) are respectively derived from (B), (A), (7.42) and (C). (8.1) is the same as (6.51). (8.6) and (8.7) have been obtained in § 7.

Saalschützian nearly-poised series in terms of well-poised series.

$$(k = 1+2a-b-c-d).$$

$$(8.1)\quad {}_5F_4\left[\begin{matrix} a,\ b,\ c,\ d,\ -m \\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w \end{matrix}\right] = \frac{(1+2k-a)_m(1+k-a)_m}{(1+k)_m(1+2k-2a)_m} \times {}_9F_8\left[\begin{matrix} k,\ 1+\tfrac{1}{2}k,\ \tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ k+b-a,\ k+c-a, \\ \qquad k+d-a,\ 1+a-w,\ -m \\ \tfrac{1}{2}k,\ 1+k-\tfrac{1}{2}a,\ \tfrac{1}{2}+k-\tfrac{1}{2}a,\ 1+a-b, \\ \qquad 1+a-c,\ 1+a-d,\ k-a+w,\ 1+k+m \end{matrix}\right],$$

$$(8.2)\quad {}_6F_5\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ -m \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w \end{matrix}\right] = \frac{(2k-a)_m(k-a)_m}{(1+k)_m(2k-2a)_m} \times {}_9F_8\left[\begin{matrix} k,\ 1+\tfrac{1}{2}k,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ 1+\tfrac{1}{2}a,\ k+b-a, \\ \qquad k+c-a,\ k+d-a,\ 1+a-w,\ -m \\ \tfrac{1}{2}k,\ \tfrac{1}{2}+k-\tfrac{1}{2}a,\ k-\tfrac{1}{2}a,\ 1+a-b,\ 1+a-c, \\ \qquad 1+a-d,\ k-a+w,\ 1+k+m \end{matrix}\right].$$

* (A) and (B) can be derived from Whipple, 1 (3.5), which can be obtained by reversing the series in (3.4) above. (C) can be derived from (7.42) by using (5.4).

Reversing the series in these relations we obtain the following formulae in which the breakdown in the equality of sums of pairs of parameters occurs with the first pair.

$$(8.11)\quad {}_5F_4\left[\begin{matrix}1+2\kappa-2a,\ b,\ c,\ d,\ -m\\ 1+\kappa-b,\ 1+\kappa-c,\ 1+\kappa-d,\ 1+\kappa+m\end{matrix}\right]$$
$$=\frac{(1+\kappa)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}{(1+a)_m(1+\kappa-b)_m(1+\kappa-c)_m(1+\kappa-d)_m}$$
$$\times {}_9F_8\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ \frac{1}{2}+a-\frac{1}{2}\kappa,\ a-\frac{1}{2}\kappa,\ 1+\kappa-a,\ -m\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ \frac{1}{2}+\frac{1}{2}\kappa,\ 1+\frac{1}{2}\kappa,\ 2a-\kappa,\ 1+a+m\end{matrix}\right],$$

$$(8.21)\quad {}_6F_5\left[\begin{matrix}2\kappa-2a,\ 1+\frac{1}{2}\kappa,\ b,\ c,\ d,\ -m\\ \frac{1}{2}\kappa,\ 1+\kappa-b,\ 1+\kappa-c,\ 1+\kappa-d,\ 1+\kappa+m\end{matrix}\right]$$
$$=\frac{(1+\kappa)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}{(1+a)_m(1+\kappa-b)_m(1+\kappa-c)_m(1+\kappa-d)_m}$$
$$\times {}_9F_8\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ \frac{1}{2}+a-\frac{1}{2}\kappa,\ 1+a-\frac{1}{2}\kappa,\ \kappa-a,\ -m\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ \frac{1}{2}+\frac{1}{2}\kappa,\ \frac{1}{2}\kappa,\ 1+2a-\kappa,\ 1+a+m\end{matrix}\right].$$

In all these formulae the parameters are restricted, so that the series on the left is Saalschützian.

Nearly-poised series, in which the sum of the denominator parameters exceeds the sum of the numerator parameters by two, in terms of well-poised series.

$$(k=1+2a-b-c-d).$$

$$(8.3)\quad {}_5F_4\left[\begin{matrix}a,\ b,\ c,\ d,\ -m\\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w\end{matrix}\right]$$
$$=\frac{(k-a)_m(1+2k-a)_{m-1}(2k-a+2m)}{(1+k)_m(2k-2a)_m}$$
$$\times {}_9F_8\left[\begin{matrix}k,\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}a,\ \frac{1}{2}a,\ k+b-a,\\ \qquad k+c-a,\ k+d-a,\ 1+a-w,\ -m\\ \frac{1}{2}k,\ \frac{1}{2}+k-\frac{1}{2}a,\ 1+k-\frac{1}{2}a,\ 1+a-b,\\ \qquad 1+a-c,\ 1+a-d,\ k-a+w,\ 1+k+m\end{matrix}\right],$$

$$(8.4)\quad {}_6F_5\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ -m\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ w\end{matrix}\right]$$
$$=\frac{(k-a-1)_m(2k-a)_{m-1}(2k-a+2m-1)}{(1+k)_m(2k-2a-1)_m}$$
$$\times {}_9F_8\left[\begin{matrix}k,\ 1+\frac{1}{2}k,\ \frac{1}{2}+\frac{1}{2}a,\ 1+\frac{1}{2}a,\ k+b-a,\\ \qquad k+c-a,\ k+d-a,\ 1+a-w,\ -m\\ \frac{1}{2}k,\ \frac{1}{2}+k-\frac{1}{2}a,\ k-\frac{1}{2}a,\ 1+a-b,\\ \qquad 1+a-c,\ 1+a-d,\ k-a+w,\ 1+k+m\end{matrix}\right].$$

Reversing these series we get

$$(8.31)\quad {}_5F_4\left[\begin{matrix}2\kappa-2a,\ b,\ c,\ d,\ -m\\ 1+\kappa-b,\ 1+\kappa-c,\ 1+\kappa-d,\ 1+\kappa+m\end{matrix}\right]$$

$$=\frac{(1+\kappa)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}{(1+a)_m(1+\kappa-b)_m(1+\kappa-c)_m(1+\kappa-d)_m}$$

$$\times {}_9F_8\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ \tfrac{1}{2}+a-\tfrac{1}{2}\kappa,\\ \qquad\qquad a-\tfrac{1}{2}\kappa,\ \kappa-a,\ -m\\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\\ \qquad\qquad \tfrac{1}{2}+\tfrac{1}{2}\kappa,\ 1+\tfrac{1}{2}\kappa,\ 1+2a-\kappa,\ 1+a+m\end{matrix}\right],$$

$$(8.41)\quad {}_6F_5\left[\begin{matrix}2\kappa-1-2a,\ 1+\tfrac{1}{2}\kappa,\ b,\ c,\ d,\ -m\\ \tfrac{1}{2}\kappa,\ 1+\kappa-b,\ 1+\kappa-c,\ 1+\kappa-d,\ 1+\kappa+m\end{matrix}\right]$$

$$=\frac{(1+\kappa)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}{(1+a)_m(1+\kappa-b)_m(1+\kappa-c)_m(1+\kappa-d)_m}$$

$$\times {}_9F_8\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ \tfrac{1}{2}+a-\tfrac{1}{2}\kappa,\ 1+a-\tfrac{1}{2}\kappa,\ \kappa-a-1,\ -m\\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ \tfrac{1}{2}+\tfrac{1}{2}\kappa,\ \tfrac{1}{2}\kappa,\ 2+2a-\kappa,\ 1+a+m\end{matrix}\right].$$

In all these formulae there is the restriction on the parameters indicated above.

A nearly-poised series in terms of a Saalschützian series

$$(8.5)\quad {}_5F_4\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ -m\\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ w\end{matrix}\right]=\frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\times {}_5F_4\left[\begin{matrix}1+\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ 1+a-b-c,\ 1+a-w,\ -m\\ \tfrac{1}{2}(3+a-w-m),\ 1+\tfrac{1}{2}(a-w-m),\ 1+a-b,\ 1+a-c\end{matrix}\right].$$

When these series are reversed we get

$$(8.51)\quad {}_5F_4\left[\begin{matrix}\kappa-a,\ 1+\tfrac{1}{2}\kappa,\ b,\ c,\ -m\\ \tfrac{1}{2}\kappa,\ 1+\kappa-b,\ 1+\kappa-c,\ 1+\kappa+m\end{matrix}\right]=\frac{(1+\kappa)_m(1+\kappa-b-c)_m}{(1+\kappa-b)_m(1+\kappa-c)_m}$$

$$\times {}_5F_4\left[\begin{matrix}\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ b,\ c,\ -m\\ 1+a,\ \tfrac{1}{2}\kappa,\ \tfrac{1}{2}+\tfrac{1}{2}\kappa,\ b+c-\kappa-m\end{matrix}\right].$$

These formulae are the natural successors to Whipple's transformations of nearly-poised series*.

* Whipple, 1.

Relations between well-poised series.

$$(8.6)\quad {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ f,\ g,\ -m \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d, \\ 1+a-e,\ 1+a-f,\ 1+a-g,\ 1+a+m \end{matrix}\right]$$

$$= \frac{(1+a)_m(1+k-e)_m(1+k-f)_m(1+k-g)_m}{(1+k)_m(1+a-e)_m(1+a-f)_m(1+a-g)_m}$$

$$\times {}_9F_8\left[\begin{matrix} k,\ 1+\tfrac{1}{2}k,\ k+b-a,\ k+c-a, \\ k+d-a,\ e,\ f,\ g,\ -m \\ \tfrac{1}{2}k,\ 1+a-b,\ 1+a-c,\ 1+a-d, \\ 1+k-e,\ 1+k-f,\ 1+k-g,\ 1+k+m \end{matrix}\right],$$

where $k = 1+2a-b-c-d$, and the parameters are subject to the restriction

$$b+c+d+e+f+g-m = 2+3a^*.$$

$$(8.7)\quad {}_9F_8\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ \tfrac{1}{2}c,\ \tfrac{1}{2}+\tfrac{1}{2}c,\ b, \\ b-\kappa-m,\ c-\kappa-m,\ 1+2a-c+m,\ -m \\ \tfrac{1}{2}a,\ 1+a-\tfrac{1}{2}c,\ \tfrac{1}{2}+a-\tfrac{1}{2}c,\ 1+a-b, \\ 1+a-b+\kappa+m,\ 1+a-c+\kappa+m,\ c-a-m,\ 1+a+m \end{matrix}\right]$$

$$= \frac{(1+a)_m(1+\kappa-b)_m(1+\kappa-c)_m(1+\kappa+a-c)_m}{(1+\kappa)_m(1+a-b)_m(1+a-c)_m(1+2a-c)_m}$$

$$\times {}_9F_8\left[\begin{matrix} \kappa,\ 1+\tfrac{1}{2}\kappa,\ b,\ c,\ 1+\kappa+a-2c, \\ \tfrac{1}{2}(1+\kappa+c-a),\ \tfrac{1}{2}(\kappa+c-a),\ 1+a-c,\ -m \\ \tfrac{1}{2}\kappa,\ 1+\kappa-b,\ 1+\kappa-c,\ 2c-a,\ \tfrac{1}{2}(1+\kappa-c+a), \\ 1+\tfrac{1}{2}(\kappa-c+a),\ \kappa+c-a,\ 1+\kappa+m \end{matrix}\right],$$

where $1+2a-b-2c+\kappa+m = 0$.

To these may be added (7.42), the sum of a well-poised ${}_7F_6$.

9. In (8.5) let† $w \to \infty$ and $m \to \infty$ so that $-m/w \to x$, and we are led to the quadratic transformation

$$(9.1)\quad {}_4F_3\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c\ ;\ x \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c \end{matrix}\right] = (1+x)(1-x)^{-a-1}$$

$$\times {}_3F_2\left[\begin{matrix} 1+\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ 1+a-b-c\ ;\ -4x/(1-x)^2 \\ 1+a-b,\ 1+a-c \end{matrix}\right].$$

* For an alternative form of (8.6), see Bailey, **3**, formula (3).

† Cf. Whipple, **1**, § 7.

The difficulty of justifying this process is easily avoided by comparing the coefficients of x^n on the two sides. On the right we get the sum of two ${}_3F_2$, and the terms can be combined so that a Saalschützian ${}_3F_2$ is obtained.

One curious result of (9 . 1) is that the sum of the first $n+1$ terms of

$$ {}_4F_3\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c\ ; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c \end{matrix}\ -1\right] $$

is

$$ \frac{(-1)^n(1+a)_n}{n!}\ {}_3F_2\left[\begin{matrix} 1+a-b-c,\ 1+a+n,\ -n \\ 1+a-b,\ 1+a-c \end{matrix}\right]. $$

10. In all the applications of the method of this paper, the quantity λ of § 2 and § 4 has been taken equal to unity. In a previous paper* some transformations are derived from formulae of the type (2 . 1) where $\lambda = 2$, but I have not made any detailed examination of such cases.

With regard to the formula (7 . 21), it seems probable that it could be used to derive a number of relations between well-poised series of the type ${}_9F_8$, corresponding to the relations between well-poised ${}_7F_6$ obtained by Whipple†.

References.

F. J. W. Whipple, **1**. "Some transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 26 (1927), 257–272.

—— **2**. "On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum", *Proc. London Math. Soc.* (2), 24 (1925), 247–263.

—— **3**. "Well-poised series and other generalized hypergeometric series", *Proc. London Math. Soc.* (2), 25 (1926), 525–544.

—— **4**. "A fundamental relation between generalized hypergeometric series", *Journal London Math. Soc.*, 1 (1926), 138–145.

W. N. Bailey, **1**. "Transformations of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 29 (1929), 495–502.

—— **2**. "Products of generalized hypergeometric series", *Proc. London Math. Soc.* (2), 28 (1929), 242–254.

—— **3**. "An identity involving Heine's basic hypergeometric series", *Journal London Math. Soc.*, 4 (1929). [In the Press.]

L. Saalschütz, **1**. "Eine Summationsformel", *Zeitschrift für Math. u. Phys.*, 35 (1890), 186–188.

A. C. Dixon, **1**. "Summation of a certain series", *Proc. London Math. Soc.* (1), 35 (1902), 284–289.

J. Dougall, **1**. "On Vandermonde's theorem, and some more general expansions", *Proc Edinburgh Math. Soc.*, 25 (1907), 114–132.

* Bailey, **1**, §§ 3 and 4.

† Whipple, **3**.

ON DARBOUX LINES

By HAROLD HILTON.

[Received 24 August, 1928.—Read 8 November, 1928.]

1. A *Darboux line* on a surface is one whose osculating sphere touches the surface at each point. Its differential equation is of the second order. This equation has been solved in the case of a conicoid and also in the case of a surface of revolution*. In this paper the solution is given when the surface is a cone; and a few other properties of Darboux lines are added.

If κ_g, τ_g, and κ_n denote the geodesic curvature, geodesic torsion, and normal curvature of a curve on a surface, while suffixes 1 and 2 denote partial differentiation with respect to the parameters u and v of this surface, and dots denote differentiation with respect to any parameter of the curve, we have

$$\begin{aligned}2(EG-F^2)^{\frac{1}{2}}(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)^{\frac{3}{2}}\kappa_g \\ = 2(EG-F^2)(\dot{u}\ddot{v}-\ddot{u}\dot{v})+(2EF_1-FE_1-EE_2)\dot{u}^3 \\ +(2EG_1-3FE_2-GE_1+2FF_1)\dot{u}^2\dot{v} \\ -(2GE_2-3FG_1-EG_2+2FF_2)\dot{u}\dot{v}^2-(2GF_2-FG_2-GG_1)\dot{v}^3. \quad (1)\end{aligned}$$

$$\begin{aligned}(EG-F^2)^{\frac{1}{2}}(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)\tau_g \\ = (LF-ME)\dot{u}^2+(LG-NE)\dot{u}\dot{v}+(MG-NF)\dot{v}^2, \quad (2)\end{aligned}$$

$$(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)\kappa_n = L\dot{u}^2+2M\dot{u}\dot{v}+N\dot{v}^2. \quad (3)$$

The Darboux lines are given by

$$\dot{\kappa}_n = (E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)^{\frac{1}{2}}\kappa_g\tau_g. \quad (4)$$

* R. E. Colomb, *Journal London Math. Soc.*, 1 (1926), 106. Other references will be found in this paper. See also W. Blaschke, *Math. Zeitschrift*, 27 (1927), 150; Enneper, *Göttingen Nachrichten* (1871), 577.

When we perform the differentiation on the left-hand side of (4) and simplify, using Codazzi's equations, we obtain the equation for the Darboux lines in the form

$$3(EG-F^2)(E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)^{\frac{3}{2}}\kappa_g\tau_g$$
$$=\{L_1(EG-F^2)+L(2FF_1-FE_2-GE_1)+M(-2EF_1+FE_1+EE_2)\}\dot{u}^3$$
$$+3\{L_2(EG-F^2)+L(FG_1-GE_2)+M(FE_2-EG_1)\}\dot{u}^2\dot{v}$$
$$+3\{N_1(EG-F^2)+M(FG_1-GE_2)+N(FE_2-EG_1)\}\dot{u}\dot{v}^2$$
$$+\{N_2(EG-F^2)+M(-2GF_2+FG_2+GG_1)+N(2FF_2-FG_1-EG_2)\}\dot{v}^3, \quad (5)$$

where κ_g and τ_g are given by (1) and (2).

If ϕ is the angle between the normal to a surface and the principal normal of a curve on the surface with curvature κ and torsion τ, then

$$\kappa_g = \kappa \sin\phi, \quad (6)$$

$$\tan\phi = \kappa_g/\kappa_n. \quad (7)$$

We see at once from (5) that *three* of the Darboux lines through a given point O of a surface have their principal normals coinciding with the normal of the surface. Their directions are given by equating to zero the right-hand side of (5). If we refer the surface to its lines of curvature as parametric réseau and put

$$F=0, \quad M=0, \quad L=Eb, \quad N=Ga, \quad (8)$$

a and b being the principal curvatures of the surface, and then use Codazzi's equations

$$(a-b)E_2 = 2b_2E \quad \text{and} \quad (b-a)G_1 = 2a_1G,$$

the resulting equation becomes

$$Eb_1\dot{u}^3+3Eb_2\dot{u}^2\dot{v}+3Ga_1\dot{u}\dot{v}^2+Ga_1\dot{v}^3=0. \quad (9)$$

The three directions are all real if

$$EG(a_2b_1-a_1b_2)^2+(Eb_2^2-Ga_1b_1)(Ea_2b_2-Ga_1^2)<0.$$

For the "Weingarten surfaces", in which $a_1b_2=a_2b_1$, we see that one or three of the directions are real according as a and b do or do not increase together as we move from O along the surface. If two of the directions given by (9) on a Weingarten surface coincide, so do all three. The osculating planes of *five* Darboux lines at O go through any line through O, which is not the normal or a tangent line to the surface at O.

From (5) and (7) we see that ϕ has a given value (not 0 or $\frac{1}{2}\pi$) for *eight* of the Darboux lines through O. From (5) and (6) we find that the locus of the centres of geodesic curvature of all the Darboux lines at O is a plane cubic with a crunode at O^*.

If ψ is the angle at which a curve cuts $v =$ constant,

$$2E(EG-F^2)^{\frac{1}{2}}\{E\dot{u}^2+2F\dot{u}\dot{v}+G\dot{v}^2)^{\frac{1}{2}}\kappa_g-\dot{\psi}\}$$
$$=(2EF_1-FE_1-EE_2)\,\dot{u}+(EG_1-FE_2)\,\dot{v}.$$

Putting $\dot{\psi}=0$ and substituting for κ_g from this equation in (5), we get a cubic equation in $\dot{v}/\dot{u}$. Hence, of the Darboux lines through O, *three* cut any singly infinite family of curves on the surface at a maximum or minimum angle at O. When the family consists of lines of curvature and $F=M=0$, these three Darboux lines are given by $b_1\dot{u}^3+a_1\dot{v}^3=0$ and only one is real.

The equations (5) of the Darboux lines take comparatively simple forms when a special parametric réseau is chosen. For instance, if the surface is referred to its lines of curvature,

$$E=e^{2\lambda},\quad F=0,\quad G=e^{2\mu},\quad L=e^{\lambda}l,\quad M=0,\quad N=e^{\mu}n,$$

with the Codazzi and Gauss equations

$$l_2=e^{\lambda-\mu}n\lambda_2,\qquad n_1=e^{\mu-\lambda}l\mu_1,$$
$$ln=e^{\lambda-\mu}(\lambda_2\mu_2-\lambda_2^2-\lambda_{22})+e^{\mu-\lambda}(\lambda_1\mu_1-\mu_1^2-\mu_{11}),$$

the equation of the Darboux lines is

$$3(e^{\lambda}n-e^{\mu}l)\,\dot{u}\dot{v}\{(\dot{u}\ddot{v}-\ddot{u}\dot{v})+(\mu_1-\lambda_1)\,\dot{u}^2\dot{v}+(\mu_2-\lambda_2)\,\dot{u}\dot{v}^2\}$$
$$+(e^{\lambda-\mu}\dot{u}^2+e^{\mu-\lambda}\dot{v}^2)\{e^{\lambda}(l_1-\lambda_1 l)\,\dot{u}^3+e^{\mu}(n_2-\mu_2 n)\,\dot{v}^3\}=0. \quad (10)$$

Again, if the parametric réseau is isometric,

$$E=e^{2t},\quad F=0,\quad G=e^{2t},\quad L=e^{t}l,\quad M=e^{t}m,\quad N=e^{t}n,$$

with the Codazzi and Gauss equations

$$mt_1+nt_2=l_2-m_1,\qquad lt_1+mt_2=n_1-m_2,\qquad t_{11}+t_{22}=m^2-ln,$$

the equation of the Darboux lines is

$$3(\dot{u}\ddot{v}-\ddot{u}\dot{v})\{-m\dot{u}^2+(l-n)\,\dot{u}\dot{v}+m\dot{v}^2\}$$
$$=\{(l_1-n_1+m_2)\,\dot{u}^3+3m_1\dot{u}^2\dot{v}+3m_2\dot{u}\dot{v}^2+(n_2-l_2+m_1)\,\dot{v}^3\}(\dot{u}^2+\dot{v}^2). \quad (11)$$

* This is due to Ribaucour, *Comptes rendus*, 73 (1871), 732.

This equation is the same for all surfaces with the same values of m and $n-l$. If the isometric réseau consists of Darboux lines, $n-l+im$ is an analytic function of u and v.

2. It is obvious that no plane curve is a Darboux line unless its plane touches the surface at every point of the curve, or unless the curve is a circle.

Again, the curvature κ and the torsion τ of a curve with arc s on a surface are given by

$$\kappa^2 = \kappa_n^2 + \kappa_g^2, \qquad \tau\kappa^2 = \kappa_g\left(\tau_g\kappa_g - \frac{d\kappa_n}{ds}\right) + \kappa_n\left(\tau_g\kappa_n + \frac{d\kappa_g}{ds}\right).$$

Along a Darboux line

$$\tau_g\kappa_g = \frac{d\kappa_n}{ds},$$

and therefore

$$\kappa^2 = \kappa_n^2 + \kappa_g^2, \qquad \tau\kappa\kappa_g = \kappa_n\frac{d\kappa}{ds}.$$

Hence no curve with constant curvature can be a Darboux line, unless it is a geodesic or a circle, and, conversely, any geodesic with constant curvature is a Darboux line.

As an illustration we may find the developable surface such that a singly infinite family of Darboux lines "develop" into parallel straight lines.

Taking $v =$ constant as these Darboux lines and their orthogonal trajectories as $u =$ constant, we have $E = G = 1$, $F = 0$ with suitable choice of u and v. Since κ_n is constant along $v =$ constant, L is a function of v. Then the Codazzi and Gauss equations give readily

$$E = 1, \quad F = 0, \quad G = 1, \quad L = \frac{a}{v}, \quad M = -\frac{au}{v^2}, \quad N = \frac{au^2}{v^3},$$

where a is a constant.

If we change to parameters r, θ, where $u = r\cos\theta$, $v = r\sin\theta$, we have

$$E = 1, \quad F = 0, \quad G = r^2, \quad L = 0, \quad M = 0, \quad N = ar\operatorname{cosec}^3\theta.$$

The surface is a cone.

3. Suppose that P is any point on a cone with vertex O and that Q is the intersection of OP with the sphere of unit radius and centre O. Take for u the arc of the locus of Q, for v the distance of P from O,

and O as origin. Then Q is (λ, μ, ν) and P is $(\lambda v, \mu v, \nu v)$, where λ, μ, ν are functions of u only and

$$\lambda^2+\mu^2+\nu^2 = 1, \quad \lambda_1^2+\mu_1^2+\nu_1^2 = 1.$$

Let k denote the geodesic curvature of the locus of Q, so that

$$k = \begin{vmatrix} \lambda_{11} & \mu_{11} & \nu_{11} \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda & \mu & \nu \end{vmatrix}.$$

Then

$$E = v^2, \quad F = 0, \quad G = 1, \quad L = kv, \quad M = 0, \quad N = 0,$$

and the equation of the Darboux lines is

$$3k\left\{v\frac{dv}{du}\frac{d^2v}{du^2}-\left(\frac{dv}{du}\right)^3\right\} = k_1 v\left\{v^2+\left(\frac{dv}{du}\right)^2\right\}.$$

This differential equation has the solution

$$v = ce^{\int \omega\, du} \quad \text{where} \quad \omega^2 = \left(\frac{k}{b}\right)^{\frac{2}{3}}-1,$$

b and c being arbitrary constants.

The Darboux lines cut the generators of the cone at constant angles if, and only if, the cone is right circular, so that $k_1 = 0$.

4. The surface obtained by screwing (with constant pitch) a straight line about a fixed axis which it meets at right angles is given by

$$x = u\cos v, \quad y = u\sin v, \quad z = av;$$

$$E = 1, \quad F = 0, \quad G = u^2+a^2, \quad L = 0, \quad M = -\frac{a}{(u^2+a^2)^{\frac{1}{2}}}, \quad N = 0.$$

The differential equation of the Darboux lines is

$$3\left\{1-(u^2+a^2)\left(\frac{dv}{du}\right)^2\right\}\frac{d^2v}{du^2}-7u\left(\frac{dv}{du}\right)^3-u(u^2+a^2)\left(\frac{dv}{du}\right)^5 = 0.$$

Putting $p \equiv dv/du$, $w \equiv p^2(u^2+a^2)$, we get

$$\frac{2dp}{p} = \frac{(7+w)\,dw}{3+6w+w^2},$$

which gives dv/du in terms of u. Hence the Darboux lines on this surface are obtained by one further quadrature.

5. For the developable formed by the tangents to a curve with curvature κ and torsion τ,

$$E = \kappa^2(v-u)^2, \quad F = 0, \quad G = 1, \quad L = \kappa\tau(v-u), \quad M = 0, \quad N = 0,$$

where u is the arc of the curve and $v-u$ is the distance from the point of contact along the tangent. The equation of the Darboux lines is

$$3(v-u)\frac{dv}{du}\frac{d^2v}{du^2} - \kappa^2(v-u)^2\left\{1+(v-u)\left(\frac{\tau_1}{\tau}-\frac{\kappa_1}{\kappa}\right)\right\}$$
$$+\left\{2-(v-u)\left(2\frac{\kappa_1}{\kappa}+\frac{\tau_1}{\tau}\right)\right\}\left(\frac{dv}{du}\right)^2 - 3\left(\frac{dv}{du}\right)^3 = 0. \quad (12)$$

Hence the Darboux lines are unaltered, if the curve is deformed in such a way that the ratio of the torsions at corresponding points is unchanged, while the length of the arc and the value of κ at each point are also unchanged, as, for instance, during the process of "development".

If a trajectory cutting the generators of the developable at a constant angle ϵ is a Darboux line, we must have along it

$$\frac{dv}{du} = \kappa(v-u)\cot\epsilon, \qquad \frac{d^2v}{du^2} = \left(\frac{dv}{du}-1\right)\kappa\cot\epsilon + (v-1)\kappa_1\cot\epsilon.$$

Then (12) gives $$(v-u)\left(\frac{\kappa_1}{\kappa}-\frac{\tau_1}{\tau}\right) = 1,$$

and this, with $dv/du = \kappa(v-u)\cot\epsilon$, gives

$$\tau = \kappa\left(a + b\int e^{-\int\kappa\cot\epsilon\,du}\,du\right),$$

where a and b are constants.

6. We will now find the surface of translation for which one system of generating curves consists of Darboux lines.

Taking the arcs of the two systems of generating curves as u and v, we have

$$E = 1, \qquad F = \cos\omega, \qquad G = 1, \qquad M = 0.$$

If $v =$ constant are Darboux lines,

$$L_1\sin\omega + L\cos\omega\,\omega_1 = 0;$$

while Codazzi's and Gauss's equations give

$$L_2\sin\omega = N\omega_1, \qquad N_1\sin\omega = L\omega_2, \qquad LN = -\sin\omega\,\omega_{12}.$$

We obtain at once

$$L\sin\omega = V,$$

where V is a function of v. Elimination of L and N from this and Codazzi's equations gives

$$\omega_{11}\left(\frac{\sin\omega}{V}\right)_2 = \omega_{12}\left(\frac{\sin\omega}{V}\right)_1,$$

so that

$$L = f(\omega_1), \qquad N = \frac{L_2 \sin\omega}{\omega_1} = \frac{f'(\omega_1)\,\omega_{12}\sin\omega}{\omega_1}.$$

Substituting in Gauss's equation, we get

$$f(\omega_1) = (a^2 - \omega_1^2)^{\frac{1}{2}}, \tag{13}$$

where a is a constant.

Now the curvature κ and torsion τ of $v =$ constant are given by

$$\kappa^2 = L^2 + \omega_1^2, \qquad \kappa^2\tau = L\cos\omega(L^2+\omega_1^2) + \sin\omega(L_1\omega_1 - L\omega_{11}),$$

when $E = G = 1$, $F = \cos\omega$, $M = 0$.

Substituting from (13), we find that $\kappa = 1/a$, $\tau = 0$. Hence, if one family of generating curves of a surface of translation are Darboux lines, they are circles.

7. We now consider whether surfaces exist whose "lines of torsion" (bisecting the angles between the lines of curvature) are Darboux lines.

Taking the lines of torsion as parametric réseau we have

$$F = 0, \qquad \frac{L}{E} = \frac{N}{G} = k \quad \text{(say)};$$

and, supposing that the réseau consists of Darboux lines,

$$ME_2 = 2EGk_1, \qquad MG_1 = 2EGk_2. \tag{14}$$

From these and Codazzi's equations it follows readily that M^2/G is a function of u and M^2/E is a function of v. Hence the réseau is isometric, and we may suppose that

$$E = G = \lambda^2, \qquad F = 0, \qquad L = N = k\lambda^2, \qquad M = a\lambda,$$

where a is constant.

Equations (14) now give

$$k = V - U, \qquad \frac{a}{\lambda} = V + U,$$

where U is a function of $u+v$, and V is a function of $v-v$.

Changing now the notation and putting u for $u+v$, v for $u-v$, we have

$$\left.\begin{aligned} E = G = \frac{a^2}{2(U+V)^2}, \quad F = 0 \\ L = \frac{a^2 V}{(U+V)^2}, \quad M = 0, \quad N = -\frac{a^2 U}{(U+V)^2} \end{aligned}\right\}, \tag{15}$$

where U is a function of u and V is a function of v.

Now, when $m = 0$, the curvature κ and torsion τ of $v =$ constant for the réseau of (11) are given by

$$e^{2t}\kappa^2 = l^2+t_2^2, \qquad e^t(l^2+t_2^2)\tau = l_1 t_2 - l t_{12}.$$

Therefore, for the surface (15), κ is a function of v and τ is zero. Hence the lines of curvature of the required surface are circles forming an isometric réseau.

Gauss's equation requires that

$$2a^2 UV+(U+V)(U_{11}+V_{22})-(U_1^2+V_2^2) = 0, \tag{16}$$

so that the desired surface is given by (15), where U and V are to be found from (16). This gives, on partial integration with respect to u,

$$U_1^2 = 2c^2 U^2+2\{V_{22}+2(a^2+c^2)V\}U+\{2V_2 V_{22}+2(a^2+c^2)V^2-V_2^2\}, \tag{17}$$

c being a function of v.

If V is a constant, U is at once obtained from (17), and the surface is an anchor-ring. If not, V must evidently be such as to make the coefficients of the various powers of U on the right-hand side of (17) all constant. This leads to $V = A+B\cos f(v+\beta)$, where A, B, f, β are constants, and so for U.

Substituting expressions of this form for U and V in (16), we obtain the necessary relations between the constants, and have

$$U = b\{1+\sqrt{2}\cos\epsilon\cos a(u+\alpha)\}, \qquad V = b\{1+\sqrt{2}\sin\epsilon\cos a(v+\beta)\},$$

where a, b, ϵ, α, β are constants.

8. Consider now the surfaces in which every oblique trajectory of an isometric réseau is a Darboux line.

Taking the réseau, as in § 1 (11), we must have m and $n-l$ constant. Now, if $(n-l)/m$ is constant, we can refer the surface to another isometric réseau cutting the former réseau at fixed angles, so that m is zero; in other words, the lines of curvature form an isometric réseau.

We may suppose then

$$m = 0, \quad n = p+q, \quad l = p-q,$$

where q is constant. Then Codazzi's equations give

$$(p+q)t_2 = p_2, \quad (p-q)t_1 = p,$$

whence

$$p = q(U-V)/(U+V), \quad t = -\log(U+V),$$

giving the same surface as in § 7.

We note that on a real minimal surface an isometric réseau cannot consist of Darboux lines.

For in such a case we should have $p = 0$, $m_2 = 2q_1$, $m_1 = -2q_2$. Solving the Codazzi equations for t_1, t_2, and substituting in $(t_1)_2 = (t_2)_1$, we should get $mq(q_1^2+q_2^2) = 0$. This would lead to E, F, G, L, M, N being all constant, which is the case only for a plane.

9. We lastly show that there is no surface applicable to a surface of revolution in which all the lines of constant geodesic curvature are Darboux lines.

For such a surface we may take

$$E = 1, \quad F = 0, \quad G_2 = 0;$$

and, if $u =$ constant are Darboux lines, $2N_2 = MG_1$; while Codazzi's equations become

$$2GM_1 - 2GL_2 + MG_1 = 0, \quad 2GM_2 - 2GN_1 + LGG_1 + NG_1 = 0.$$

Elimination of L and N gives

$$2GM_{22} + M(G_1^2 - GG_{11}) = 0,$$

and therefore

$$M = f(u)\cos pv + \phi(u)\sin pv,$$

where

$$2Gp^2 = G_1^2 - GG_{11}.$$

Now, from $2N_2 = MG_1$ and the first Codazzi equation, we get L and N. Substitution in the Gauss equation

$$4(LN-M^2)G + 2GG_{11} - G_1^2 = 0,$$

then readily gives p constant. Hence the surface, if it exists, must have constant Gaussian curvature. It will then have a triply infinite number of lines of constant geodesic curvature, and these cannot all be Darboux lines, which are only doubly infinite in number.

We may show also that, on the surface with constant curvature, the lines whose points are at a constant geodesic distance from a fixed point of the surface cannot all be Darboux lines. For, if they were, we should have the above equations with $p = 1$, $G = \sinh^2 u$. Then

$$L = -(\phi' + \phi \coth u) \cos v + (f' + f \coth u) \sin v + \theta(u),$$

$$N = -\sinh u \cosh u\, \phi \cos v + \sinh u \cosh u\, f \sin v + \psi(u).$$

When we substitute in Gauss's equation, these give

$$f = a \cosh u \operatorname{cosech}^2 u, \qquad \phi = b \cosh u \operatorname{cosech}^2 u,$$

$$\theta\psi = (a^2 + b^2) \cosh^2 u \operatorname{cosech}^4 u - \sinh^2 u, \qquad \psi = \sinh^2 u \cosh^2 u\, \theta;$$

and they prove to be inconsistent with the second Codazzi equation which requires that

$$\sinh^2 u \cosh^2 u\, \theta = \sinh u\, \psi' - \cosh u\, \psi.$$

ON A GENERAL EXPANSION IN THE THEORY OF THE TIDES

By J. PROUDMAN.

[Received 2 October, 1928.—Read 8 November, 1928.]

In recent papers* on the oscillations of a rotating mechanical system of infinite freedom, a certain transformation of coordinates and their speeds was evolved and used. Now the oceans constitute a mechanical system of infinite freedom, rotating with the earth, and the tides consist in oscillations of that system. But the theory of the tides is not to be deduced immediately from the theory worked out in the papers quoted, because the system there considered possesses no zero coefficients of stability, while such zero coefficients are possessed by the oceans. It is possible, however, to use the restricted investigation to formulate the corresponding theorem for the tides, and then to seek independent proof of its validity. This is what is done in the present paper.

The general expansion-theorem is enunciated in § 2 in reference to an ocean which is completely surrounded by land. By means of it any possible tidal state and motion of the ocean are expressed in terms of infinite series of the states and motions appropriate to the fundamental free oscillations, explicit formulae being given for the coefficients in these series. As corollaries from the theorem, expressions are obtained for the free tides arising out of any initial conditions, and for the forced tides maintained by any distribution of periodic disturbing potential.

When the angular speed of the mechanical system vanishes the transformation reduces to that of normal coordinates. Now for a sea near the equator the effective component of the earth's angular velocity

* J. Proudman and F. E. Mercer, *Proc. London Math. Soc.* (2), 27 (1926), 1–38; 28 (1927), 102–108; see also 28 (1927), 109–120, for a simple analogue of the theorem of the present paper.

vanishes, and in this case the expansion-theorem may be deduced* from that of Hilbert. But when the angular speed of rotation does not vanish the proof of validity is not so easily supplied, and in the present paper is only given for what may be regarded as the simplest case. This is the case of a flat circular sea in which the depth and motion are symmetrical about the centre, and the expansion may then be reduced to Sturm-Liouville type. The proof may be extended immediately to the cases of an annular sea in which the depth and motion are symmetrical about the centre, and of a channel whose section and motion are uniform along its length.

1. *Notation and fundamental equations.*

Let a denote the radius of the earth, Ω the angular speed of its rotation, g the acceleration of its gravity, θ, χ the co-latitude and east longitude of any point on its surface, and h the depth of water below any point of the mean surface of the ocean. Let t denote the time, ζ the elevation at any time of any point of the surface of the ocean, u, v the southerly and easterly components of tidal current at any time and place, and $-g\bar{\zeta}$ the potential of the disturbing forces of the moon and sun.

Then the equation of continuity takes the form

$$\frac{1}{a \sin\theta}\left\{\frac{\partial}{\partial\theta}(h \sin\theta u) + \frac{\partial}{\partial\chi}(hv)\right\} + \dot{\zeta} = 0, \tag{1.11}$$

and is satisfied over the whole ocean. Let P denote any point on the normal to the coastline at any point Q of it, and let ν denote the component along PQ of the current at P; then the coastal condition takes the form

$$h\nu \to 0 \quad \text{as} \quad P \to Q, \tag{1.12}$$

and is satisfied for all positions of Q. The dynamical equations may be

* The proof of this is virtually contained in an earlier paper, *Proc. London Math. Soc.* (2), 18 (1917), 21-68. It applies to any non-rotating basin, and the expansion is used to express the tides of a rotating basin in terms of such free oscillations as would exist if the same basin did not rotate. To connect the formulae of the present paper with those of the earlier one, we write

$$\sigma_n^2 = \nu_n^2, \quad \Phi_n = \Phi_{-n} = \frac{1}{|\sigma_n|}\phi_n, \quad \Psi'_n = \Psi'_{-n} = 0,$$

$$\sigma_{(n)} = 0, \quad \Phi_{(n)} = 0, \quad \Psi'_{(n)} = \sqrt{2}\,\psi_n,$$

$$\xi_n = \tfrac{1}{2}\sigma_n p_n - \tfrac{1}{2}i\dot{p}_n, \quad \xi_{-n} = \tfrac{1}{2}\sigma_n p_n + \tfrac{1}{2}i\dot{p}_n, \quad \xi_{(n)} = \dot{p}_{-n}/\sqrt{2}.$$

The series (2.4) below are then absolutely convergent.

written

$$\left.\begin{aligned} \dot{u}-2\Omega\cos\theta\, v &= -\frac{g}{a}\frac{\partial}{\partial\theta}(\zeta-\bar{\zeta}) \\ \dot{v}+2\Omega\cos\theta\, u &= -\frac{g}{a\sin\theta}\frac{\partial}{\partial\chi}(\zeta-\bar{\zeta}) \end{aligned}\right\} \qquad (1.13)$$

and they are satisfied over the whole ocean.

Now let ζ', u', v', $\bar{\zeta}'$ refer to another possible state and motion. Multiply the first of equations (1.13) by hu', the second by hv', and integrate over the whole surface of the ocean. We obtain, on using Green's theorem and (1.11),

$$\begin{aligned} \iint h(\dot{u}u'+\dot{v}v')\,dS+2\Omega\iint h\cos\theta(uv'-vu')\,dS & \\ &= -\frac{g}{a}\iint h\left\{u'\frac{\partial}{\partial\theta}(\zeta-\bar{\zeta})+\frac{v'}{\sin\theta}\frac{\partial}{\partial\chi}(\zeta-\bar{\zeta})\right\}dS \\ &= -g\iint\zeta\dot{\zeta}'\,dS+g\iint\bar{\zeta}\dot{\bar{\zeta}}'\,dS. \qquad (1.21) \end{aligned}$$

Similarly we may obtain

$$\begin{aligned} \iint h(\dot{u}'u+\dot{v}'v)\,dS+2\Omega\iint h\cos\theta(u'v-v'u)\,dS & \\ &= -g\iint\zeta'\dot{\zeta}\,dS+g\iint\bar{\zeta}'\dot{\zeta}\,dS, \qquad (1.22) \end{aligned}$$

and then, on adding (1.21) and (1.22), we have

$$\begin{aligned} \iint h(\dot{u}u'+\dot{v}v')\,dS+\iint h(\dot{u}'u+\dot{v}'v)\,dS & \\ +g\iint\zeta\dot{\zeta}'\,dS+g\iint\zeta'\dot{\zeta}\,dS &= g\iint\bar{\zeta}\dot{\zeta}'\,dS+g\iint\bar{\zeta}'\dot{\zeta}\,dS. \qquad (1.23) \end{aligned}$$

The equation (1.23) we shall call the general *energy relation*.

When $\bar{\zeta}=0$, so that the motion is *free*, the equations (1.11), (1.12), and (1.13) possess an infinite sequence of linearly independent solutions of the form

$$\zeta=\zeta_n e^{i\sigma_n t},\quad u=u_n e^{i\sigma_n t},\quad v=v_n e^{i\sigma_n t}, \qquad (1.31)$$

where ζ_n, u_n, v_n are functions of θ, χ, but are independent of t. Corresponding to every solution (1.31) there is also a solution of the form

$$\zeta=\zeta_{-n}e^{i\sigma_{-n}t},\quad u=u_{-n}e^{i\sigma_{-n}t},\quad v=v_{-n}e^{i\sigma_{-n}t}, \qquad (1.32)$$

where $\sigma_{-n}=-\sigma_n$, and ζ_{-n}, u_{-n}, and v_{-n} are respectively conjugate to ζ_n, u_n, v_n. The expressions (1.31) are undetermined to the extent of a common factor, and when more than one solution corresponds to the same value of σ_n they are undetermined to a greater extent.

If, in the general energy relation (1 . 23), we take

$$\bar{\zeta} = 0, \quad \zeta = \zeta_m e^{i\sigma_m t}, \quad u = u_m e^{i\sigma_m t}, \quad v = v_m e^{i\sigma_m t};$$

$$\bar{\zeta}' = 0, \quad \zeta' = \zeta_n e^{i\sigma_n t}, \quad u' = u_n e^{i\sigma_n t} \quad v' = v_n e^{i\sigma_n t};$$

we obtain*

$$(\sigma_m+\sigma_n)\left\{\iint h(u_m u_n+v_m v_n)\,dS+g\iint \zeta_m \zeta_n\,dS\right\} = 0.$$

When $\sigma_m+\sigma_n \neq 0$, we deduce that

$$\tfrac{1}{2}\iint h(u_n u_n+v_m v_n)\,dS+\tfrac{1}{2}g\iint \zeta_m \zeta_n\,dS = 0, \tag{1 . 33}$$

and even when $\sigma_m+\sigma_n = 0$ we may arrange† that (1 . 33) is still satisfied provided that $m+n \neq 0$. We may also take

$$\tfrac{1}{2}\iint h(u_n u_{-n}+v_n v_{-n})\,dS+\tfrac{1}{2}g\iint \zeta_n \zeta_{-n}\,dS = 1, \tag{1 . 34}$$

for all the values of n, where, as throughout, the surface integrals are taken over the whole ocean.

To any distribution of the functions ζ, u, v satisfying (1 . 11) and (1 . 12) there correspond functions ϕ, $\dot{\psi}$ determined by the equations

$$\left.\begin{aligned} \frac{1}{a^2\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(h\frac{\partial\phi}{\partial\chi}\right)\right\} &= \zeta, \\ \frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial\dot{\psi}}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\chi}\left(\frac{1}{h}\frac{\partial\dot{\psi}}{\partial\chi}\right)\right\} &= \frac{\partial}{\partial\theta}(\sin\theta v)-\frac{\partial u}{\partial\chi} \end{aligned}\right\} \tag{1 . 41}$$

over the ocean, with

$$h\frac{\partial\phi}{\partial\nu}\to 0, \quad \frac{\dot{\psi}}{h}\to 0, \tag{1 . 42}$$

at the coastline, $\partial/\partial\nu$ denoting differentiation along the normal to the coastline and the limits being taken as in (1 . 12). Then‡

$$\left.\begin{aligned} u &= -\frac{1}{a}\frac{\partial\dot{\phi}}{\partial\theta}-\frac{1}{ha\sin\theta}\frac{\partial\dot{\psi}}{\partial\chi} \\ v &= -\frac{1}{a\sin\theta}\frac{\partial\dot{\phi}}{\partial\chi}+\frac{1}{ha}\frac{\partial\dot{\psi}}{\partial\theta} \end{aligned}\right\}. \tag{1 . 43}$$

Let $\Phi_n e^{i\sigma_n t}, \quad \Psi'_n e^{i\sigma_n t}$

---

* Cf. *Proc. London Math. Soc.* (2), 13 (1913), 295–296.

† Cf. p. 13 of first paper quoted on p. 527.

‡ See p. 53 of paper quoted on p. 528.

denote the functions ϕ, $\dot{\psi}$ corresponding to (1 . 31), so that Φ_n, Ψ'_n are independent of time.

2. *The expansion-theorem.*

For any distribution of the functions ζ, u, v satisfying (1 . 11) and (1 . 12) take

$$\xi_n = \tfrac{1}{2}\iint h(uu_{-n}+vv_{-n})\,dS+\tfrac{1}{2}g\iint \zeta\zeta_{-n}\,dS, \tag{2.1}$$

for all the different values of n, so that there is one value of ξ_n for each fundamental free oscillation (1 . 31). Then, if $\sum_n$ denotes summation over all values of n, positive and negative, the expansion-theorem asserts that

$$\left.\begin{aligned} \zeta &= \sum_n \xi_n\zeta_n, \quad \dot{\zeta} = \sum_n i\sigma_n\xi_n\zeta_n \\ u &= \sum_n \xi_n u_n, \quad v = \sum_n \xi_n v_n \end{aligned}\right\} \tag{2.2}$$

for such of these series as are convergent, together with

$$\tfrac{1}{2}\iint h(u^2+v^2)\,dS+\tfrac{1}{2}g\iint \zeta^2\,dS = \sum_n \xi_n\xi_{-n}, \tag{2.3}$$

this series being always absolutely convergent.

To prove the absolute convergence of the series (2 . 3) we take ζ', u', v' to be the conjugates respectively of ζ, u, v. Then, if ξ_n becomes ξ'_n when ζ, u, v in (2 . 1) are replaced by ζ', u', v', it follows that ξ'_n is conjugate to ξ_{-n} for all values of n. Let Σ denote summation over any definite selection of a finite number of values of n; then, on using (1 . 33), (1 . 34), and (2 . 1), we find

$$\begin{aligned} \tfrac{1}{2}\iint h\{(u-\Sigma\xi_n u_n)(u'-\Sigma\xi_{-n}u_{-n})+(v-\Sigma\xi_n v_n)(v'-\Sigma\xi_{-n}v_{-n})\}\,dS& \\ +\tfrac{1}{2}g\iint(\zeta-\Sigma\xi_n\zeta_n)(\zeta'-\Sigma\xi_{-n}\zeta_{-n})\,dS& \\ = \tfrac{1}{2}\iint h(uu'+vv')\,dS+\tfrac{1}{2}g\iint\zeta\zeta'\,dS-\Sigma\xi_n\xi'_{-n}.& \end{aligned}$$

This means that

$$\tfrac{1}{2}\iint h(|u|^2+|v|^2)\,dS+\tfrac{1}{2}g\iint|\zeta|^2\,dS \geqslant \sum_n|\xi_n|^2,$$

and the absolute convergence follows.

In terms of the functions ϕ and $\dot{\psi}$ the expansion-theorem becomes

$$\left.\begin{aligned} \phi &= \sum_n \xi_n\Phi_n \\ \dot{\phi} = \sum_n i\sigma_n\xi_n\Phi_n, &\quad \dot{\psi} = \sum_n \xi_n\Psi'_n \end{aligned}\right\}. \tag{2.4}$$

Now suppose that, at $t = 0$,

$$\zeta = \zeta^0, \quad u = u^0, \quad v = v^0, \tag{2.51}$$

and take

$$k_n = \tfrac{1}{2}\iint h(u^0 u_{-n} + v^0 v_{-n})\, dS + \tfrac{1}{2} g \iint \zeta^0 \zeta_{-n}\, dS \tag{2.52}$$

for all the values of n. Then the subsequent *free motion* will be given by

$$\left.\begin{aligned} \phi &= \sum_n k_n \Phi_n e^{i\sigma_n t} \\ \dot{\psi} &= \sum_n k_n \Psi'_n e^{i\sigma_n t} \end{aligned}\right\}, \tag{2.53}$$

since the series (2.53) represent a combination of free oscillations, and at $t = 0$ the state and motion become (2.51) in view of (2.52) and the expansion.

Next suppose that $\bar{\zeta}$ contains the time only through the factor $e^{i\sigma t}$, so that ζ, u, v will also contain the time only through this factor. If, in the general energy relation (1.23), we take

$$\dot{\zeta} = i\sigma\zeta, \quad \dot{u} = i\sigma u, \quad \dot{v} = i\sigma v,$$

$$\bar{\zeta}' = 0, \quad \zeta' = \zeta_{-n} e^{-i\sigma_n t}, \quad u' = u_{-n} e^{-i\sigma_n t}, \quad v' = v_{-n} e^{-i\sigma_n t},$$

we obtain

$$(\sigma - \sigma_n)\left\{\iint h(uu_{-n} + vv_{-n})\, dS + g\iint \bar{\zeta}\zeta_{-n}\, dS\right\} = -\sigma_n g \iint \bar{\zeta}\zeta_{-n}\, dS,$$

so that, on comparing with (2.1), we have

$$\zeta_n = \tfrac{1}{2}\frac{-\sigma_n}{\sigma - \sigma_n} g \iint \bar{\zeta}\zeta_{-n}\, dS. \tag{2.61}$$

The formulae (2.4) then give

$$\left.\begin{aligned} \phi &= \tfrac{1}{2}\sum_n \frac{-\sigma_n}{\sigma - \sigma_n} g \iint \bar{\zeta}\zeta_{-n}\, dS \,.\, \Phi_n \\ \dot{\psi} &= \tfrac{1}{2}\sum_n \frac{-\sigma_n}{\sigma - \sigma_n} g \iint \bar{\zeta}\zeta_{-n}\, dS \,.\, \Psi'_n \end{aligned}\right\}, \tag{2.62}$$

and these* we shall call the *resonance equations*.

* For a non-rotating basin the formulae (2.62) become

$$\phi = \sum_n{}' \frac{-\sigma_n^2}{\sigma^2 - \sigma_n^2} g \iint \bar{\zeta}\zeta_n\, dS \,.\, \Phi_n, \quad \dot{\psi} = 0.$$

where $\sum_n{}'$ denotes summation over positive values of n only. When the basin is one of uniform depth bounded by two meridians all the elements are known and the formulae provide a solution of the problem of finding the forced tides in such a basin, a problem which has hitherto been solved only for particular depths. See J. Proudman and A. T. Doodson, *M. N. Roy. Astron. Soc., Geophys. Supp.*, 1 (1927), 468–483.

It will be noticed that none of the formulae of this section contain explicitly a, θ, or χ, so that they require no modification when applied to a sea over which the curvature of the earth may be neglected and for which other coordinates are used.

3. *Flat circular sea.*

We shall now restrict our attention to a flat circular sea of radius c, and write ω for $\Omega\cos\theta$, as this may now be regarded as constant. We shall denote by r the distance from the centre of the sea, and suppose that all our variables are functions only of r and t. We shall now use u, v to denote the radial and transverse components of tidal current. Then, instead of the equations (1 . 11), (1 . 12), and (1 . 13), we have

$$\frac{1}{r}\frac{\partial}{\partial r}(hru)+\dot{\xi}=0, \tag{3 . 11}$$

$$hu\to 0, \quad \text{as} \quad r\to c, \tag{3 . 12}$$

$$\dot{u}-2\omega v=-g\frac{\partial}{\partial r}(\xi-\bar{\xi}), \quad \dot{v}+2\omega u=0. \tag{3 . 13}$$

For free periodic motion with the time-factor $e^{i\sigma t}$ we deduce, from (3 . 13),

$$u=\frac{i\sigma}{\sigma^2-4\omega^2}g\frac{\partial\xi}{\partial r}, \quad v=\frac{-2\omega}{\sigma^2-4\omega^2}g\frac{\partial\xi}{\partial r}, \tag{3 . 21}$$

and then, on substituting into (3 . 11) and (3 . 12),

$$\frac{1}{r}\frac{\partial}{\partial r}\left(hr\frac{\partial\xi}{\partial r}\right)+\frac{\sigma^2-4\omega^2}{g}\xi=0, \tag{3 . 22}$$

with

$$h\frac{\partial\xi}{\partial r}\to 0 \quad \text{as} \quad r\to c. \tag{3 . 23}$$

It is known* that the equations (3 . 22) and (3 . 23) are satisfied only for a definite infinite sequence of positive values of σ^2, say

$$\sigma_1^2, \ \sigma_2^2, \ \ldots, \ \sigma_n^2, \ \ldots,$$

each of which is greater than $4\omega^2$. We therefore take

$$\xi_n=k_nZ_n, \quad Z_{-n}=Z_n,$$

where Z_n satisfies the equations (3 . 22) and (3 . 23), and k_n is a constant to be chosen so as to satisfy (1 . 34). It is easily proved that

$$2\pi\int_0^c h\left(\frac{dZ_n}{dr}\right)r\,dr=2\pi\frac{\sigma_n^2-4\omega^2}{g}\int_0^c Z_n^2 r\,dr,$$

* See, for instance, E. L. Ince, *Ordinary differential equations*, Chap. 10.

and we shall fix the magnitude of Z_n by taking each of these quantities to be unity. On substituting into (1 . 34) we find

$$k_n^2 = \frac{(\sigma_n^2 - 4\omega^2)^2}{\sigma_n^2 g^2},$$

so that

$$\left.\begin{aligned} \xi_n &= \frac{\sigma_n^2 - 4\omega^2}{|\sigma_n| g} Z_n, \\ u_n = \frac{i\sigma_n}{|\sigma_n|}\frac{dZ_n}{dr}, \quad v_n &= -\frac{2\omega}{|\sigma_n|}\frac{dZ_n}{dr}, \end{aligned}\right\} \tag{3 . 24}$$

for $\pm n = 1, 2, 3, \ldots$.

The equations (3 . 11), (3 . 12), and (3 . 13), however, are also satisfied by

$$\bar{\xi} = 0, \quad u = 0, \quad v = \frac{g}{2\omega}\frac{d\xi}{dr}, \tag{3 . 31}$$

where ζ is any constant function of r, and these expressions represent an infinite variety of free steady motions. To bring these free motions under the form of § 1 we take

$$\sigma_{(n)} = 0, \quad \xi_{(n)} = k_{(n)} Z_n,$$

for $n = 1, 2, 3, \ldots$, where Z_n is the function already defined, but $k_{(n)}$ is a new constant. On substituting into (1 . 34), we find that

$$k_{(n)}^2 = \frac{8\omega^2}{g^2}\left(1 - \frac{4\omega^2}{\sigma_n^2}\right),$$

so that

$$\left.\begin{aligned} \xi_{(n)} &= \frac{2\sqrt{2}\omega}{g}\left(1 - \frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}} Z_n, \\ u_{(n)} = 0, \quad v_{(n)} &= \sqrt{2}\left(1 - \frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}}\frac{dZ_n}{dr}, \end{aligned}\right\} \tag{3 . 32}$$

for $n = 1, 2, 3, \ldots$.

It is easily proved that

$$\int_0^c h\frac{dZ_m}{dr}\frac{dZ_n}{dr} r\,dr = 0, \quad \int_0^c Z_m Z_n r\,dr = 0,$$

when $m \neq n$, and, on using these relations, we find that the equation (1 . 33) is satisfied by the combined assemblage of free motions (3 . 24) and (3 . 32).

Now suppose that ζ, u, v are any functions satisfying (3 . 11) and (3 . 12), and, in view of (2 . 1), take

$$\xi_n = -\frac{\pi}{|\sigma_n|}\int_0^c h(i\sigma_n u + 2\omega v)\frac{dZ_n}{dr} r\,dr + \pi\frac{(\sigma_n^2 - 4\omega^2)}{|\sigma_n|}\int_0^c \zeta Z_n r\,dr \tag{3 . 41}$$

for $\pm n = 1, 2, 3, \ldots$, with

$$\dot{\xi}_{(n)} = \sqrt{2\pi}\left(1-\frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}}\left\{\int_0^c hv\frac{dZ_n}{dr}r\,dr+2\omega\int_0^c \zeta Z_n r\,dr\right\} \qquad (3.42)$$

for $n = 1, 2, 3, \ldots$.

The series in (2.2) may be written respectively

$$\sum_{n=1}^{\infty}\left\{\frac{\sigma_n^2-4\omega^2}{|\sigma_n|g}(\dot{\xi}_n+\dot{\xi}_{-n})+\frac{2\sqrt{2}\omega}{g}\left(1-\frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}}\dot{\xi}_{(n)}\right\}Z_n,$$

$$\sum_{n=1}^{\infty}\frac{i\sigma_n}{|\sigma_n|}\frac{\sigma_n^2-4\omega^2}{g}(\dot{\xi}_n-\dot{\xi}_{-n})Z_n,$$

$$\sum_{n=1}^{\infty}\frac{i\sigma_n}{|\sigma_n|}(\dot{\xi}_n-\dot{\xi}_{-n})\frac{dZ_n}{dr},$$

$$\sum_{n=1}^{\infty}\left\{-\frac{2\omega}{|\sigma_n|}(\dot{\xi}_n+\dot{\xi}_{-n})+\sqrt{2}\left(1-\frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}}\dot{\xi}_{(n)}\right\}\frac{dZ_n}{dr},$$

and, on substituting from (3.41), (3.42), and using (3.11), they reduce to

$$2\pi\sum_{n=1}^{\infty}\left\{\frac{\sigma_n^2-4\omega^2}{g}\int_0^c \dot{\zeta}Z_n r\,dr\right\}Z_n,$$

$$2\pi\sum_{n=1}^{\infty}\left\{\frac{\sigma_n^2-4\omega^2}{g}\int_0^c \dot{\zeta}Z_n r\,dr\right\}Z_n,$$

$$2\pi\sum_{n=1}^{\infty}\left\{\int_0^c hu\frac{dZ_n}{dr}r\,dr\right\}\frac{dZ_n}{dr},$$

$$2\pi\sum_{n=1}^{\infty}\left\{\int_0^c hv\frac{dZ_n}{dr}r\,dr\right\}\frac{dZ_n}{dr},$$

respectively. These are the Sturm-Liouville expansions for ζ, $\dot{\zeta}$, u, v respectively, so that we have established (2.2) for the particular case of the present section.

The series in (2.3) becomes

$$\sum_{n=1}^{\infty}(2\dot{\xi}_n\dot{\xi}_{-n}+\dot{\xi}_{(n)}^2),$$

and, on substitution from (3.41) and (3.42), it reduces to

$$2\pi^2\sum_{n=1}^{\infty}\left[\left\{\int_0^c hu\frac{dZ_n}{dr}r\,dr\right\}^2+\left\{\int_0^c hv\frac{dZ_n}{dr}r\,dr\right\}^2+(\sigma_n^2-4\omega^2)\left\{\int_0^c \dot{\zeta}Z_n r\,dr\right\}^2\right].$$

By Parseval's theorem this is equal to

$$\pi\int_0^c h(u^2+v^2)\,r\,dr+\pi g\int_0^c \dot{\zeta}^2\,r\,dr,$$

so that we have established (2.3) for the particular case taken

The equations (1 . 41) and (1 . 42) give

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(hr\frac{\partial\phi}{\partial r}\right) = \xi, \quad \dot{\psi} = -\int_r^c hv\,dr, \\ h\frac{\partial\phi}{\partial r} \to 0 \quad \text{as} \quad r \to c, \end{aligned}\right\} \tag{3 . 5}$$

and, from (3 . 24) and (3 . 32), we deduce that

$$\Phi_n = -\frac{1}{|\sigma_n|}Z_n, \quad \Psi'_n = \frac{2\omega}{|\sigma_n|}\int_r^c h\frac{dZ_n}{dr}\,dr,$$

for $\pm n = 1, 2, 3, \ldots$, with

$$\Phi_{(n)} = \frac{-2\sqrt{2}\omega}{|\sigma_n|(\sigma_n^2-4\omega^2)^{\frac{1}{2}}}Z_n, \quad \Psi'_n = -\sqrt{2}\left(1-\frac{4\omega^2}{\sigma_n^2}\right)\int_r^c h\frac{dZ_n}{dr}\,dr,$$

for $n = 1, 2, 3, \ldots$. The series in (2 . 4) may then be written respectively

$$-\sum_{n=1}^{\infty}\left\{\frac{1}{|\sigma_n|}(\xi_n+\xi_{-n})+\frac{2\sqrt{2}\omega}{|\sigma_n|(\sigma_n^2-4\omega^2)^{\frac{1}{2}}}\xi_{(n)}\right\}Z_n,$$

$$-\sum_{n=1}^{\infty}\frac{i\sigma_n}{|\sigma_n|}(\xi_n-\xi_{-n})\,Z_n,$$

$$\sum_{n=1}^{\infty}\left\{\frac{2\omega}{|\sigma_n|}(\xi_n+\xi_{-n})-\sqrt{2}\left(1-\frac{4\omega^2}{\sigma_n^2}\right)^{\frac{1}{2}}\xi_{(n)}\right\}\int_r^c h\frac{dZ_n}{dr}\,dr,$$

and, on substituting from (3 . 41), (3 . 42), and using (3 . 11), they reduce to

$$-2\pi\sum_{n=1}^{\infty}\left\{\int_0^c \xi Z_n\,r\,dr\right\}Z_n,$$

$$-2\pi\sum_{n=1}^{\infty}\left\{\int_0^c \dot{\xi} Z_n\,r\,dr\right\}Z_n,$$

$$-2\pi\sum_{n=1}^{\infty}\left\{\int_0^c hv\frac{dZ_n}{dr}\,r\,dr\right\}\int_r^c h\frac{dZ_n}{dr}\,dr,$$

respectively. These series are known to be absolutely convergent, and, by using (3 . 5), the first two may also be written

$$2\pi\sum_{n=1}^{\infty}\left\{\frac{\sigma_n^2-4\omega^2}{g}\int_0^c \phi Z_n\,r\,dr\right\}Z_n,$$

$$2\pi\sum_{n=1}^{\infty}\left\{\frac{\sigma_n^2-4\omega^2}{g}\int_0^c \dot{\phi} Z_n\,r\,dr\right\}Z_n,$$

so that the three series are the Sturm-Liouville expansions for ϕ, $\dot{\phi}$ and $\dot{\psi}$ respectively. This establishes (2 . 4) for the particular case considered.

TRANSFORMATIONS OF LIMITED VARIATION

By P. J. DANIELL.

[Received 23 October, 1928.—Read 8 November, 1928.]

This paper studies the differential character of two-dimensional transformations. This study leads, on the one hand (cf. 2 . 3), to a form of integral in one dimension more general than the Stieltjes integral, but, roughly, of the same type ; and, on the other hand (cf. 4 . 1), to a Stieltjes integral in two dimensions with respect to a pair of functions. S. Banach* has defined a two-dimensional transformation of limited variation, and, in one sense, the present paper consists of a deeper analysis of that problem. The results obtained enable us to go further than Banach, and the definition of limited variation is also somewhat different. Recently Schauder† has obtained some valuable results in this same field, but these results do not overlap those obtained here.

One dimension.

Before we proceed to consider the more difficult problem we will give a rapid sketch of the case of one dimension.

1 . 1. Let $x = f(u)$ be defined and continuous in the interval $J(0 \leqslant u \leqslant 1)$. This determines a transformation of points in J to an image set in $H(J)$.

Given an interval $K(a \leqslant u \leqslant b)$ we define

$$\begin{aligned} \phi(s) &= +1 \quad \text{if} \quad f(a) < s < f(b) \\ &= -1 \quad \text{if} \quad f(a) > s > f(b) \\ &= 0 \ \text{otherwise.} \end{aligned}$$

* S. Banach, "Sur les lignes rectifiables, etc.", *Fundamenta Math.*, 7 (1925), 231.

† J. Schauder, "Ueber stetige Abbildungen", *Fundamenta Math.*, 12 (1928), 47.

Then $\phi(s)$ is measurable in the sense of Borel, and in all cases

$$\int \phi(s)\,ds = f(b)-f(a),$$

$$\int |\phi(s)|\,ds = |f(b)-f(a)| = |\int \phi(s)\,ds|,$$

On J lay a dense net system having, at the stage n, the meshes K_i, and let H_0 be the image of the points of division. H_0 is at most countable and is therefore of zero measure. At stage n of the sub-division define

$$\omega_n(s) = \Sigma_i |\phi_i(s)|$$

for all the meshes K_i of stage n. We also observe that, if s is not on H_0, $\phi(s)$ relative to the whole of J satisfies the equation

$$\phi(s) = \Sigma_i \phi_i(s).$$

Then $\omega_n(s)$ forms a non-decreasing sequence and has a limit $\omega(s)$, which may be $+\infty$. Again

$$\int \omega_n(s)\,ds = \Sigma_i \int |\phi_i(s)|\,ds = \Sigma_i |f(b_i)-f(a_i)|.$$

1.2. Hence a necessary and sufficient condition that $f(u)$ be of limited variation is that

$$\omega(s) = \lim \omega_n(s)$$

be finite nearly everywhere and integrable. Also the total variation V is given by

$$V = \lim \int \omega_n(s)\,ds = \int \omega(s)\,ds.$$

1.3. If $\omega(s) = k$ is finite for a particular s, since $\phi(s)$ is additive for intervals, there will be a stage in the sub-division such that, beyond this stage, $\phi(s)$ will remain equal to $+1$ or else to -1 in each of k distinct meshes. These meshes will form k sequences, each converging on a particular value of u. If $u_1, u_2, \ldots, u_k$ are these values of u, each will have corresponding to it a number $I(u)$ equal to ± 1 and

$$\phi(s) = I(u_1)+I(u_2)+\ldots+I(u_k),$$

$$\omega(s) = |I(u_1)|+|I(u_2)|+\ldots+|I(u_k)|.$$

The number $I(u)$ will be independent of the net system used, and, in fact, if $I(u_i) = +1$, $f(u) < s$ on the left and $f(u) > s$ on the right of u_i within a sufficiently small interval enclosing u_i, so that u_i is a *point of increase*. Similarly, if $I(u_i) = -1$, $f(u) < s$ on the right and $f(u) > s$ on the left and u_i is a *point of decrease*.

1.4. THEOREM. *If $f(a) < f(b)$, and if there is a value of s between $f(a)$ and $f(b)$ for which $\omega(s)$ is finite, then there is a point $u (a < u < b)$ which is a point of increase; in other words, there is a point at which the lower derived number of $f(u)$ is non-negative for all methods of defining the derived number, i.e.*

$$\underset{h,k>0;\ h,k\to 0}{\text{lower lim}} \frac{f(u+h)-f(u-k)}{h+k} \geqslant 0.$$

For this theorem it is necessary only for $f(u)$ to be continuous. If it is of limited variation also, then, by (1.2), $\omega(s)$ is finite for nearly all s and the condition of the hypothesis will certainly be satisfied.

1.5. By a well known property of continuous functions, if for some interval $\phi(s)$ is not 0, there is a value of u within this interval for which $f(u) = s$. Hence, if $N_n(s)$ is Banach's* function, that is the number of meshes at stage n for which

$$\min f(u) \leqslant s \leqslant \max f(u),$$

then

$$\omega_n(s) \leqslant N_n(s).$$

Hence, in the limit, except on H_0,

$$\omega(s) \leqslant N(s),$$

where $N(s)$ is the number of solutions of $f(u) = s$. But

$$\int N_n(s)\, ds = \Sigma_i |\max f - \min f| \text{ on } K_i,$$

$$\int N(s)\, ds = V = \int \omega(s)\, ds.$$

Hence $\omega(s) = N(s)$ for nearly all s.

1.6. THEOREM. *If $f(u)$ is of limited variation all the points u are points of increase or points of decrease except for a set of points whose s-image is of zero measure.*

1.7. Let $f(u)$ be of limited variation (and continuous) and let $g(u)$ be continuous in J. At stage n of the division define

$$A_n(s) = \Sigma_i \phi_i(s)\, g(a_i),$$

where a_i is the left-hand end of mesh K_i. Then $A_n(s)$ is Borel measurable and

$$\int A_n(s)\, ds = \Sigma_i\, g(a_i)[f(b_i) - f(a_i)].$$

Now

$$|A_n(s)| \leqslant \omega(s) \times \max |g(u)|,$$

* Banach, *loc. cit.*

and if $\omega(s) = k$ is finite for a particular s,

$$A(s) = \lim A_n(s) = I(u_1)\, g(u_1) + \ldots + I(u_k)\, g(u_k).$$

Hence the ordinary Stieltjes integral

$$\int g(u)\, df(u) = \lim \int A_n(s)\, ds = \int A(s)\, ds,$$

where $A(s)$ is defined as

$$A(s) = I(u_1)\, g(u_1) + \ldots + I(u_k)\, g(u_k).$$

Two-dimensional curve.

2.1. Consider the continuous closed curve C,

$$x = f(u), \quad y = g(u) \quad (0 \leqslant u \leqslant 1),$$

$$f(1) = f(0), \quad g(1) = g(0).$$

The points of this curve form a closed set of points $E(C)$. If s, t is not on $E(C)$ it is at a finite distance away, and we can choose continuous functions $r(u)$, $\psi(u)$, so that

$$f(u) - s = r \cos\psi, \quad g(u) - t = r \sin\psi,$$

$$r^2 = (f-s)^2 + (g-t)^2 \quad (r \geqslant \delta > 0).$$

Since C is a closed curve,

$$\cos\psi(1) = \cos\psi(0), \quad \sin\psi(1) = \sin\psi(0),$$

$$\psi(1) - \psi(0) = 2k\pi,$$

where k is a positive or negative integer or zero and is independent of the initial choice of ψ at a particular point of C. The function equal to zero on $E(C)$ and equal to k elsewhere we will call $\phi(s, t)$.

The complement of $E(C)$ is an at most countable set of open connected region σ_i, in each of which $\phi(s, t)$ is continuous and an integer and therefore constant. Let us call it ϕ_i, and let the measure of σ_i be called σ_i.

2.2. If a closed curve is the "sum" of several others in the sense that common but reversed parts of distinct curves can be wiped out on addition we see that, except on the curves themselves, $\phi(s, t)$ is additive for such sums of curves. Now for a triangle $\phi = 0$ except within the triangle, and there it is $+1$ or -1 according to the direction round the triangle. Thus for a triangle,

$$\int \phi(s, t)\, ds\, dt = \pm\sigma = \int x\, dy = -\int y\, dx$$

taken round the triangle. Hence, for any polygon,

$$\Sigma_i \phi_i \sigma_i = \int \phi(s, t)\, ds\, dt = -\int y\, dx.$$

2 . 3. *Definition. The pair of continuous functions*

$$x = f(u), \quad y = g(u) \quad (0 \leqslant u \leqslant 1)$$

is said to form an extended Stieltjes pair if the change in ψ *(as defined in* § 2 . 1*) along the curve from* 0 *to* 1 *is integrable over the values of* s, t *contained in a rectangle* T *completely enclosing the curve and if, in addition, the set of points forming this curve is of zero measure.*

If P_0N_0 is the ordinate at $f(0)$, $g(0)$, and P_1N_1 that at $f(1)$, $g(1)$, while N_0N_1 is the segment of the s-axis between these two ordinates, we *define*

$$\int_0^1 g(u)\, df(u) = -\Sigma_i \phi_i \sigma_i = -\int \phi(s, t)\, ds\, dt,$$

where ϕ is defined, as in § 2 . 1, for the closed curve $P_0P_1N_1N_0P_0$ consisting of the curve itself closed by means of the polygonal curve $P_1N_1N_0P_0$.

2 . 4. Within the rectangle T the change in ψ along $P_1N_1N_0P_0$ cannot exceed 2π numerically, so that, if the conditions of § 2 . 3 are satisfied, $\Sigma \phi_i \sigma_i$ is an absolutely convergent series for the curve thus closed and the definition of the integral is consistent. Furthermore, if we close the curve by any polygonal curve P_1QP_0,

$$\int g(u)\, df(u) = -\int \phi_1(s, t)\, ds\, dt + \text{algebraic area of } P_0N_0N_1P_1QP_0,$$

where ϕ_1 refers to the new closed curve.

2 . 5. In particular, if we interchange the rôles of the two axes and close by means of P_1K_1 and K_0P_0 parallels to the x-axis and K_1K_0 a segment of the y-axis,

$$\begin{aligned}\int g(u)\, df(u) &= -\int f(u)\, dg(u) + \text{algebraic area of } P_0N_0N_1P_1K_1K_0P_0 \\ &= -\int f(u)\, dg(u) + f(1)\, g(1) - f(0)\, g(0).\end{aligned}$$

This is the formula for *integration by parts.*

2 . 6. We now return to the case of a closed curve and take up a different point of view.

The line $s = f(u) =$ constant cuts the curve in a closed set of points $E(s)$ and the solutions for u of $f(u) = s$ form a closed set whose complement is an at most countable set of open intervals $\delta_1, \delta_2, \ldots$. Let

$\delta = (u_1, u_2)$ $(u_1 < u_2)$ be one of these, and let $\theta = (t_1, t_2)$ be the *open* interval where $t_1 = g(u_1)$, $t_2 = g(u_2)$. To θ can be assigned a number $\lambda(\theta)$ which is $+\frac{1}{2}$ or $-\frac{1}{2}$ according to whether the sign of $f(u)-s$ throughout δ is of the same sign as t_2-t_1 or of the opposite sign. If $t_1 = t_2$ the open interval will disappear. Any point s, t (s fixed) on the complement of $E(s)$ can belong only to a finite number of θ intervals. For if it should belong to an infinite number there will be a corresponding infinity of δ intervals, and among these there will be a sequence of intervals such that both end-values of u tend to a single limit u_1, let us say. Then $f(u_1) = s$, $g(u_1) = t$, which is contrary to the hypothesis. Hence to every point of this complementary set $CE(s)$ we can assign a number $B(s, t)$, which is the sum of the values of $\lambda(\theta)$ for the intervals θ to which s, t belongs. Moreover, the curve is closed, so that the number of such θ intervals is even and $B(s, t)$ will be an integer (positive, negative, or zero). On $E(s)$ itself we define $B(s, t)$ to be 0. Then

$$B(s, t) = \phi(s, t),$$

as defined in § 2.1. For whenever $\lambda(\theta) = +\frac{1}{2}$ (or $-\frac{1}{2}$) ψ increases (decreases) by the angle π if t belongs to θ, while, if t does not belong to θ, ψ returns to its original value, and the total change in ψ over the remaining (finite set of) intervals in u is 0. To any interval, Δ_j say, of the complement of $E(s)$ and to an interval θ_i corresponds a fraction m_{ij} equal to the ratio of the length of Δ_j to that of θ_i if Δ_j is included in θ_i and equal to 0 otherwise. The integral of ϕ with respect to t over Δ_j will be

$$-A_j(s) = -\Sigma_i m_{ij}[r(a_i)\, g(a_i) + l(b_i)\, g(b_i)],$$

where δ_i is the interval $a_i < u < b_i$, $r(a) = \frac{1}{2}$ or $-\frac{1}{2}$ according as $f(u) > s$ or $< s$ when $u > s$ ($u-s$ sufficiently small) and $l(b) = \pm\frac{1}{2}$ according as $f(u) \lesseqgtr s$ when $u < s$. This sum is absolutely convergent, since m_{ij} is 0 except for a finite number of values of i.

If $f(u)$, $g(u)$ form an extended Stieltjes pair, $\phi(s, t)$ is integrable over t for nearly all s and

$$\int g(u)\, df(u) = -\iint \phi(s, t)\, ds\, dt = \int \Sigma_j A_j(s)\, ds,$$

and since $|A_j(s)|$ is the integral of $|\phi|$ over Δ_j, $\Sigma_j A_j(s)$ is absolutely convergent for nearly all s.

Conversely, if the points of the closed curve $s = f(u)$, $t = g(u)$ form a set of plain measure zero, f, g form an extended Stieltjes pair if

$$\int \Sigma_j |A_j(s)|\, ds$$

is finite.

Finally, if $f(u)$ is of limited variation and $\omega(s)$ is finite and equal to $N(s)$, the set $E(s)$ consists of the images of a finite set of u-points, and at any of these, say u_i,

$$2r(u_i) = 2l(u_i) = r(u_i) + l(u_i) = I(u_i).$$

Then u_i is a point a_i for one δ interval, b_i for another, and

$$\Sigma_j A_j(s) = \Sigma_i I(u_i)\, g(u_i) = A(s),$$

the expression in § 1 . 7. Also

$$\Sigma_j |A_j(s)| \leqslant \Sigma_i |I(u_i)|\, |g(u_i)| = \omega(s) \times \max |g|.$$

2 . 7. THEOREM. *When $f(u)$ is of limited variation the extended Stieltjes integral defined in § 2 . 3 exists and coincides with the ordinary Stieltjes integral.*

Proof. The proof has been practically given in § 2 . 6. It only remains to add that the ordinary Stieltjes integral $\int g\,df$ over the broken line $P_1N_1N_0P_0$ which completes the closed curve, must be zero, since over P_1N_1 and N_0P_0 f is constant, while over N_1N_0 g is 0.

It follows as a corollary that, if f, g forms a continuous closed rectifiable curve, our definition of $\int g\,df$ coincides with the usual one and is what is ordinarily defined as the "algebraic area" of that curve except for the minus sign required by the usual two-dimensional convention.

2 . 8. THEOREM. *If $u = u(\lambda)$ $0 \leqslant \lambda \leqslant 1$ is any continuous function of λ, then*

$$\int_{u(0)}^{u(1)} g(u)\, df(u) = \int_0^1 g[u(\lambda)]\, df[u(\lambda)],$$

in the sense that both integrals exist in the extended Stieltjes sense if either does and they are then equal.

Proof. This follows from the fact that in the definition we depend only on (*a*) the set of points through which the curve passes and (*b*) the total change in ψ as we pass along the curve.

In particular, if $g(u) = F[f(u)]$,

where F is a single-valued continuous function defined for the interval of values of $f(u)$,

$$\int_0^1 g(u)\, df(u) = \int_{t_0}^{t_1} F(t)\, dt,$$

where $t_0 = f(0), \quad t_1 = f(1).$

From the form of the integral $\int \Sigma_j A_j(s)\, ds$,

$$A_j(s) = \Sigma_i m_{ij}[r(a_i)\, g(a_i) + l(b_i)\, g(b_i)],$$

it follows that if $(f,\ g_1)$, $(f,\ g_2)$ form extended Stieltjes pairs so does $(f,\ g_1,$ and $g_2)$ and

$$\int (g_1+g_2)\, df = \int g_1\, df + \int g_2\, df.$$

From the formula of integration by parts it is also true that

$$\int g\, d(f_1+f_2) = \int g\, df_1 + \int g\, df_2.$$

Similarly, if c is a constant,

$$\int (cg)\, df = c\int g\, df = \int g\, d(cf).$$

But in this extended integral the limit of an integral will not, in general, equal the integral of the limit. This equality would depend on inequalities which are no longer valid.

2 . 9. THEOREM. *If the angle ψ, defined in* § 2 . 1, *for the continuous closed curve $x = f(u)$, $y = g(u)$ is of limited variation for one point s_0, t_0 not on the curve, then $f(u)$, $g(u)$ form an extended Stieltjes pair and*

$$\int g(u)\, df(u) = -\tfrac{1}{2}\int r^2 d\psi.$$

Proof. Since s_0, t_0 is not on the curve we can choose $\delta > 0$, so that no point of the curve is within the circle of radius δ with centre at s_0, t_0. Since ψ is of limited variation,

$$\tfrac{1}{2}\int r^2 d\psi = \tfrac{1}{2}\int A(\sigma)\, d\sigma,$$

$$A(\sigma) = I(u_1)\, r^2(u_1) + \ldots + I(u_k)\, r^2(u_k),$$

where $u_1,\ u_2,\ \ldots,\ u_k$ are the solutions of $\psi(u) = \sigma$.

The angle ψ ranges through a finite interval, and we can add together the integrals with respect to σ over the ranges

$$-2m\pi \text{ to } -2(m-1)\,\pi,\ \ldots,\ 2(l-1)\,\pi \text{ to } 2l\pi.$$

If we put in each of these $\psi = 2j\pi + \theta$ $(0 \leqslant \theta < 2\pi)$ we can express the integral in the form

$$\tfrac{1}{2}\int r^2 d\psi = \int c(\theta)\, d\theta,$$

$$c(\theta) = \tfrac{1}{2}\Sigma_j[\Sigma_i I(u_{ij})\, r^2(u_{ij})].$$

On the other hand, at any point (ρ, θ) on a radius θ $(\neq 0)$ for which $\Sigma_j N(2j\pi+\theta)$ is finite, $\phi(s, t)$ is equal (except on the curve itself) to

$$\phi(s, t) = \Sigma_{ij} I(u_{ij})\, \eta(r_{ij}-\rho),$$

where $\eta(x) = 1$ if $x > 0$ and zero if $x \leqslant 0$. Then

$$|\phi(s, t)| \leqslant \Sigma_j N(2\pi j+\theta),$$

$$\int |\phi|\, \rho\, d\rho < \Sigma_j N(2\pi j+\theta) \times \max r^2,$$

and the integral $\int |\phi|\, \rho\, d\rho\, d\theta$ exists. Hence f, g form an extended Stieltjes pair. Again

$$\int \phi\rho\, d\rho = \Sigma_{ij} I(u_{ij}) \int_0^{r_{ij}} \rho\, d\rho = \tfrac{1}{2}\Sigma_{ij} I(u_{ij})\, r_{ij}^2 = C(\theta).$$

Therefore
$$\int g(u)\, df(u) = -\int \phi(s, t)\, ds\, dt = -\int \phi(s, t)\, \rho\, d\rho\, d\theta$$
$$= -\int C(\theta)\, d\theta = -\tfrac{1}{2}\int r^2 d\psi.$$

W. H. Young* defines the area of the curve as the limit of

$$\tfrac{1}{2}\Sigma\left[x(u_{i-1})\, y(u_i) - x(u_i)\, y(u_{i-1})\right]$$

if this limit exists for all dense net-systems on $a \leqslant u \leqslant b$. J. C. Burkill† in his Theorem 3.1 states that if the angle ψ, measured from a point so far outside the curve that the whole curve is within a quadrant taking this point as origin, is of limited variation the curve has an area in his sense. It can be shown that this is also true if ψ is measured from *any point* not on the curve. Our previous theorem shows that in this latter case the area exists also in our sense (and the two areas are equal). In many cases, however, the two definitions will not be equivalent. By Theorem 2.8 we can replace u by a non-monotonic transformation of u without altering our area. With Young's definition the order of u is essential. On the other hand, Young's area depends only on the existence of a limit, while ours depends on $|\phi|$ for its existence.

Burkill gives an example‡ of a simple closed curve which does not possess an area in Young's sense. Its area, according to our definition, is given immediately.

* W. H. Young, "On a formula for an area", *Proc. London Math. Soc.* (2), 18 (1919), 339–345 (344).

† J. C. Burkill, "The expression of an area as an integral", *Proc. London Math. Soc.* (2), 22 (1923), 311–336 (323).

‡ Burkill, *loc. cit.*, 327, Example 2.

2.91. *Definition.* We can define the *absolute area of the closed curve C* as

$$\int |\phi(s, t)|\, ds dt.$$

It follows that $f(u)$, $g(u)$ form an extended Stieltjes pair if, and only if, they define a finite absolute area for the curve (closed, if necessary, by $P_1 N_1 N_0 P_0$).

2.92. THEOREM. *If $f(u)$ is of limited variation V, and f, g form a closed curve C, the absolute area of C does not exceed*

$$\tfrac{1}{2}(\max g - \min g)\, V.$$

Proof. By § 2.6, the absolute area of C is

$$\int \Sigma_j |A_j(s)|\, ds \leqslant \max |g| \times \int \omega(s)\, ds = \max |g| \times V.$$

But, for a closed curve, the absolute area is unchanged if we add a constant value throughout to g, and we can thus obtain the given inequality.

2.921. Similarly it can be shown that, in the case of Theorem 2.9, the absolute area of C does not exceed

$$\tfrac{1}{2} \max r^2 \times \text{variation of } \psi.$$

2.922. THEOREM. *The absolute area of a rectifiable closed curve does not exceed*

$$\tfrac{1}{2} \times \textit{diameter} \times \textit{length of curve.}$$

Proof. It is necessary to use only § 2.92, remembering that

$$\max g - \min g \leqslant \text{diameter},$$

$$\text{variation of } f \leqslant \text{length}.$$

The factor $\frac{1}{2}$ could probably be reduced.

2.93. THEOREM. *If A is the* absolute area *of a closed curve $f(u)$, $g(u)$, and A_n that of the closed curve $f_n(u)$, $g_n(u)$, then, if $f_n \to f$, $g_n \to g$ uniformly,*

$$A \leqslant \underline{\lim}\, A_n.$$

Proof. If s, t is not on the f, g curve there will be a stage beyond which it is not on the curve f_n, g_n, so that

$$\phi_n(s, t) = \phi(s, t) \quad \{n \geqslant n_0(s, t)\}.$$

Hence at points not on the curve, that is nearly everywhere,

$$\lim |\phi_n| = |\phi|.$$

The theorem then follows directly from the fact* that

$$\int |\phi| \, ds \, dt \leqslant \underline{\lim} \int |\phi_n| \, ds \, dt.$$

2 . 94. THEOREM. *If $f(u)$, $g(u)$ form a rectifiable closed curve C, and if C_n refers to its approximating polygon, then*

$$A = \lim A_n.$$

Proof. Let $u_{i-1}u_i$ be a mesh of a dense net-system laid on the range of u at the stage n. Let γ_i refer to the curve C from u_{i-1} to u_i closed by the returning chord belonging to C_n, but reversed. Given $\epsilon > 0$, we can choose n so great that

$$\max \operatorname{diam} \gamma_i < \epsilon,$$

$$\phi \text{ for } C_n = \phi \text{ for } C - \Sigma_i \phi \text{ for } \gamma_i,$$

$$A_n \leqslant A + \Sigma_i A(\gamma_i),$$

$$\leqslant A + \epsilon l,$$

where l is the length of C, using § 2 . 922. This inequality, together with § 2 . 93, completes the proof.

Two dimensions.

3 . 1. Let $$x = f(u, v), \quad y = g(u, v)$$

be defined and continuous in the square J $(a \leqslant u, v \leqslant b)$. This pair of functions can be said to determine a transformation from J to its image $H(J)$ of values of x and y.

The boundary of any rectangle R contained in J can be traced out as a simple closed curve C,

$$u = u(\lambda), \quad v = v(\lambda) \quad (0 \leqslant \lambda \leqslant 1),$$

$$u(1) = u(0), \quad v(1) = v(0),$$

and is to be taken in the usual counter clockwise direction. The H-image of this will be a continuous closed curve

$$x = x(\lambda) = f[u(\lambda), v(\lambda)], \quad y = y(\lambda) = g[u(\lambda), v(\lambda)],$$

whose points form a closed set $E(C)$.

* S. Pollard, "On Parseval's theorem", *Proc. London Math. Soc.* (2), 25 (1926), 237-246 (240).

As in § 2.1 and § 2.6 we can define, corresponding to this curve, a function $\phi(s, t)$ which is the change in $\psi(\lambda)$ as we pass round the curve divided by 2π. This function is B-measurable if we add the definition that on the curve itself $\phi = 0$.

If a rectangle R is divided up into a finite number of sub-rectangles $R_1, R_2, \ldots, R_p$, we can regard the boundary C of R as a "sum" of the boundaries $C_1, C_2, \ldots, C_p$ of the sub-rectangles each taken in the same order. If s, t is not on any of the image sets $E(C_1), \ldots, E(C_n)$, we can define ψ so as to be continuous over the whole set of curves and obtain the result that

$$\phi(s, t) = \phi_1(s, t) + \ldots + \phi_p(s, t).$$

On J lay a dense net-system having, at the stage n, the meshes K_i, and let H_0 be the image of the dividing lines of the net. At a stage n of the subdivision define

$$\omega_n(s, t) = \Sigma_i |\phi_i(s, t)|,$$

$$l_n(s, t) = \Sigma_i |\operatorname{sgn} \phi_i|.$$

Furthermore, ϕ for the whole of J satisfies

$$\phi(s, t) = \Sigma_i \phi_i(s, t).$$

Then l_n, ω_n form non-decreasing sequences, and have limits $l(s, t)$ and $\omega(s, t)$. Also

$$l(s, t) \leqslant \omega(s, t).$$

If $l(s, t)$ is finite there will be a stage in the division beyond which $\phi(s, t)$ will be zero for all except l meshes, and these l meshes will form l sequences defining l points $(u_1, v_1), \ldots, (u_l, v_l)$. To each of these corresponds a final number

$$I(u_1, v_1), \ldots, I(u_l, v_l),$$

such that, for all meshes beyond a certain stage enclosing (u_1, v_1), $\phi(s, t) = I(u_1, v_1)$. Hence

$$\omega(s, t) = \lim \omega_n(s, t) = |I(u_1, v_1)| + \ldots + |I(u_l, v_l)|,$$

$$\phi(s, t) = I(u_1, v_1) + \ldots + I(u_l, v_l).$$

It is clear that, for every point (u, v) whose s, t image has a finite value of l, the number $I(u, v)$ will be independent of the particular net-system used provided that s, t is not on the image of the bounding divisions of that net-system.

Taking the given square J we can reflect it, first in each of its sides, and then again to form effectively 9 squares, of which J is the centre

square. We can then define f, g to have the same values at corresponding points in these 9 squares. We can thus extend the definition outside J and obtain net-systems whose bounding lines do not include the boundaries of J. Points s, t can be classified as (i) those which lie on the images of the divisions of *all* net-systems, (ii) those for which $l(s, t)$ is infinite for all net-systems such that s, t does not lie on the images of their divisions, and (iii) those for which $l(s, t)$ has the same finite value for all appropriate net-systems. Points s, t of types (i) and (ii) can be called *intractable*. To points s, t of type (iii) will correspond l points (u, v) for which $I(u, v)$ is not zero and possibly other points for which

$$f(u, v) = s, \quad g(u, v) = t,$$

but for which $\phi(s, t)$ is zero for all sufficiently small meshes surrounding them. These points s, t and the corresponding points u, v will be called *tractable* points.

3.2. *Definition.* Tractable points u, v for which $I(u, v) = \pm 1$ we shall call *regular points* and also *points of simple positive (or negative) transformation.*

Those for which $|I(u, v)| > 1$ we shall call *branch points.*

Finally, those for which $I(u, v) = 0$ we can call *fold points.*

The nature of a point is independent of the particular net-system used, and if $I(u_1, v_1) = r$ for a tractable point u_1, v_1 then, when any simple continuous closed curve is traced round u_1, v_1 in the positive direction, the angle ψ for s_1, t_1 $[s_1 = f(u_1, v_1), t_1 = g(u_1, v_1)]$ will increase algebraically by $2\pi r$ provided that the curve is within a sufficiently small square and does not pass through any u, v for which

$$f(u, v) = s_1, \quad g(u, v) = t_1.$$

We can further reduce somewhat the intractable u, v points by saying that u_1, v_1 is *locally tractable* if a rectangle R can be found enclosing u_1, v_1 strictly, and such that u_1, v_1 would be tractable if J were replaced by R.

3.3. THEOREM. *If, for a rectangle R, $\phi(s, t)$ differs from 0, and if s, t is tractable locally with respect to R, then there is at least one regular point or branch point within R for which*

$$f(u, v) = s, \quad g(u, v) = t.$$

Proof. By the reasoning of § 3.1 there will be at least one point (u_0, v_0) such that, for any mesh sufficiently small enclosing it,

$\phi(s, t) = n \neq 0$. Hence, on the boundary of this mesh there is at least one point Q_1 at which $\psi \equiv 0 \bmod 2\pi$, so that at Q_1, $g(u, v) = t$; and at least one point Q_2 at which $\psi \equiv \pi/2 \bmod 2\pi$, so that at Q_2, $f(u, v) = s$. Taking a sequence of meshes tending to u_0, v_0 the points Q_1, Q_2 will both approach u_0, v_0. Since f, g are continuous

$$f(u_0, v_0) = s, \quad g(u_0, v_0) = t,$$

and this point is a regular point or a branch point.

3.4. Consider now a regular point or a branch point u_0, v_0 for which

$$I(u_0, v_0) = n \geqslant 1.$$

[The case in which $I = -n$, $n \geqslant 1$, is exactly parallel and need not be given special treatment.]

We proceed, as before, with the division until u_0, v_0 is within a square (or rectangle) R such that, within R, $\phi(s_0, t_0)$ is zero for any sub-mesh except one enclosing u_0, v_0 and such that s_0, t_0 is not on the image $H(C)$ of the boundary C of R.

Let $\delta > 0$ be the least distance between s_0, t_0, and points of $H(C)$, and let (s_1, t_1) be *any* tractable point whose distance from s_0, t_0 is less than δ. These two points can be joined by a continuous line which has no point on $H(C)$ and along it $\phi(s, t)$ is continuous. Therefore

$$\phi(s_1, t_1) = \phi(s_0, t_0) = n.$$

By § 3.3 there is at least one point (α, β) within R at which

$$f(\alpha, \beta) = s_1, \quad g(\alpha, \beta) = t_1.$$

From the interior of R cut out u_0, v_0, together with a mesh so small that the maximum distance of s_0, t_0 from the image of all points within or on it is less than

$$r_1 = \sqrt{[(s_1 - s_0)^2 + (t_1 - t_0)^2]}.$$

There remains a closed rectangular ring which we shall suppose cut so as to become simply connected. Throughout the ring

$$r = \sqrt{[(s - s_0)^2 + (t - t_0)^2]} \geqslant \eta > 0.$$

We can therefore define ψ as a continuous function of u, v, so that

$$f(u, v) - s_0 = r \cos\psi, \quad g(u, v) - t_0 = r \sin\psi.$$

Choose a point P_0 on the boundary of R and ψ_0 as one of the possible values of ψ. Since $\phi(s, t) = n$ there will be at least n points

$P_0, P_1, \ldots, P_{n-1}$ on the boundary such that at P_q $(q = 0, 1, \ldots, n-1)$,

$$\psi_q = \psi_0 + 2q\pi.$$

Define $$z = (f - s_0) + i(g - t_0), \quad i = \sqrt{(-1)},$$

and choose $\zeta_q = \xi_q + i\eta_q$, so that throughout the ring

$$\zeta_q^n = z,$$

and so that at P_q, $\arg \zeta_q = \psi_0/n$.

There will be no discontinuity in ζ_q at the cut, since ψ changes by $2n\pi$ and $\arg \zeta_q$ by 2π across the cut. At another of the points P_s $(s \neq q)$,

$$\arg \zeta_q = \frac{1}{n}\psi_0 + \frac{s-q}{n} 2\pi \neq \arg \zeta_s.$$

Therefore ζ_q, ζ_s can *never* be equal within the ring. Now define $\sigma_1 + i\tau_1$ as one solution of

$$(\sigma + i\tau)^n = (s_1 - s_0) + i(t_1 - t_0).$$

Within R, ξ_q, η_q form a continuous function pair, and since $\sqrt{(\sigma_1^2 + \tau_1^2)}$ is less than the least value of $|\zeta_q|$ on the boundary of R, the argument of

$$(\xi_q - \sigma_1) + i(\eta_q - \tau_1)$$

increases by 2π as we pass round R. Also as we pass round any sufficiently small curve enclosing (α, β) the change in the argument for $\xi + i\eta$ will be the same as that for $s + it$, so that if s_1, t_1 is tractable for f, g, so also is σ_1, τ_1 for ξ_q, η_q.

Hence, by § 3 . 3, there is, at least, one regular or branch point (u_q, v_q) for which

$$\xi_q(u_q, v_q) = \sigma_1, \quad \eta_q(u_q, v_q) = \tau_1,$$

and this point is not within the excluded core of the ring.

For different q and s the points (u_q, v_q), (u_s, v_s) cannot coincide, for if they did we should have $\zeta_q = \zeta_s$ at this point, which we have already shown to be impossible.

There are therefore at least n distinct points within R, at each of which

$$(f - s_0) + i(g - t_0) = \zeta_q^n = (s_1 - s_0) + i(t_1 - t_0),$$

that is, at which $$f(u, v) = s_1, \quad g(u, v) = t_1,$$

and which are regular points or branch points.

3 . 5. THEOREM. *If s_0, t_0 is a tractable point for which $\omega(s_0, t_0) = k$ there exists a circle with centre at (s_0, t_0) such that to every tractable*

point (s_1, t_1) within this circle except its centre, there correspond at least k points (u, v) for which

$$f(u, v) = s_1, \quad g(u, v) = t_1,$$

and which are regular points or branch points.

Proof. Proceed with the net-division until the l solutions of $f = s_0$, $g = t_0$ are separated and choose a mesh enclosing each in turn to be the R in the previous paragraph. For the radius of the circle of the theorem choose the least of the values of δ for these l meshes and use the reasoning for each given in § 3 . 4.

3 . 6. THEOREM. *The set of branch points is at most countable.*

Proof. Define ω and l as in § 3 . 1, and define

$$\omega - l = \rho,$$

and let $E(l, \rho)$ be the set of points s, t for which l and ρ have certain values. Now, if l_0 and ρ_0 are positive, each point of $E(l_0, \rho_0)$ is surrounded, according to § 3 . 5, by a circle such that, for all tractable points of this circle except its centre,

$$l(s, t) \geqslant l_0 + \rho_0 > l_0.$$

If
$$F(l_0) = E(l_0, 1) + E(l_0, 2) + \ldots,$$

we see that the points of $F(l_0)$ are separated from one another and form an at most countable set. The set of branch points is $F(1) + F(2) + \ldots$ and is also an at most countable set, so that the theorem is proved.

3 . 7. It is an immediate corollary that

$$\iint \omega(s, t)\, ds\, dt = \iint l(s, t)\, ds\, dt$$

in the sense that these two integrals are both infinite, or else both finite and equal (substituting 0 for l and ω whenever s, t is intractable).

3 . 8. *Definition.* The continuous function-pair

$$s = f(u, v), \quad t = g(u, v) \quad (0 \leqslant u,\ v \leqslant 1)$$

is said to determine a transformation of limited variation in the sense D if the set of intractable points (s, t) is of zero measure and if $l(s, t)$ is integrable with respect to s, t. We then define

$$V_D = \iint l(s, t)\, ds\, dt = \iint \omega(s, t)\, ds\, dt.$$

3.9. S. Banach* gives a definition of transformations of limited variation (in the sense B, let us say) if $N(s, t)$ is finite nearly everywhere and integrable, where $N(s, t)$ is the number of solutions of the equations

$$f(u, v) = s, \quad g(u, v) = t.$$

He also defines the variation

$$V_B = \iint N(s, t)\, ds\, dt.$$

Banach shows that if the conditions are satisfied it is possible to choose the net-system so that the image of its divisions is of zero measure.

From § 3.3 it follows that

$$l_n(s, t) \leqslant N_n(s, t),$$

where $N(s, t)$ is the number of meshes of stage n within or on the boundary of which there is at least one solution of

$$f(u, v) = s, \quad g(u, v) = t.$$

Hence $$l(s, t) = \lim l_n(s, t) \leqslant \lim N_n(s, t) = N(s, t).$$

It follows that if Banach's condition is satisfied so also is ours. The analogy of the case of one dimension suggests that $l = N$ nearly everywhere, that is, that the image of the fold points is of zero measure. This the author has been able neither to prove nor to disprove.

Two-dimensional integral.

4.1. Banach's function $N(s, t)$ cannot be used for more than a consideration of the total variation. Our functions $\phi(s, t)$, $I(u, v)$ enable us to define a Stieltjes integral of $h(u, v)$ with respect to the pair $f(u, v)$, $g(u, v)$.

Let $h(u, v)$ be continuous and $f(u, v)$, $g(u, v)$ form a pair of limited variation in the sense D. Then

$$\iint h(u, v)\, d[f(u, v), g(u, v)] = \iint A(s, t)\, ds\, dt,$$

$$A(s, t) = \Sigma_i r(u_i, v_i)\, h(u_i, v_i),$$

where $(u_1, v_1), \ldots, (u_l, v_l)$ are the $l = l(s, t)$ regular and branch-point solutions of

$$f(u, v) = s, \quad g(u, v) = t,$$

and where $$r(u_i, v_i) = \operatorname{sgn} I(u_i, v_i).$$

* S. Banach, *loc. cit.*

This definition is legitimate since $l(s, t)$ is finite nearly everywhere and since the image of the set of branch points is of zero measure. It therefore makes no difference whether we take $r(u_i, v_i)$ or $l(u_i, v_i)$ in the above definition.

It will be observed that

$$\iint h\, d[f, g] \leqslant \max|h| \times V_D.$$

The integral thus defined is a two-dimensional Stieltjes integral, which can be written in the form

$$\int h(u, v)\, dm(e),$$

if $m(e)$ is the mass-distribution such that, for a rectangle R,

$$m(R) = \iint \phi(s, t)\, ds\, dt,$$

$\phi(s, t)$ corresponding to the boundary of R.

This follows from the additive character of our integral and from the inequality stated above.

4.2. From the definition in § 2.3 it is clear that, provided that the simple curve C, $u = u(\lambda)$, $v = v(\lambda)$ is properly chosen, if $f(u, v)$, $g(u, v)$ is a pair of limited variation in sense D,

$$f[u(\lambda), v(\lambda)], \quad g[u(\lambda), v(\lambda)]$$

form an extended Stieltjes pair and

$$m(S) = -\int_C g[u(\lambda), v(\lambda)]\, df[u(\lambda), v(\lambda)],$$

where S is the region enclosed by C. In particular, if $f(u, v)$, $g(u, v)$ traces a rectifiable curve whenever $u(\lambda)$, $v(\lambda)$ traces a rectangle R, then

$$m(R) = \tfrac{1}{2}\int (f\, dg - g\, df).$$

This very special case has been considered by Nalli and Andreoli*, but it was considered earlier by W. H. Young†.

4.3. THEOREM. *If V is the variation (in our sense) within J of the pair $f(u, v)$, $g(u, v)$ and V_n that of f_n, g_n in J, then if $f_n \to f$, $g_n \to g$ uniformly in J,*

$$V \leqslant \underline{\lim}\, V_n.$$

---

* P. Nalli and G. Andreoli, "Sull' area di una superficie, etc.", *Rend. di Palermo*, 52 (1928), 30.

† W. H. Young, *loc. cit.*

Proof. Lay a dense net-system on J and consider the stage m of this subdivision. Let K_i refer to a typical mesh at this stage. Then, by § 2 . 93,

$$\begin{aligned}\int \omega_m \, ds\, dt &= \Sigma_i \int |\phi_i| \, ds\, dt \leqslant \Sigma_i \varliminf \int |\phi_{in}| \, ds\, dt \\ &\leqslant \varliminf \Sigma_i \int |\phi_{in}| \, ds\, dt \\ &= \varliminf \int \omega_{m,n} \, ds\, dt.\end{aligned}$$

But
$$\omega_{m,n} \leqslant \lim_{m=\infty} \omega_{m,n} = \omega_n.$$

Therefore
$$\begin{aligned}\int \omega_m \, ds\, dt &\leqslant \varliminf \int \omega_n \, ds\, dt \\ &= \varliminf V_n.\end{aligned}$$

We now make m increase indefinitely and the theorem is proved.

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

Bailey, W. N. Transformations of generalized hypergeometric series, 495.

——————— Some identities involving generalized hypergeometric series, 503.

Berwick, W. E. H. On soluble sextic equations, 1.

Bessel-Hagen, E. Bemerkungen zur Behandlung des major Arc bei der Anwendung der Hardy-Littlewood'schen Methode auf das Waring'sche Problem, 382.

Cooper, R. The behaviour of certain series associated with limiting cases of elliptic theta-functions (II), 111.

Daniell, P. J. Transformations of limited variation, 537.

Estermann, T. On the representation of a number as the sum of three products, 453.

Ferrar, W. L. Conditionally convergent double series, 322.

Fox, C. A generalization of the Fourier-Bessel integral transform, 401.

Grimshaw, M. E. A contribution to the theory of uniqueness of representation by trigonometrical integrals, 243.

Hardy, G. H., and Littlewood, J. E. The approximate functional equation for $\zeta(s)$ and $\zeta^2(s)$, 81.

Hill, M. J. M. Determinantal form of the formula for expressing a rational integral function of a single variable in terms of two other such functions which are mutually prime, 490.

Hilton, H. On Darboux lines, 517.

Hobson, E. W. The addition theorems for general Legendre functions, 355.

Hodge, W. V. D. Linear systems of algebraic curves on a plane and on a cone, 145.

Love, A. E. H. Biharmonic analysis, especially in a rectangle, and its applications to the theory of elasticity, 189.

Mulholland, H. P. Some theorems on Dirichlet series with positive coefficients and related integrals, 281.

Pepper, Echo D. On density distribution in stellar space, 98.

Poole, E. G. C. Dirichlet's principle for a flat ring, 342.

Proudman, J. On a general expansion in the theory of the tides, 527.

Stanley, G. K. On the representation of a number as a sum of squares and primes, 122.

Steen, S. W. P. On Fermat's last theorem, 331.

Temple, G. The computation of characteristic numbers and characteristic functions, 257.

Titchmarsh, E. C. On conjugate functions, 249.

Watson, G. N. Theorems stated by Ramanujan (V): Approximations connected with e^x, 293.

Wishart, J. A problem in combinatorial analysis giving the distribution of certain moment statistics, 309.

Wilton, J. R. A series of Bessel functions connected with the theory of lattice points, 168.

Wood, F. M. Reciprocal integral formulae, 29.

Young, R. C. On "Riemann" integration with respect to an additive function of sets, 479.

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Algebra: Berwick, W. E. H., 1; Hill, M. J. M., 490.
Approximations: Watson, G. N., 293.
Bessel Functions: Fox, C., 401; Wilton, J. R., 168.
Biharmonic Analysis: Love, A. E. H., 189.
Characteristic Numbers and Functions: Temple, G., 257.
Combinatorial Analysis: Wishart, J., 309.
Dirichlet Series: Mulhollaud, H. P., 281.
Dynamics, Theoretical: Proudman, J., 527.
Elasticity: Love, A. E. H., 189.
Equations of Mathematical Physics: Love, A. E. H., 189; Poole, E. G. C., 342.
Equations, Theory of: Berwick, W. E. H., 1.
Exponential Function: Watson, G. N., 293.
Fermat's Last Theorem: Steen, S. W. P., 331.
Fourier Series and Integrals: Grimshaw, M. E., 243; Titchmarsh, E. C., 49; Wood, F. M., 29.
Functional Equations: Hardy, G. H., and Littlewood, J. E., 81.
Functions of a Complex Variable: Cooper, R., 111.
Functions of a Real Variable: Daniell, P. J., 537; Fox, C., 401; Titchmarsh, E. C., 49; Wood, F. M., 29; Young, R. C., 479.
Geometry, Analytical: Hilton, H., 517.
Geometry, Synthetic: Hodge, W. V. D., 145.
Harmonic Analysis: Love, A. E. H., 189; Poole, E. G. C., 342.
Hypergeometric Functions: Bailey, W. N., 495, 503; Fox, C., 401.
Integral Calculus: Daniell, P. J., 537; Grimshaw, M. E., 243; Young, R. C., 479.
Integral Equations: Temple, G., 257.
Lattice Points: Wilton, J. R., 168.
Legendre Functions: Hobson, E. W., 355.
Numbers, Theory of: Bessel-Hagen, E., 382; Estermann, T., 453; Stanley, G. K., 122; Steen, S. W. P., 331.
Probability: Pepper, Echo D., 98.
Series, Theory of: Cooper, R., 111; Ferrar, W. L., 322; Fox, C., 401; Mulholland, H. P., 281.
Statistics: Pepper, Echo D., 98; Wishart, J., 309.
Theta Functions: Cooper, R., 111.
Tides, Theory of: Proudman, J., 527.
Trigonometrical Series and Integrals: Grimshaw, M. E., 243; Titchmarsh, E. C., 49; Wood, F. M., 29.
Waring's Problem: Bessel-Hagen, E., 382.
Zeta Function: Hardy, G. H., and Littlewood, J. E., 81.

CORRIGENDUM: C. Fox.

A generalization of the Fourier-Bessel integral transform:

Page 449, line 7, *for* $-\frac{1}{2} \leqslant 2\alpha + \nu < \frac{5}{2}$ *read* $-\frac{1}{2} \leqslant 2\alpha + \nu < \frac{3}{2}$.

BIBL.
LONDIN.
UNIV.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 29.—PART 1.

CONTENTS.

	PAGE
ON SOLUBLE SEXTIC EQUATIONS: W. E. H. BERWICK	1
RECIPROCAL INTEGRAL FORMULAE: F. M. WOOD	29
ON CONJUGATE FUNCTIONS: E. C. TITCHMARSH	49

Issued December 10th, 1928.

LONDON:

PRINTED BY C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.4.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of two guineas is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is two guineas. The subscription due from a newly elected member for his first year of membership is one guinea if his election takes place between March and June inclusive: any Member may compound for his *annual* subscriptions by the payment of twenty-five guineas in one sum, or in two equal annual instalments. In certain cases the composition fee is reduced. Members of the Society who are also members of the American Mathematical Society are entitled to a reduction of subscription as regards one of the Societies. Details of the arrangements may be obtained from the Secretaries or the Treasurer of either Society.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the first Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County, Westminster and Parr's Bank, 1 St. James's Square, London, S.W.1, or to the Treasurer, Dr. A. E. Western, 35 Essex Street, Strand, London, W.C.2.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting in November.

The Officers are a President, Vice-Presidents, a Treasurer, a Librarian, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1928–29 are on November 8 and December 13, 1928, and January 10, February 14, March 14, April 25, May 16, and June 13, 1929. Cards giving the Programmes of the meetings can be had by Members a few days beforehand on application to C. F. Hodgson & Son, Ltd., 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings* or the *Journal*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers of more than about eight pages in length intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained, for the use of the Council and Referees, not for publication. Such an Abstract should be extremely brief; as a rule its length should not exceed the length of half a page of the *Proceedings*.

Communications for the Secretaries may be forwarded to them at the following addresses:—

The University, Birmingham—G. N. Watson, Sc.D., F.R.S.
St. John's College, Cambridge—F. P. White, M.A.
London Mathematical Society, Burlington House, W.1.

PRICES OF BOUND VOLUMES, PARTS, &c., REVISED JULY 1st, 1920.

First Series (cloth, 8vo).*

Vol.	†I.	Nos.	1-8	...	Jan.	1865 to Nov.	1866	...	20s.
	†II.	,,	9-20	...	Nov.	1866 ,, ,,	1869	...	25s.
,,	III.	,,	21-40	...	,,	1869 ,, ,,	1871	...	30s.
,,	IV.	,,	41-66	...	,,	1871 ,, ,,	1873	...	45s.
,,	V.	,,	67-75	...	,,	1873 ,, ,,	1874	...	25s.
,,	VI.	,,	76-88	...	,,	1874 ,, ,,	1875	...	30s.
,,	VII.	,,	89-103	...	,,	1875 ,, ,,	1876	...	30s.
,,	VIII.	,,	104-123	...	,,	1876 ,, ,,	1877	...	30s.
,,	IX.	,,	124-140	...	,,	1877 ,, ,,	1878	...	30s.
,,	X.	,,	141-155	...	,,	1878 ,, ,,	1879	...	25s.
,,	XI.	,,	156-165	...	,,	1879 ,, ,,	1880	...	17s. 6d.
,,	XII.	,,	166-179	...	,,	1880 ,, ,,	1881	...	22s. 6d.
,,	XIII.	,,	180-194	...	,,	1881 ,, ,,	1882	...	25s.
,,	XIV.	,,	195-214	...	,,	1882 ,, ,,	1883	...	30s.
,,	XV.	,,	215-232	...	,,	1883 ,, ,,	1884	...	30s.
,,	XVI.	,,	233-252	...	,,	1884 ,, ,,	1885	...	30s.
,,	XVII.	,,	253-279	...	,,	1885 ,, ,,	1886	...	35s.
,,	XVIII.	,,	280-304	...	,,	1886 ,, ,,	1887	...	35s.
,,	XIX.	,,	305-342	...	,,	1887 ,, ,,	1888	...	42s.
,,	XX.	,,	343-369	...	,,	1888 ,, ,,	1889	...	35s.
,,	XXI.	,,	370-398	...	,,	1889 ,, ,,	1890	...	35s.
,,	XXII.	,,	399-429	...	,,	1890 ,, ,,	1891	...	40s.
,,	XXIII.	,,	430-449	...	,,	1891 ,, ,,	1892	...	30s.
,,	XXIV.	,,	450-474	...	,,	1892 ,, ,,	1893	...	30s.
,,	XXV.	,,	475-499	...	,,	1893 ,, ,,	1894	...	30s.
,,	XXVI.	,,	500-534	...	,,	1894 ,, ,,	1895	...	42s.
,,	XXVII.	,,	535-574	...	,,	1895 ,, ,,	1896	...	50s.
,,	XXVIII.	,,	575-611	...	,,	1896 ,, ,,	1897	...	45s.
,,	XXIX.	,,	612-657	...	,,	1897 ,, ,,	1898	...	Pt. I. 30s. Pt. II. 30s.
,,	XXX.	,,	658-678	...	,,	1898 ,, Mar.	1899	...	30s.
,,	XXXI.	,,	679-709	...	April	1899 ,, Dec.	1899	...	30s.
,,	XXXII.	,,	710-737	...	Jan.	1900 ,, June	1900	...	30s.
,,	XXXIII.	,,	738-763	...	Nov.	1900 ,, Feb.	1901	...	30s.
,,	XXXIV.	,,	764-789	...	Mar.	1901 ,, April	1902	...	30s.
,,	XXXV.	,,	790-819	...	May	1902 ,, Jan.	1903	...	30s.

Postage 1s. per volume extra.

* *Complete Index* (112 pages), Vols I-XXX, 5s. (by post, 5s. 3d.).

† Vols. I and II of the first Series will not be sold except to purchasers of not less than twenty volumes of that Series.

Second Series (cloth, Super Royal 8vo, about 450 pages).

Vol. I of the Second Series was issued in 1903, since when 27 volumes have been issued to date (1928). Price 35s. per Volume (by post, 36s.).

N.B.—The public may subscribe to the current Volume of "Proceedings"—35s. payable in advance, and can receive it in Parts *(about five to Volume)* as issued, *or, when ready, as a bound Volume—post free.*

Separate Parts, 10s. each (by post, 10s. 6d.).

Special Discount of 5% is allowed to purchasers of complete sets of "Proceedings," otherwise the prices quoted above are net.

Binding Cases (either Series), 2s. each (by post, 2s. 3d.).

List of Members (current issue only), 5s. each (by post, 5s. 3d.).

N.B.—Members may purchase separate Parts or Volumes to complete their own sets of "Proceedings" at a discount of 10% off the above prices.

FRANCIS HODGSON (Publisher to the London Mathematical Society), 89 Farringdon Street, E.C.4.
Telephone: City 4871. Bankers: Barclays.

Reprint of Authors' Papers.

Authors of Papers printed in the *Proceedings* are entitled to receive, free of cost, up to 50 separate copies of their communications, stitched in a paper cover.* They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	75 copies.†	100 copies.†	125 copies.†
	s. d.	s. d.	s. d.
4 pages	4 6	6 6	8 0
8 ,, ...	6 0	8 6	10 6
12 ,, ...	9 6	14 0	17 0
16 ,, ...	10 6	16 0	20 0
32 ,, ...	18 6	28 0	34 6

* Copies of Papers not exceeding 4 pp. are supplied without covers.

† Including 50 Free Copies.

If **any** *separate copies are required, an Order Form must be filled in and returned to the Printers with proofs.*

Out-of-Print Books & Others

¶ We have a very efficient department for finding the unusual, the rare, and the out-of-print book.

¶ We have a very highly organized department for the supply to Librarians, Institutions, and others, of New Books and Periodicals, both English and Foreign. We can guarantee promptitude and intelligent handling of all orders entrusted to us.

Librarians, Executors, and others, desirous of either disposing of or buying Scientific Books or Periodicals, or Books in General Literature, are invited to communicate with us. Valuers for Probate. Catalogues frequently issued and sent post free.

Catalogue 309.—Scientific Books and Journals.

We will purchase the following:

BOOKS:

Archimedes, all Works by.
Bromwich. Infinite Series.
Cayley. Mathematical Papers.
Euclid, Geometrie, Billingsley.
Galilei, all Works by.
Gilbert. On the Magnet. All editions.
Gow. Greek Mathematics.
Green. Mathematical Papers.
Hamilton. Quaternions.
Harkness & Morley. Functions.
Maxwell. Scientific Papers.
Montucla. Histoire des Mathématiques.
Napier (John), all Books by.
Newton. Opera Omnia.
„ Principia. All editions.
Oppolzer. Bahnbestimmung der Planeten.
Recorde, all Books by.
Russell. Principles of Mathematics.
Salmon. Higher Algebra.
„ Higher Plane Curves.
Stokes. Mathematical Papers.
Todhunter. Theory of Probability.
„ Laplace's Functions.
„ Elasticity.
„ Attraction.
Wallis, Opera Mathematica.

PERIODICALS: Complete Sets or Long Runs.

Acta Mathematica.
Annalen der Physik.
Archiv der Mathematik und Physik.
Astrophysical Journal.
Cambridge Philosophical Society Transactions.
Comptes Rendus de l'Académie.
Jahrbuch der Mathematik.
Journal de Mathématiques. Liouville.
Journal de Physique.
Journal für Mathematik. Crelle.
Journal (Quarterly) of Mathematics.
LONDON MATHEMATICAL SOCIETY PROCEEDINGS.
Mathematical Gazette.
Maths. from "The Educational Times."
Mathematische Annalen.
Mathematische Encyklopädie.
Mathematische Zeitschrift.
Messenger of Mathematics.
Philosophical Magazine.
Physical Society Proceedings.
R.S. Philosophical Transactions.
Science Abstracts. Series A and B.
Zeitschrift für Mathematik und Physik.

And all standard Mathematical Publications.

W. HEFFER & SONS, Ltd., CAMBRIDGE ENGLAND.

Telephone: 862. Telegrams and Cables: "Heffer, Cambridge."

1956-1
1957